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On Equality and Inequality in Stationary Populations

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Abstract. Although it is an analytic construct important in its own right, a stationary population is an integral component of a life table. Using this perspective, we discuss well-known and not-so-well known equalities that are found a stationary population as well as a set of inequalities. There are two parts to the set of inequalities we discuss. The first (theorem 1) is that at any given age x, the sum of mean years lived and mean years remaining exceeds life expectancy at birth when x is greater than zero and less than the maximum lifespan (When x = zero or x =maximum lifespan, then the sum of mean years lived and mean years remaining is equal to life expectancy at birth). The second inequality (theorem 2) is a generalization of the first, namely that for the entire population, the sum of mean years lived and mean years remaining exceeds life expectancy at birth. It may be that the inequality we identify as Theorem 1 is common knowledge in some circles. However, we have found no formal description of it and believe that Theorem 1 represents a contribution to the literature. Similarly, it may be the case that one would expect that Theorem 2 would hold, given Theorem 1, but we also have not found a formal description of this in the literature and believe that it also represents a contribution. Finally, we note we have not found any discussion of an equality we found embedded in Theorem 1 (when age = 0 and when age = ω, then $\lambda_x + e_x = e_0$) and believe that the identification of this equality represents a contribution. We provide illustrations of the two inequalities and discuss them as well as selected equalities.

Keywords. Carey's Equality Theorem, Two Inequality Theorems, Mean years lived, mean years remaining, life expectancy at birth, sum or mean years lived and mean years remaining, mean age at death, variance in age at death

1. INTRODUCTION

Although many of them are apparent and some that are not so apparent have been described, equalities represent a defining characteristic of stationary populations (Kintner 2004). In addition to the obvious equalities such as the crude birth rate and crude death rate, research has revealed that: (1) mean years lived is equal to mean years remaining; and (2) the distribution of age composition is equal to the distribution of remaining lifetimes(Carey et al. 2008; Rao and Carey 2014, Vaupel 2009). To these equalities, the following can be added: (1) mean age is equal to mean years lived (Rao and Carey 2014); and (2) mean age is equal to mean years remaining (Kim and Aron 1989).

As we show in this paper, mean age can be expressed as a function of total years lived by the stationary population and its life expectancy at birth, which implies that for a given stationary population, its mean age can be expressed as a function of its crude birth rate as well as its crude death rate. In turn, because mean age is equivalent to mean years lived and mean years remaining, it also can be expressed as a function of total years lived and, respectively, life expectancy at birth, the crude birth rate and the crude death rate.

To these equalities, we add a set of inequalities by demonstrating: (1) that at any given age x, the sum of mean years lived and mean years remaining exceeds life expectancy at birth in a given stationary population, where $0 < x < \omega$ (maximum lifespan); and (2) that for a stationary population as a whole, the sum of mean years lived and mean years remaining exceeds life expectancy at birth. We discuss this set of inequalities and provide an empirical illustrations of them.

Before proceeding, it is worth noting that while a stationary population is an analytic construct in its own part, it is an integral component of a life table [1]. As such, the equalities and inequalities we identify and discuss apply to life tables and their construction. As our main findings, we offer: (1) Theorem 1 and provide a proof for it that shows that for a given age x, the sum of mean years lived ($\boldsymbol{\lambda_x}$) and mean years remaining ($\boldsymbol{e_x}$) exceeds life expectancy at birth where $0 < x < \omega$; (2) Theorem 2 as a generalization of Theorem 1 to all ages and provide a proof for it; and (3) an equality we found embedded in Theorem 1, namely that when age = 0 or when age = ω, then $\boldsymbol{\lambda_x} + \boldsymbol{e_x} = \boldsymbol{e_0}$.

1.1 Equalities in a Stationary Population

Let the size of a stationary population be $\boldsymbol{T_o}$

where

$\boldsymbol{T_0} = \boldsymbol{ke_0}$

and

$\boldsymbol{k}$ = radix of the life table (i.e., $\boldsymbol{k}$ = 100,000) = $\boldsymbol{l_0}$

$\boldsymbol{e_0}$= life expectancy at birth (Mean years remaining at birth)

Using the notation used by Vaupel (2009) as a starting point, the age distribution of a stationary population of size $\boldsymbol{T_o}$ can be described by: (1) the probability density function $\boldsymbol{c(a)}$, the distribution of years lived; (2) the probability density function $\boldsymbol{\lambda(a)}$; and (3) the distribution of years remaining be described by the probability density function $\boldsymbol{r(a)}$. Note that by definition, $\boldsymbol{c(a)} = \boldsymbol{\lambda(a)}$. Using this notation, we can define the total number of years lived by individuals currently alive in the stationary population ($\boldsymbol{T_\lambda}$) and the total number of years remaining to them ($\boldsymbol{T_r}$), respectively, as:

$$(1)\ \boldsymbol{T_\lambda} = \int_n^\omega \alpha\, c(\alpha) = \boldsymbol{T_0\mu_\lambda}$$
$$(2)\ \boldsymbol{T_r} = \int_0^\omega \alpha\, r(\alpha) = \boldsymbol{T_0\mu_r}$$

Because, as we noted earlier, $\boldsymbol{c(\alpha)} = \boldsymbol{\lambda(\alpha)}$,

then $$\boldsymbol{T_c} = \int_0^\omega \alpha\, c(\alpha) = \boldsymbol{T_\lambda} = \int_0^\omega \alpha\, \lambda(\alpha)$$

Kim and Aron (1989) provide a proof that mean age in a stationary population is equal to mean expected years remaining. Because Vaupel (2009) demonstrated that that the mean number of years lived in a stationary population is equal to the mean expected years remaining, we can see that the three means are equivalent, using the notation just described:

(3) $\boldsymbol{\mu_c} = \boldsymbol{\mu_r} = \boldsymbol{\mu_\lambda}$

where

$\boldsymbol{\mu_c}$ = mean age = $\int_0^\omega \alpha\, c(\alpha)\, da$
$\boldsymbol{\mu_r}$ = mean years remaining = $\int_0^\omega \alpha\, r(\alpha)\, da$

and

$\boldsymbol{\mu_\lambda}$ = mean years lived = $\int_0^\omega \alpha\, \lambda(\alpha)\, da$

Because $\boldsymbol{T_0} = \boldsymbol{ke_0}$, then it follows that

(4) $\boldsymbol{T_c}/\boldsymbol{T_0} = \boldsymbol{\mu_c}$

Because $\boldsymbol{\mu_c} = \boldsymbol{\mu_r} = \boldsymbol{\mu_\lambda}$, then it follows that

(5) $\boldsymbol{T_c}/\boldsymbol{T_0} = \boldsymbol{\mu_r} = \boldsymbol{\mu_\lambda}$

And because $\boldsymbol{T_0} = \boldsymbol{ke_0}$, $\boldsymbol{\mu_c}$ can be expressed as

(6) $\boldsymbol{\mu_c} = \boldsymbol{T_c}/\boldsymbol{ke_0}$

then it follows that

(7) $\boldsymbol{T_c} = \boldsymbol{\mu_c ke_0}$

and

(8) $\boldsymbol{T_c}/\boldsymbol{k} = \boldsymbol{\mu_c e_0}$

In verbal terms, equation (8) states that when divided by the radix of the life table, $\boldsymbol{k}$, the total number of years lived by those alive in the stationary population, $\boldsymbol{T_c}$, is equal to the product of the mean age of the stationary population, $\boldsymbol{\mu_c}$, and its life expectancy at birth, $\boldsymbol{e_0}$. When divided by the radix of the life table, the total number of years lived by those alive in the stationary population also is equal to: (1) the product of the mean number of years lived by those alive in the stationary population, $\boldsymbol{\mu_\lambda}$, and life expectancy at birth, $\boldsymbol{e_0}$; and (2) the product of the mean number of years remaining to those alive in the stationary population, $\boldsymbol{\mu_r}$, and life expectancy at birth, $\boldsymbol{e_0}$.

Further,

(9) $\boldsymbol{e_0} = \boldsymbol{T_c}/\boldsymbol{k\mu_c}$

and because $\boldsymbol{1/e_0} = \boldsymbol{b} = \boldsymbol{d}$

where

$\boldsymbol{b}$ = the crude birth rate in the stationary population ($\boldsymbol{k}/\boldsymbol{T_0}$)

$\boldsymbol{d}$ = the crude death rate in the stationary population ($\boldsymbol{k}/\boldsymbol{T_0}$)

then it follows that the relationship, $\boldsymbol{\mu_c} = \boldsymbol{T_c}/\boldsymbol{ke_0}$ can be expressed as

(10) $\boldsymbol{\mu_c} = (\boldsymbol{T_c b})/\boldsymbol{k}$

In verbal terms, equation (9) states that when divided by the radix of the life table, $\boldsymbol{k}$, the product of the total number of years lived by those alive in the stationary population, $\boldsymbol{T_c}$, and the population's crude birth rate, $\boldsymbol{b}$, is equal to the mean age of the individuals currently alive in the stationary population. This equality is the product of the force of fertility and the total years lived by those alive. Because $\boldsymbol{b} = \boldsymbol{d}$, the equality can also be viewed as the product of the force of mortality and the total years lived by those alive. These equalities should not be surprising because for a population to be stationary, the force of increments is equal to the force of decrements. Similarly, it should not be surprising that specific values of mean years lived, $\boldsymbol{\mu_\lambda}$, and mean years remaining, $\boldsymbol{\mu_r}$, also result from the specific equality of the force of increments and the force of decrements acting in concert with the total years lived in a given stationary population.

1.2 A Set of Inequalities

Theorem 1

when $0 < x < \omega$, then $\boldsymbol{\lambda_x} + \boldsymbol{e_x} > \boldsymbol{e_0}$

Definition

$\boldsymbol{\lambda_x} = (\boldsymbol{T_0} - \boldsymbol{T_x})/\boldsymbol{l_0}$ = mean years lived to age x

and

$\boldsymbol{e_x} = \boldsymbol{T_x}/\boldsymbol{l_x}$ = mean years remaining at age x

Corollary

when x =0 then $\lambda_x + e_x = e_0$ since

$(T_0 - T_0)/l_0 + T_0/l_0 = 0 + e_0 = e_0$

and when x = ω then $\lambda_x + e_x = e_0$ since

$(T_0 - T_\omega)/l_0 + T_x/l_x = (T_0 - T_\omega)/l_0 + T_\omega/l_\omega = (T_0 - 0)/l_0 + 0 = e_0 + 0 = e_0$

Proof

Let $\lambda_x = (T_0 - T_x)/l_0 = (e_0 l_0 - T_x)/l_0 = e_0 - T_x/l_0$

then $\lambda_x + e_x = e_0 - T_x/l_0 + T_x/l_x$

and except when x= 0, so that $T_x/l_0 = T_0/l_0 = e_0$

and when $T_x/l_x = T_0/l_0$ so that $e_0 - T_0/l_0 + T_0/l_0 = 0 + e_0 = e_0$

and except when x = ω, so that $T_x/l_0 = T_\omega/l_0$

and when $T_x/l_x = T_\omega/l_\omega$, so that $e_0 - T_\omega/l_0 + T_\omega/l_\omega = e_0 - 0/l_0 + 0/0 = e_0 - 0 + 0 = e_0$

then $T_x/l_0 < T_x/l_x$ because $l_0 > l_x$ when x >0

Thus, $\lambda_x + e_x > e_0$ because

$e_0 - T_x/l_0 + T_x/l_x > e_0$

Theorem 2

$\mu_\lambda + \mu_r > e_0$

Proof

Because $\mu_c = \mu_r = \mu_\lambda$

then it follows that $\mu_\lambda + \mu_c = 2\mu_c = 2\mu_r = 2\mu_\lambda$

Because $e_0 = Tc/k\mu_c$

then it follows that

$e_0/2 = T_c/k2\mu_c$

and since $e_0/2 < e_0$

then

$(\mu_\lambda + \mu_r) > e_0$

Once we have T_c and μ_c, both of which are easily obtained when $c(\alpha)$ is determined, we can determine life expectancy at birth by dividing total years in the stationary population by the product of k (remember $k = l_0$) and the mean age of the population. Because of the equalities shown earlier, e_0 also can be determined when either $r(\alpha)$ or $\lambda(\alpha)$ is found. And, of course, once e_0 is obtained, b and d can be determined, as can T_0.

It is useful to note here that Pressat (1972) examined the relationship between mean age of a stationary population and life expectancy at birth and found (in the notation we use):

$$(11) \qquad \mu_c = \tfrac{1}{2}(e_0 + (\sigma^2/e_0))$$

where

μ_c = mean age of the stationary population

e_0= life expectancy at birth

and

σ^2 = variance in age at death

Pressat's (1972: 408) identification of equation (11) was independently re-discovered by Morales (1989) and identified as a re-discovery by Preston (199).

Equation (11) is particularly useful here because it provides the basis for an interpretation of the inequality given in Theorem 2, namely that $\mu_\lambda + \mu_r > e_0$. First, recall that as shown earlier, the mean age of the stationary population is equal to mean years lived and to mean years remaining: $\mu_c = \mu_r = \mu_\lambda$ and, therefore = $2\mu_c = 2\mu_r = 2\mu_\lambda$. Thus, if we multiply μ_c by 2, then equation (11) can be restated as

$$(12) \qquad 2\boldsymbol{\mu_c} = 2(½(\boldsymbol{e_0} + (\boldsymbol{\sigma}^2 / \boldsymbol{e_0}))) = \boldsymbol{e_0} + (\boldsymbol{\sigma}^2/\boldsymbol{e_0})$$

Because $2\boldsymbol{\mu_c}$ = mean years lived ($\boldsymbol{\mu_\lambda}$) plus mean years remaining ($\boldsymbol{\mu_r}$) and because $2\boldsymbol{\mu_c} = \boldsymbol{e_0} + (\boldsymbol{\sigma}^2/\boldsymbol{e_0})$, we can see that the sum of mean years lived and mean years remaining is equal to the sum of life expectancy at birth and the ratio of variance in age at death to life expectancy at birth: $\boldsymbol{\mu_\lambda} + \boldsymbol{\mu_r} = \boldsymbol{e_0} + (\boldsymbol{\sigma}^2/\boldsymbol{e_0})$. And since $\boldsymbol{e_0} + (\boldsymbol{\sigma}^2/\boldsymbol{e_0}) > \boldsymbol{e_0}$, it follows that $(\boldsymbol{\mu_\lambda} + \boldsymbol{\mu_r}) > \boldsymbol{e_0}$. Because we also know that life expectancy at birth is equivalent to mean age at death, we also can state equation (12) as:

$$(13) \qquad 2\boldsymbol{\mu_c} = \boldsymbol{\mu_d} + (\boldsymbol{\sigma}^2/\boldsymbol{\mu_d})$$

where

$\boldsymbol{\mu_d}$ = mean age at death and $\boldsymbol{\mu_c}$ and $\boldsymbol{\sigma}^2$ are defined as before

Because $2\boldsymbol{\mu_c} = \boldsymbol{\mu_\lambda} + \boldsymbol{\mu_r}$ we can re-express (13) as:

$$(14) \qquad \boldsymbol{\mu_\lambda} + \boldsymbol{\mu_r} = \boldsymbol{\mu_d} + (\boldsymbol{\sigma}^2/\boldsymbol{\mu_d})$$

where

all of the terms are as previously defined

Thus, the sum of mean years lived and mean years remaining is equal to mean age at death plus the ratio of the variance in age at death to mean age at death.

1.2.1 Illustration of Theorem 1

Using a 1990 USA Life Table (both sexes combined) from the Human Mortality Database (2009) as an illustration of a stationary population, we examine $\boldsymbol{\lambda_x}$, $\boldsymbol{e_x}$, and $\boldsymbol{\lambda_x}+\boldsymbol{e_x}$ by age, where ω= 110.5 (which we set as the maximum life span; nobody lives beyond this age). Our examination is displayed by Figure 1, which provides a scatterplot of the relationship between age (x axis) and $\boldsymbol{\lambda_x}+\boldsymbol{e_x}$, the sum of mean years lived and mean years remaining (y axis). Life expectancy at birth for this population is 75.40 years. As shown in Figure 1, when age (x) = 0, $\boldsymbol{\lambda_x}+\boldsymbol{e_x} = \boldsymbol{e_0}$ and when age (x) = 110.5, $\boldsymbol{\lambda_x}+\boldsymbol{e_x} = \boldsymbol{e_0}$. The scatterplot shows that $\boldsymbol{\lambda_x}+\boldsymbol{e_x}$ rises non-monotonically from 75.40 years ($\boldsymbol{e_0}$) when age = zero, reaches a maximum of 79.82 years at age 78.5, remains at this maximum to age 79.5, then monotonically declines back to 75.40

(e_0), at the maximum possible age, 110.5. As it increases, the curve is steepest from age 45 to age 79 and the decline from age 79 is steep all the way to age 110.5.

(FIGURE 1 ABOUT HERE)

1.2.2 Illustration of Theorem 2

In order to empirically illustrate the inequality provided by Theorem 2 and the relationship linking it to variance in age at death (see equations (11) through (14)), we selected a (non-random) sample of complete USA life tables for years ending in zero and five from the Human Mortality Database (2009), which has an online collection of these life tables annually from 1933 to 2013. Table 1 provides these 16 empirical examples of this inequality, $\mu_\lambda + \mu_r > e_0$.

(TABLE 1 ABOUT HERE)

As can be seen in Table 1, the difference between $\mu_\lambda + \mu_r$, on the one hand, and e_0, on the other, declines (although not monotonically) as e_0 increases from 1935 to 2010. The mean difference over all 16 observations is 5.37 years, with a standard deviation of 1.90. Because of Theorem 2 we know that the difference will remain positive from the re-expressed form of equation (12), namely, $\mu_\lambda + \mu_r = e_0 + (\sigma^2/e_0)$. The trend in the sample confirms that the relationship is curvilinear as expected from this same re-expressed equation. To empirically illustrate this, we constructed scatter plots of different equations and variable transformations that seemed promising using the NCSS package, version 8 (2016) and found that a quadratic model of the following form fit well: (difference2) = A + B*(ln(e_0)) + C*(ln(e_0))2, where A = 25498.4. B = -11685.8 and C = 1339.6, with R^2 = .9965. This model was estimated in 21 iterations with a random seed of 2695. A scatterplot of the relationship between difference and e_0 along with the fitted model's trend line is shown in Figure 2.

In verbal terms, the explanation for the empirical illustration of the relationship found in Figure 2 and specified in the non-linear equation given by $\mu_\lambda + \mu_r = e_0 + (\sigma^2/e_0)$, is that the sum of mean years lived (μ_λ) and mean years remaining (μ_r) is equal to the mean age at death (μ_d) plus the ratio of the variance in age at death to mean age at death (σ^2/μ_d). Recalling that mean age at death is equal to life expectancy at birth (e_0), we can see that if the variance in age at death remained relatively constant (or, relatively speaking, did not increase as much as life expectancy) from 1935 to 2010 while life expectancy increased, then the difference, μ_λ

+ μ_r - e_0 , would decrease during the same period, which is what is shown in Figure 2. To some extent, the trend found in Figure 2 likely reflects this because other than the initial effect of the baby boom (1946-64), the US population aged between 1935 and 2010 and holding all else constant, one would expect that variance in age at death would not increase as a population ages because deaths become more concentrated in the older population, which, in turn, would be reflected in life tables constructed from such a population.

(FIGURE 2 ABOUT HERE)

2. RESULTS AND DISCUSSION

Using Carey's equality Theorem (Carey et al. 2008, Rao and Carey 2014, Müller et al. 2004) and a 2005 life table for the United States, Vaupel (2009) estimates that more than 48 percent are 41 years or older, which implies that nearly half of the life table population will be alive in 2050, assuming that the 2005 life table holds to 2009. Using the same US life table and corresponding stationary population, we find that on average the population lived 40.60 years and will live another 40.60 years on average. If we assume that the 2005 life table applied to 2009 as did Vaupel, then on average the members will live to almost 2050, which is in agreement with Vaupel's estimate. Even without such an assumption, it is the case that on average the 2005 population lived 40.6 years and will, on average, live an another 40.6 years, or 81.3 years in total, which is 3.67 years more than their life expectancy at birth of 77.63 years. While the actual differences may vary, the proof shown earlier for Theorem 2 shows that mean years lived + mean years remaining is greater than life expectancy at birth ($\mu_{\lambda} + \mu_r > e_0$). If we apply this line of reasoning to the actual 2010 US life table, we find that on average the 2010 population lived 41.14 years and will, on average, live another 41.14 years, or 82.28 years in total, which is 3.43 years longer than this population's life expectancy at birth of 78.85. Notice that as shown in Figure 2, that this difference is less than the difference found for the 2005 life table, which is consistent with the model shown in Figure 2 and discussed at the end of the preceding section.

Vaupel (2009) notes that in regard to work by Müller et al. (2004) and Müller et al. (2007)

on wildlife population dynamics, Carey's equality Theorem could be used to estimate population age structure. In regard to this application, we add that if a representative age structure is obtained for a stationary population (or one that can be made stationary with adjustments suggested by Müller et al. (2004) and Müller et al. (2007), through Vaupel's suggestion or from another method, such as a sample, then its mean age, mean years lived, and mean years remaining can be determined as can its life expectancy at birth, its crude birth rate and its crude death rate. If a representative age structure is obtained from a random sample then interval estimates of these parameters can be constructed for the stationary population in question.

In the form of $\boldsymbol{\lambda_x}$ and $\boldsymbol{e_x}$, Carey's Equality Theorem also manifests itself in the data displayed as Figure 3, although somewhat imperfectly because the data are discrete rather than continuous.[1]As can be seen in Figure 3, the plotted values of $\boldsymbol{\lambda_x}$ by age are nearly a mirror image of the plotted values of $\boldsymbol{e_x}$ by age. The two curves cross at 39.75 years, which is the average number of years lived for this population and, also, the average number of years remaining.

(FIGURE 3 ABOUT HERE)

Theorem 1 shows that for a given age x, the sum of mean years lived ($\boldsymbol{\lambda_x}$) and mean years remaining ($\boldsymbol{e_x}$) exceeds life expectancy at birth where $0 < x < \omega$. Theorem 2 generalizes Theorem 1 to all ages. As shown in equations (12) through (14) and the discussion directly related to these equations, we have an explanation for the inequality demonstrated in theorem 2, which is linked to the variance in age at death. For example, if variance in age at death is held constant and life expectancy (mean age at death) increases then the inequality described by theorem 2 decreases; if variance in age at death increases and life expectancy is held constant then the inequality described by theorem 2 increases.

The explanation provided for the inequality described by theorem 2 can be extended to theorem 1 by looking at the variance in age at death up to and including a given age. For example, if we are interested in the inequality found at age x, we will find that if variance in age at death up to and including age x is held constant and life expectancy (mean age at death) increases, then the inequality described by theorem 1 decreases; if variance in age of death up to and including age x increases and life expectancy is held constant then the inequality described by theorem 1 increases.

One implication of these two related theorems is that the average longevity of all of the "living" members of a given stationary population exceeds the average number of years lived expected at birth. This implication suggests that life when a life table is used for planning the future, it is worthwhile to keep in mind that life expectancy at birth understates average longevity for the "living" members of the life table population.[2] As such, it may be preferable to use the sum of mean years lived and mean years remaining instead of life expectancy at birth in some applications. This also suggests that at a given age, it may be preferable to use the sum of mean years lived to that age and mean years remaining at that age instead of simply using life expectancy at the age in question. Although it does not directly take into account the inequalities we have demonstrated here, work by others such as Canudas-Romo and Zarulli (2016) and Canudas-Romo and Engelman (2016) recognizes similar implications involving years lived and years remaining.

3. ENDNOTES

1. Villavicencio and Riffe (2016) provide a complete and formal proof of Carey's equality in a discrete-time framework.
2. In addition to Pressat (1972), Morales (1989), and Preston (1991), among others, Canudas-Romo and Engelman (2016) have examined the sum of mean years lived and mean years remaining. However, none of these authors describes the inequalities demonstrated here in the forms of theorems 1 and 2.

4. REFERENCES

Canudas-Romo, V., and V. Zarnulli. (2016). Am I halfway? Years Lived = Expected Life. pp. 33 – 50 in R. Schoen (Ed.) *Dynamic Demographic Analysis*. Springer. Dordrecht, The Netherlands.

Canudas-Romo, V., and M. Engelman. (2016). Maximum life expectancies: Revisiting the best practice trends. *Genus* 65 (1): 59-79.

Carey, J.R., R., Papadopoulos, H-G Müller, B. Katsoyannos, B., N. Kouloussis, J-L Wang, K. Wachter, W. Yu, and P. Liedo. (2008). Age structure and extraordinary life span in wild medfly populations. *Aging Cell* 7: 426-437.

Human Mortality Database. (2009) University of California, Berkeley, and Max Planck Institute for Demographic Research. www.mortality.org.

Kintner, H. (2004). The Life Table. pp. 301-340 in J. Siegel and D. Swanson (Eds.) *The Methods and Materials of Demography, 2nd Edition.* San Diego, CA: Elsevier Academic Press.

Rao, A. and J. Carey. (2014). Generalization of Carey's equality and a Theorem on stationary population." *Journal of Mathematical Biology* DOI 10.1007/s00285-014-0831-6. http://entomology.ucdavis.edu/files/203430.pdf

Kim, Y. and J. Aron. (1989). On the equality of average age and average expectation of remaining life in a stationary population. *SIAM Review* 31 (1): 110-113.

Morales, V. (1989). Mean age and life expectancy at birth in stationary populations: Research Note. *Social Biology* 36 (1-2): 114

Müller, H-G., J-l Wang, J. Carey, E. Caswell-Chen, C. Chen, N. Papadoupoulos and F. Yao. (2004). Demographic window to aging in the wild: Constructing life tables and estimating survival functions from marked individuals of unknown age. *Aging Cell* 3(3): 125–131.

Müller, H-G., J-L Wang, W. Yu, A. Delaigle, and J. Carey. (2007) Survival in the wild via residual demography. *Theoretical Population Biology* 72 (4): 513–522.

NCSS (2016). "General Description of NCSS 8 and its system requirements." http://www.ncss.com/download/ncss/updates/ncss-8/

Pressat, R. (1972). *Demographic Analysis: Methods, Results, Applications*. Aldine-Atherton: Chicago.

Preston, S. (1991). Mean age and life expectancy at birth in stationary populations: Comment. *Social Biology* 38 (1-2): 154

Vaupel, J. (20. 09)Life lived and left: Carey's equality. (2009). *Demographic Research* 20: 7-10.

Villavicencio, F., and T. Riffe. (2016). Symmetries between life lived and left in finite stationary populations. *Demographic Research* 35: 381-398.

TABLE 1. DIFFERENCE BETWEEN THE SUM OF MEAN YEARS LIVED & MEAN YEARS REMAINING AND LIFE EXPECTANCY AT BIRTH: SELECTED USA LIFE TABLES FOR BOTH SEXES COMBINED, 1935 TO 2010 (N=16)					
YEAR	E_0 (1)	MEAN YRS LIVED (2)	MEAN YRS REMAINING (3)	TOTAL MEAN YRS LIVED & REMAINING (4)	DIFFERENCE: (4) - (1)
1935	60.89	35.47	35.47	70.94	10.05
1940	63.23	35.86	35.86	71.72	8.49
1945	65.58	36.55	36.55	73.10	7.52
1950	68.07	37.12	37.12	74.24	6.17
1955	69.56	37.62	37.62	75.24	5.68
1960	69.83	37.66	37.66	75.32	5.49
1965	70.24	37.81	37.81	75.62	5.38
1970	70.74	38.00	38.00	76.00	5.26
1975	72.54	38.67	38.67	77.34	4.80
1980	73.74	39.09	39.09	78.18	4.44
1985	74.67	39.39	39.39	78.78	4.11
1990	75.40	39.75	39.75	79.50	4.10
1995	75.89	39.90	39.90	79.80	3.91
2000	76.86	40.20	40.20	80.40	3.54
2005	77.63	40.60	40.60	81.20	3.57
2010	78.85	41.14	41.14	82.28	3.43

Source of data discussed in text. Calculations by authors.

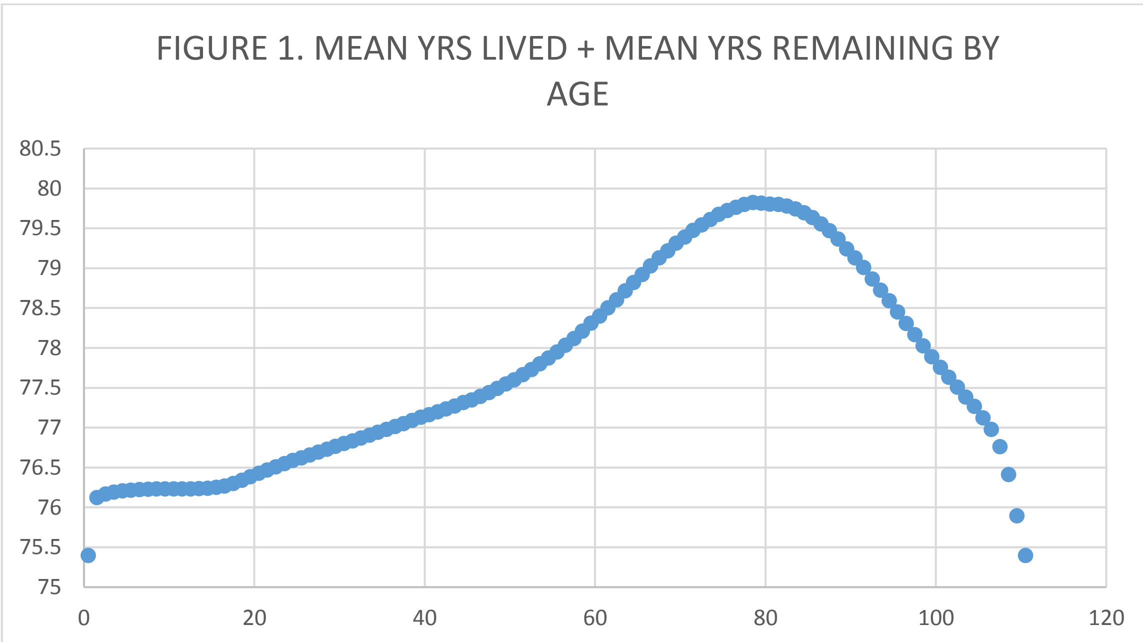
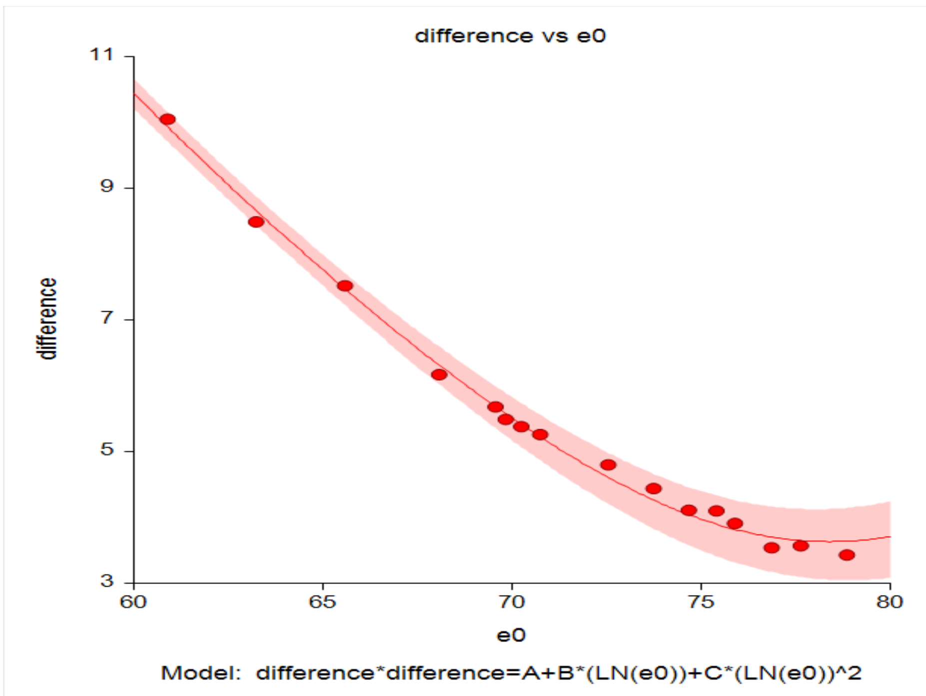


FIGURE 1. MEAN YRS LIVED + MEAN YRS REMAINING BY AGE
80.5
80
79.5
79
78.5
78
77.5
77
76.5
76
75.5
75
0
20
40
60
80
100
120

FIGURE 2. The Difference between the sum of mean years lived + mean years remaining and e_o (sum – e_0) by e_0



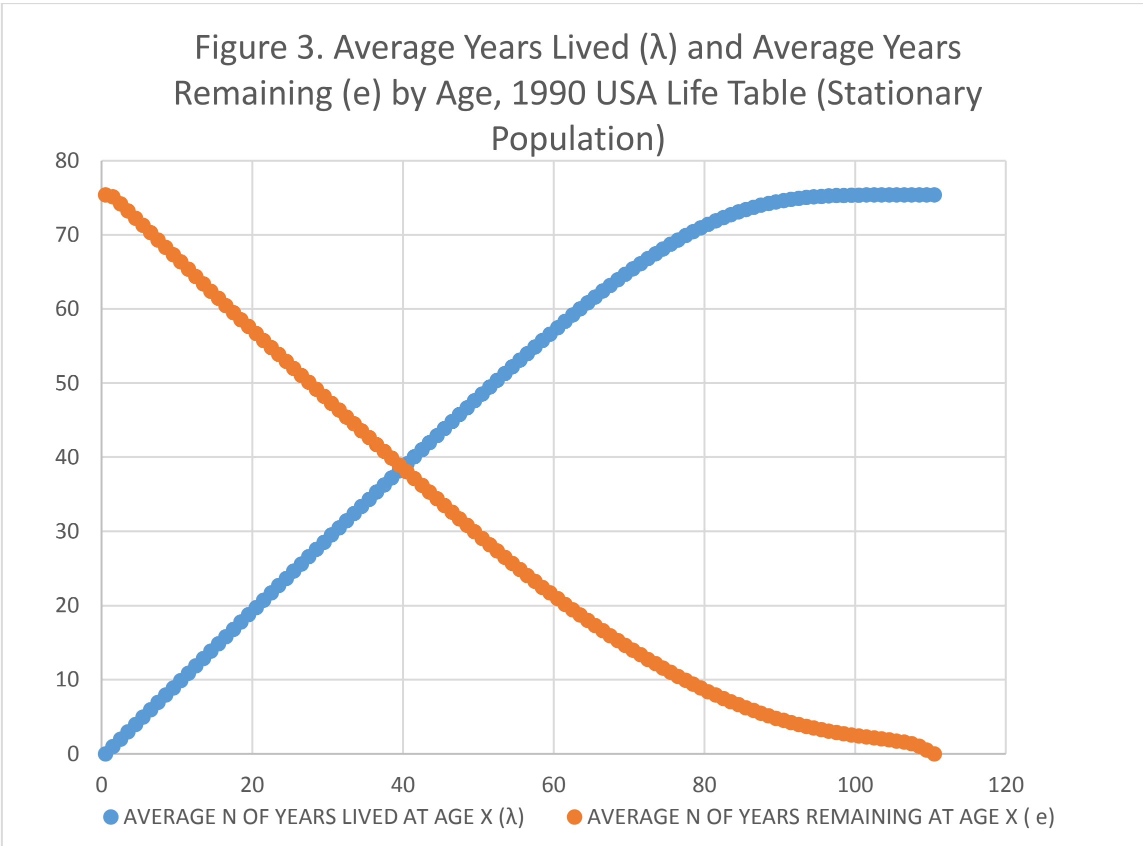


Figure 3. Average Years Lived (λ) and Average Years Remaining (e) by Age, 1990 USA Life Table (Stationary Population)
80
70
60
50
40
30
20
10
0
0
20
40
60
80
100
120
AVERAGE N OF YEARS LIVED AT AGE X (λ)
AVERAGE N OF YEARS REMAINING AT AGE X (e)