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# Implementing Fuzzy Random Variables
# -- Some Preliminary Observations

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## Abstract

Three definitions of fuzzy random variables (FRVs) have been cited in the current literature: the first is due to Kwakernaak (1978), who viewed a FRV as a vague perception of a crisp but unobservable RV; the second is due to Puri and Ralescu (1986), who regarded FRVs as random fuzzy sets; and the third is due to Liu and Liu (2003), whose notion of FRV was based on a concept they called a credibility measure. Given the different rationales, three questions arise: (1) how is each of these views of FRVs conceptualized; (2) what are the differences and similarities between the metrics for each of these views, and (3) how are the metrics for these three views implemented. The purpose of this article is to present some preliminary observations with respect to the answers to these questions.

# 1. Introduction

Actuarial models involve two important sources of uncertainty: randomness and fuzziness. Randomness relates to the stochastic variability of all possible outcomes of a situation. Fuzziness, on the other hand, can be traced to incomplete knowledge regarding the situation. Randomness and fuzziness can be merged to formulate a fuzzy random variable (FRV), that is, a function that assigns a fuzzy subset to each possible random outcome [Shapiro (2009)].

Anecdotal evidence suggests that actuaries are receptive to the notion of FRVs. They generally recognize that there are sources of uncertainty that random variables cannot capture and they are used to hearing that fuzziness is a key component of that uncertainty. Consequently, since random variables are at the core of actuarial concepts and fuzziness permeates every aspect of actuarial modeling and analysis, one would expect to see FRVs implemented often, both in the actuarial literature and in practice. But this is rarely the case.

A plausible explanation of why FRVs are not being implemented more often by potential actuarial users is that many of these users have a problem conceptualizing FRVs and/or they are not sufficiently familiar with FRV methodology. This state of affairs likely is exasperated by the fact that there are three different definitions of FRVs cited in the current literature, and each of these definitions is associated with a different set of metrics. The first is due to Kwakernaak (1978), who viewed a FRV as a vague perception of a crisp but unobservable RV; the second is due to Puri and Ralescu (1986), who regarded FRVs as random fuzzy sets; and the third is due to Liu and Liu (2003), whose notion of FRV was based on a concept they called credibility measure.

Assuming the foregoing explanation is valid, the purpose of this ongoing study is to help alleviate the situation. As part of this effort, this article is a synopsis of the three different definitions of FRVs cited in the current literature. The article begins with an overview of probability, possibility and credibility spaces, since they are pertinent to this discussion. The Kwakernaak FRV, the Puri and Ralescu FRV, and Liu and Liu FRV are discussed next. The focus is on their definitions and how they are conceptualized, their expected values, and their variances. Finally, there is a brief summary.

# 2. Probability, Possibility, and Credibility[1]

In this section, we briefly review three notions that are pertinent to our discussion: probability, possibility, and credibility spaces. Basic features of these spaces are summarized in Table 1 and the discussion that follows.

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[1] See Liu (2005), Chapters 2 and 3, for a more detailed discussion of the topics of this section.

Table 1: Probability, Possibility, and Credibility Spaces

| |
|---|
| Probability Space |
| $(\Omega, \mathcal{A}, \Pr)$ is a probability space |
| $\Omega$: sample space |
| $\mathcal{A}$: $\sigma$-algebra of subsets of $\Omega$ |
| Pr: probability measure on $\Omega$ |
| |
| Possibility Space |
| $(\Theta, \mathcal{P}(\Theta), \mathrm{Pos})$ is a possibility space |
| $\Theta$: sample space |
| $\mathcal{P}(\Theta)$: power set of $\Theta$ |
| Pos: possibility measure on $\Theta$ |
| |
| Credibility Space |
| $(\Theta, \mathcal{P}(\Theta), \mathrm{Cr})$ is a credibility space |
| $\Theta$: sample space |
| $\mathcal{P}(\Theta)$: power set of $\Theta$ |
| Cr: credibility measure on $\Theta$ |
| |

2.1 Probability

For benchmarking purposes, we begin with a description of a probability space. As indicated in Table 1, a probability space is defined as the 3-tuple $(\Omega, \mathcal{A}, P)$, where $\Omega = \{\omega_1, \omega_2, ..., \omega_N\}$ is a sample space, $\mathcal{A}$ is the $\sigma$-algebra of subsets of $\Omega$, and Pr, a probability measure on $\Omega$, that satisfies:

$\Pr\{\Omega\} = 1$
$\Pr\{\varnothing\} = 0$
$0 \leq \Pr\{A\} \leq 1$ for any $A \in \mathcal{A}$,
For every countable sequence of mutually disjoint events $\{A_i\}$, i=1, 2, ...

$$\Pr\left\{\bigcup_{i=1}^{\infty} A_i\right\} = \sum_{i=1}^{\infty} \Pr\{A_i\}$$

Since a probability measure is based on binary (Boolean) logic, it satisfies the law of excluded middle (which requires that a proposition be either true or false), the law of contradiction (which requires that a proposition cannot be both true and false), and the law of truth conservation (which requires that the truth values of a proposition and its negation should sum to unity).[2]

2.2 Possibility

As one contrast to the probability space of Table 1, a possibility space is defined as the 3-tuple ($\Theta$, $\mathcal{P}(\Theta)$, Pos), where $\Theta=\{\theta_1, \theta_2, ..., \theta_N\}$ is a sample space, $\mathcal{P}(\Theta)$, also denoted as $2^{\Theta}$, is the power set of $\Theta$, that is, the set of all subsets of $\Theta$, and Pos is a possibility measure defined on $\Theta$. Pos{A}, the possibility that A will occur, satisfies:

$\mathrm{Pos}\{\Theta\} = 1$
$\mathrm{Pos}\{\varnothing\} = 0$
$0 \leq \mathrm{Pos}\{A\} \leq 1$, for any A in $\mathcal{P}(\Theta)$
$\mathrm{Pos}\{\cup_i A_i\} = \sup_i \mathrm{Pos}\{A_i\}$ for any collection $\{A_i\}$ in $\mathcal{P}(\Theta)$

As an example, the heavy (red) line of Figure 1 shows a representation of the possibility of a fuzzy event characterized by $\xi \geq x$, where $\xi = (a, b, c)$, $a < b < c$, is a triangular fuzzy variable.[3]

$$\mathrm{Pos}\{\xi \geqslant x\} = \begin{cases} 1, & x \leqslant b, \\ \dfrac{c-x}{c-b}, & b \leqslant x \leqslant c, \\ 0, & x \geqslant c, \end{cases}$$

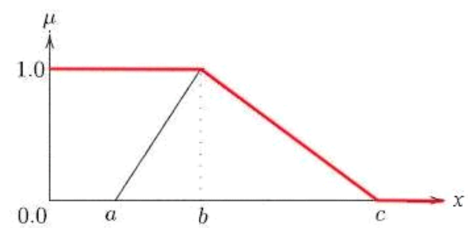


Figure 1: Possibility that $\xi$ is greater than x

Note that the possibility of an event is determined by its most favorable case only, in contrast to the probability of an event, where all favorable cases are accumulated.

By its very nature, the possibility measure is inconsistent with the law of excluded middle and the law of contradiction, and does not satisfy the law of truth conservation. [Liu (2012: 377)]

[2] Liu (2012: xiii-xiv)

[3] Adapted from Huang (2006).

2.3 Necessity[4]

The necessity measure of a set A often is defined as the impossibility of the opposite set $A^c$.[5] [Liu (2005: 80)]

Formally, [Zadeh (1979)] let ($\Theta$, $\mathcal{P}(\Theta)$, Pos) be a possibility space, and A a set in $\mathcal{P}(\Theta)$. Then the necessity measure of A is defined by

$$\text{Nec}\{A\} = 1 - \text{Pos}\{A^c\}.$$

The heavy (red) line of Figure 2 shows a representation of Nec{$\xi \geq x$}.

$$\text{Nec}\{\xi \geqslant x\} = \begin{cases} 1, & x \leqslant a, \\ \dfrac{b-x}{b-a}, & a \leqslant x \leqslant b, \\ 0, & x \geqslant b, \end{cases}$$

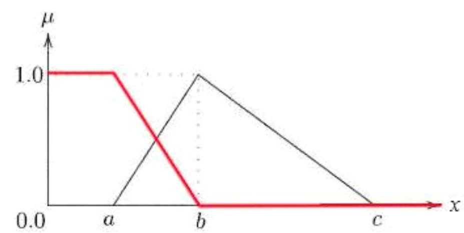


Figure 2: Necessity that $\xi$ is greater than or equal x

Notice that Nec{ $\xi \geq x$ } = 1 - Pos{ $\xi < x$ }.

2.4 Credibility

Given the limitations of the possibility measure mentioned previously, Liu and Liu (2002) suggested replacing it with what they termed a credibility measure. This credibility measure takes the form

$$\text{Cr}\{X \leq r\} = \frac{1}{2}\left(\text{Pos}\{X \leq r\} + \text{Nec}\{X \leq r\}\right)$$

or, what is equivalent

$$\text{Cr}\{X \leq r\} = \frac{1}{2}\left(\sup_{t \leq r} \mu_X(t) + 1 - \sup_{t > r} \mu_X(t)\right).$$

The set Cr on the power set $\mathcal{P}$ is called a credibility measure if it satisfies the following four axioms: [Liu (2007: 81-2)]

[4] Possibility and necessity measures were used as plausibility and belief measures by Shafer (1976, 1987) and with respect to fuzzy sets by Zadeh (1978). Possibility theory, which is based on these two measures, was extensively covered in Dubois and Prade (1988). [Wang and Klir (1992: 68)]

[5] This need not be the case. Liu and Liu (2005: 281), for example, first defines the conditions for a necessary measure, and then validates that Nec(A) is the dual of Pos($A^c$).

$Cr\{\Theta\} = 1$
$Cr\{A\} \le Cr\{B\}$ whenever $A \subset B$
$Cr\{A\} + Cr\{A^c\} = 1$ for any event A
$Cr\{\cup_i A_i\} = \sup_i Cr\{A_i\}$ for any events $\{A_i\}$ with $\sup_i Cr\{A_i\} < 0.5$[6]

Notice that the credibility measure is a special type of non-additive measure with self-duality. [Liu (2007: 82)] Also, a fuzzy event may fail even though its possibility achieves 1, and may hold even though its necessity is 0. However, the fuzzy event must hold if its credibility is 1, and fail if its credibility is 0. [Liu (2005: 81)]

The heavy (red) line of Figure 3 shows a representation of the credibility value of the fuzzy event characterized by $\xi \ge x$.

$$Cr\{\xi \geqslant x\} = \begin{cases} 1, & x \leqslant a, \\ \dfrac{2b-a-x}{2(b-a)}, & a \leqslant x \leqslant b, \\ \dfrac{c-x}{2(c-b)}, & b \leqslant x \leqslant c, \\ 0, & x \geqslant c. \end{cases}$$

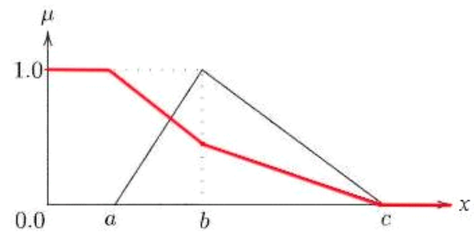


**Figure 3: Credibility that $\xi \ge$ x**

## 3. The Kwakernaak FRV Model

Kwakernaak (1978) conceptualized a FRV as a vague perception of a crisp but unobservable RV, which is referred to as the "original". As an example of his view, consider the task of projecting pension wealth at retirement for a cohort of active plan participants. Their actual pension wealth, X, say, is an ordinary RV on the positive real line. However, we might only perceive a random variable, $\xi$, say, through a set of "windows" like “less than average pension wealth,” “average pension wealth,” and “more than average pension wealth,” which we refine according to our information. That is, fuzzy sets are perceived as observation results since the underlying X is not observable.

Figure 4 shows a representation of the Kwakernaak FRV trajectory. As indicated, realizations are assigned to each elementary event $\omega_i \in \Omega$. Real valued realizations lead to real RVs, $X(\omega_i)$, while fuzzy perceptions of the realizations lead to FRVs, $\xi(\omega_i)$. Moreover, if $X(\omega_i) \in \xi(\omega_i)$, then $X(\omega_i)$ constitutes an "original" of $\xi(\omega_i)$.

[6] An alternate criterion is given by Liu (2005: 82)

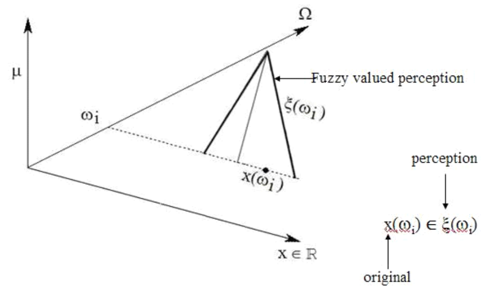


Figure 4: Kwakernaak FRV Trajectory

3.1 The Kwakernaak FRV

Formally, let $(\Omega, \mathcal{A}, P)$ be a probability space and $F(\mathbb{R})$ denote the set of all fuzzy numbers in $\mathbb{R}$, the set of real numbers. Specifically, $F(\mathbb{R})$ denotes the class of the normal convex fuzzy subsets of $\mathbb{R}$ having compact $\alpha$-levels for $\alpha \in [0, 1]$. This is the class of mappings $U: \mathbb{R} \rightarrow [0, 1]$, that is, $U(u) \in [0, 1]$, for all $u \in \mathbb{R}$, such that $U_\alpha$ is a nonempty compact interval, where

$$
\begin{aligned}
U_\alpha &= \{x \in \mathbb{R} \mid U(x) \geq \alpha\} && \text{if } \alpha \in (0,1] \\
&= \mathrm{cl}(\mathrm{supp}\ U) && \text{if } \alpha = 0.
\end{aligned}
$$

Then, a FRV is a mapping $\xi: \Omega \rightarrow F(\mathbb{R})$ such that for any $\alpha \in [0, 1]$ and all $\omega \in \Omega$, the real valued mapping[7]

$\inf \xi_\alpha: \Omega \rightarrow \mathbb{R}$, satisfying $\inf \xi_\alpha(\omega) = \inf (\xi(\omega))_\alpha$, and

$\sup \xi_\alpha: \Omega \rightarrow \mathbb{R}$, satisfying $\sup \xi_\alpha(\omega) = \sup (\xi(\omega))_\alpha$,

are real valued RVs.

Given ω, the unique characteristic of the Kwakernaak FRV is captured by its α-cut, which is depicted Figure 5:

[7] Adapted from Kwakernaak (1978), Kruse and Meyer (1987: 64-65), Gil (2004: 11) and Coppi et al (2006).

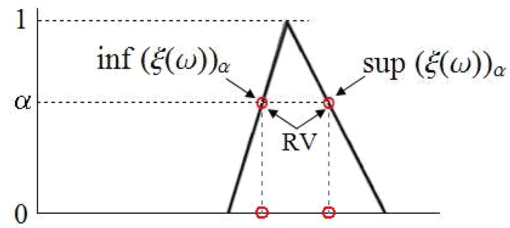


Figure 5: α-cut of a Kwakernaak FRV

Summarizing, the Kwakernaak FRV takes the form of a mappings from Ω to the left and right-hand side of the fuzzy target F(ℝ), where the latter are real-valued random variables.

### 3.2 Kwakernaak FRV Metrics

Let X be a FRV and $\mathcal{U}_{\mathcal{A}}$ the collection of all $\mathcal{A}$-measurable RVs of Ω. Then [Körner (1997: 31)]

(i) The k-th moment $EX^k$ of a FRV X is a fuzzy set on ℝ with

$$\mu_{\mathbb{E}X^k}(x) = \sup\{\mu_X(U) \mid U \in \mathcal{U}_{\mathcal{A}}, \mathbb{E}U^k = x\},\ \ x \in \mathbb{R}$$

(ii) The fuzzy variance of X is a fuzzy set $\mathrm{Var}_K(x)$ on [0,∞) with

$$\mu_{\mathrm{Var}_K(X)}(\sigma^2) = \sup\{\mu_X(U) \mid U \in \mathcal{U}_{\mathcal{A}}, D^2U = \sigma^2\},\ \ \sigma^2 \in [0,\infty).$$

## 4. The Puri and Ralescu FRV Model

Puri and Ralescu expressed concern over two limitation of the Kwakernaak model: its mapping to the real line, rather than an Euclidean n-space; and its notion of measurability, since it was limited to fuzzy numbers in ℝ. To overcome both these concerns they proposed the concept of FRVs whose values are fuzzy subsets of $\mathbb{R}^n$, that is, they conceptualized a FRV as a fuzzification of a random set [8] (therefore, sometimes called a random fuzzy sets). [9]

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[8] The study of random sets can be traced back to Robbins (1944), although Matheron (1975) is credited with rigorously defining the concept.

[9] It often is mentioned (see, for example, Nather (2001: 71)) that the embedding of the concept of a FRV into the concept of random sets, under the Puri and Ralescu approach, avoids the measurability issues of the Kwakernaak approach. In contrast, however, Kruse and Meyer (1987: 188), who advocated the Kwakernaak approach, countered that the Puri and Ralescu approach relies heavily on probability techniques in Banach spaces, the key tools being

Figure 6 is a simple representation of the Puri & Ralescu FRV trajectory.

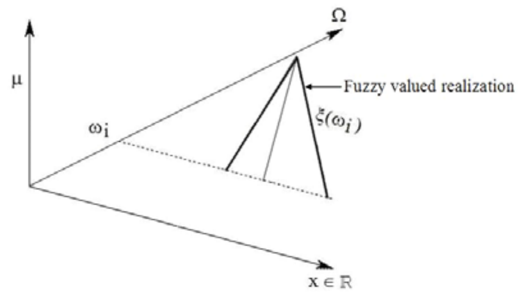


Figure 6: Puri & Ralescu FRV Trajectory

As indicated in the figure, under the Puri and Ralescu approach, the FRV is directly generated as a fuzzy-valued variable.

## 4.1 The Puri and Ralescu FRV

Let $(\Omega, \mathcal{A}, P)$ be a probability space, $F(\mathbb{R}^n)$ denote the set of fuzzy subsets, u: $\mathbb{R}^n \rightarrow [0,1]$, X: $\Omega \rightarrow F(\mathbb{R}^n)$ be defined by $X_\alpha(\omega) = \{ x \in \mathbb{R}^n: X(\omega)(x) \geq \alpha\}$, and $\mathcal{B}$ denote the Borel subsets of $\mathbb{R}^n$. A Puri and Ralescu FRV is a function X: $\Omega \rightarrow F(\mathbb{R}^n)$ such that, for every $\alpha \in [0,1]$ [Puri and Ralescu (1986: 413)]:

$$\{(\omega,x): x \in X_\alpha(\omega)\} \in \mathcal{A} \times \mathcal{B}$$

Given ω, Figure 7 depicts a representation of the α-cut of a Puri and Ralescu FRV.

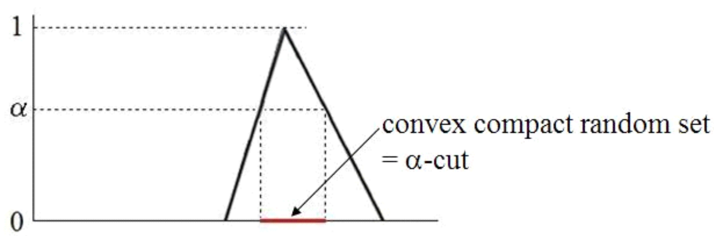


Figure 7: α-cut of a random fuzzy set

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embedding theorems, which meant that the Puri and Ralescu result cannot be directly implemented in a software tool, unlike the Kwakernaak approach.

4.2 Expected Value of a Puri and Ralescu FRV

When quantifying the central tendency of the distribution of a FRV in the Puri and Ralescu sense, the most common measure is the Aumann-type mean [(Aumann (1965)], which extends the mean of a real-valued random variable and preserves its main properties and behavior. [Sinova et al (2011)]

Given the probability space $(\Omega, \mathcal{A}, P)$, $\xi$ an integrably bounded[10] FRV associated with $(\Omega, \mathcal{A}, P)$, and S(F) a nonempty bounded set with respect to the $L^1(P)$-norm, the expected value of $\xi$ is the unique fuzzy set $\tilde{E}(\xi \mid P)$ of $\mathbb{R}^n$ such that [Diaz and Gil (1999: 31)]

$$(\tilde{E}(\xi \mid P))_\alpha = \int_\Omega \xi_\alpha dP \text{ for all } \alpha \in [0, 1],$$

where

$$\int_\Omega \xi_\alpha dP = \left\{ \int_\Omega f dP \mid f \in S(\xi_\alpha) \right\}$$

is the Aumann integral of $\xi_\alpha$ with respect to P.

As noted by Diaz and Gil (1999: 29), Puri and Ralescu defined the EV of a FRV as a generalization of the EV.

Operationally, when a fuzzy random variable $\xi : \Omega \to F(\mathbb{R})$ is integrably bounded, the EV of $\xi$ is unique and, for all $\alpha \in [0,1]$, is given by the compact interval [Lubiano et al (2000: 308)]

$$[E(\inf \xi_\alpha), E(\sup \xi_\alpha)].$$

4.3 Variance of a Puri and Ralescu FRV

Feng et al (2001) argued that, as in the case of real-valued random variables, the variance should be used to measure the spread or dispersion of the FRV around its EV. Accordingly, they defined the scalar variance of a Puri & Ralescu FRV as:

$$V(\tilde{X}) = \frac{1}{2}\int_0^1 \left[ V(\underline{X}_\alpha) + V(\overline{X}_\alpha) \right] d\alpha$$

Other proposals for scalar variance are equivalent to considering first a representative (numerical) element of every fuzzy realization of the fuzzy random variable (the midpoint of the

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[10] A FRV $\xi$ is said to be an integrably bounded FRV associated with the probability space $(\Omega, \mathcal{A}, P)$ if and only if $\|\xi_0\| \in L^1(\Omega, \mathcal{A}, P)$, where, for the function f, $L^1(\Omega, \mathcal{A}, P) = \{f \mid f: \Omega \to \mathbb{R}, \mathcal{A}\text{-measurable}, \int |f|^1 dP < \infty\}$.

support, for instance) and then calculating the dispersion of these numerical values. Körner (1997: 12), for example, used Steiner points in this way.

## 5. The Liu and Liu FRV Model

Liu and Liu (2002, 2003) expressed concern that both the Kwakernaak and Puri and Ralescu FRV models were based on the possibility measure, and, as such, did not obey the law of truth conservation and were inconsistent with the law of excluded middle and the law of contradiction. To overcome these perceived shortcomings, they based their FRV on the credibility measure,

$$\mathrm{Cr}\{A\} = \tfrac{1}{2}(\mathrm{Pos}\{A\} + 1 - \mathrm{Pos}\{A^c\})$$

which they contended plays the role of probability measure more appropriately than either the possibility and necessity measures.[11] Finally, their FRV model incorporated a scalar, rather than a fuzzy, expected value, since they viewed the latter as problematic from an implementation perspective.

### 5.1 Definition of the Liu and Liu FRV

Focusing on the credibility measure, Liu (2006: 399) defines a fuzzy random variable as a function $\xi$ from a probability space $(\Omega, \mathcal{A}, \Pr)$ to the set of fuzzy variables such that $\mathrm{Cr}\{\xi(\omega) \in \mathcal{B}\}$ is a measurable function of $\omega$ for any Borel set $\mathcal{B}$ of $\mathbb{R}$.

Thus, for example, the trajectory of a triangular fuzzy random variable $\xi$, given $\omega$, is a triangular fuzzy variable $\xi_\omega$, which may be denoted by $(X_1(\omega), X_2(\omega), X_3(\omega))$, where the $X_i$'s are random variables defined on the probability space $\Omega$. The randomness of $\xi$ is attributable to the random variables $X_i$ , i = 1, 2, 3. [Hao and Liu (2009: 13)]

### 5.2 Expected Value of the Liu and Liu FRV

As mentioned previously, Liu and Liu (2003) define the expected value of an FRV as a scalar value rather than a fuzzy number. In particular, in their model, if $\xi$ is a fuzzy random variable defined on the probability space $(\Omega, \mathcal{A}, \Pr)$, then the expected value of $\xi$ is defined by [Liu (2005 105)]

$$E[\xi] = \int_0^\infty \mathrm{Cr}\{\xi \ge x\}dx - \int_{-\infty}^0 \mathrm{Cr}\{\xi \le x\}dx$$

provided that at least one of the two integrals is finite.

[11] In this regard, Kuchta (2008: 53) commented that the possibility measure is very weak, in the sense that it is very easy for a fuzzy number to be smaller than a given value to a high degree. For example, $\mathrm{Pos}(\tilde{A} \le r) = 1$ for each $r \ge b$. He noted that the necessity measure is significantly stronger.

Alternatively, the expected value of $\xi$ can be conceptualized as [Li et al (2006: 210)]

$$E[\xi] = \int_{\Omega} \left[ \int_{0}^{\infty} Cr\{\xi(\omega) \geq r\}dr - \int_{-\infty}^{0} Cr\{\xi(\omega) \leq r\}dr \right] Pr(d\omega)$$

provided that at least one of the two integrals is finite, and, in the event that $\xi$ is a nonnegative fuzzy random variable, it expected value can be written as: [Li et al (2006: 210)]

$$E[\xi] = \int_{\Omega} \int_{0}^{\infty} Cr\{\xi(\omega) \geq r\}dr \ Pr(d\omega).$$

5.3 Variance of the Liu and Liu FRV

Let $\xi$ be a FRV with finite expected value $E[\xi]$. Liu & Liu (2003: 154) define the variance, $Var[\xi]$, of $\xi$ as the expected value of the FRV $(\xi - E[\xi])^2$. That is

$$Var[\xi] = E[(\xi - E[\xi])^2].$$

6. Comments

The purpose of this article was to give a synopsis of the three different types of FRVs cited in the current literature. To this end, their conceptualization, definition, expected value and variance were discussed. The three methodologies can be differentiated on the basis of whether their expected value and variance take the form of a fuzzy variable or a scalar, as shown in Table 2.

Table 2: Type of expected value and variance for the FRVs

| Researcher(s) | Expected Value | Variance |
|---|---|---|
| | | |
| Kwakernaak | fuzzy | fuzzy |
| Puri & Relescu | fuzzy | scalar |
| Liu & Liu | scalar | scalar |
| | | |

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