On the absolute ruin problem in a Sparre Andersen risk model with constant interest

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Outline

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1. Definition of the absolute ruin model featuring interest.
2. Gerber-Shiu function for Erlang(n) IAT with Matrix Exponential claim amounts.
3. Closed-form solutions of the absolute ruin probability for Erlang(2) IAT and exponential claims.
4. Conclusions and further extensions.

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We extend the Compound Poisson ruin model:

The surplus process is $U(t) = u + ct - \sum_{j=1}^{N(t)} Y_j$ where

- u is the initial capital
- ct stands for the premiums assumed to arrive continuously over time
- $S(t) = \sum_{j=1}^{N(t)} Y_j$ is the aggregate-claims process, which is a compound Poisson process with rate $\beta > 0$ and i.i.d. claim amounts $\{Y_1, Y_2, \dots\}$ with c.d.f. $F(y)$ and p.d.f. $f(y), y > 0$
- A positive relative security loading θ is charged

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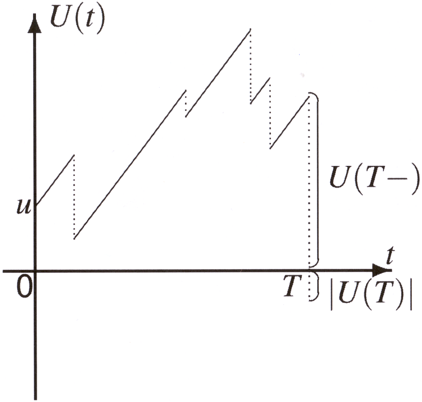
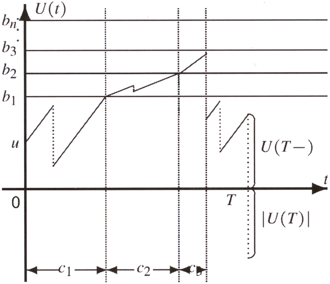
Sample path:



Changing premium rates or earning dividends:



The insurer's surplus at time t satisfies

$$dU(t) = \begin{cases} c_1 dt - dS(t), & b_0 \le U(t) < b_1 \\ \vdots & \\ c_n dt - dS(t), & b_{n-1} \le U(t) < b_n \\ c_{n+1} dt - dS(t), & b_n \le U(t) \end{cases}$$

Multi-threshold Compound Poisson Surplus Process with Interest

The insurer's surplus at time t satisfies

$$dU(t) = \begin{cases} c_0 dt + r_0 U(t)dt - dS(t), & b_{-1} = -c_0/r_0 < U(t) < b_0 \\ c_1 dt + r_1 U(t)dt - dS(t), & b_0 \leq U(t) < b_1 \\ \vdots & \\ c_n dt + r_n U(t)dt - dS(t), & b_{n-1} \leq U(t) < b_n \\ c_{n+1} dt + r_{n+1} U(t)dt - dS(t), & b_n \leq U(t) \end{cases}$$

Motivation:

- Ruin models incorporating multiple thresholds allow the insurer to change the premium rate charged depending on the current surplus level.
- In addition, interest might be earned on the liquid reserves. Conversely, if the surplus drops below zero, the amount of the deficit might be borrowed under a known in advance interest rate.
- It seem to be realistic to consider models which allow more flexibility upon claims, beyond the Poisson case. We consider a Markovian Arrival Process (MAP) with an underlying continuous time Markov chain with m states (later restricted to a renewal process).

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The Gerber-Shiu function for MAP(m) processes featuring interest

The $MAP(\underline{\alpha}, D_0, D_1)$ model incorporating interest

Imagine a CTMC controlling arrivals and claims amounts.

- Let $J = \{1, 2, \ldots, m\}$ the underlying CTMC,
- $\underline{\alpha}$ the initial probability vector,
- $\mathbf{D_0} = (d_{ij})_{i,j=1,\ldots,m}$ = matrix of transitions with no claims,
- $\mathbf{D_1} = (D_{ij})_{i,j=1,\ldots,m}$ = matrix of transitions at the instant of a claim.
- Remark: $(\mathbf{D_0} + \mathbf{D_1}) \times \underline{1} = \underline{0}$, i.e.,

$$d_{ii} = -(\sum_{j \neq i} d_{ij} + \sum_{j=1}^{m} D_{ij}).$$

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The (vector) Gerber-Shiu function

- $\underline{\Phi}(u) = (\Phi_1(u), \ldots, \Phi_m(u))$, where
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$$\Phi_i(u) = E[e^{-\delta\tau} w(U(\tau-), |U(\tau)|) I(\tau < \infty) | U(0) = u, J(0) = i],$$
- $\tau = \inf\left\{t \geq 0 | U(t) \leq -\frac{c}{r}\right\}$
- Suppose the claim size X_{ij} depends on both, previous state i and subsequent state j. Let $B_{ij}()$ and $b_{ij}()$ be its cdf and pdf, respectively.

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$$\Phi_i(u) = (1 + d_{ii}dt)e^{-\delta dt}\Phi_i(ue^{rdt} + c\bar{s}^{(r)}_{\overline{dt}|})$$
$$+ \sum_{j \neq i} d_{ij}dte^{-\delta dt}\Phi_j(ue^{rdt} + c\bar{s}^{(r)}_{\overline{dt}|})$$
$$+ \sum_{j=1}^{m} D_{ij}dte^{-\delta dt}\left[\int_0^{ue^{rdt}+c\bar{s}^{(r)}_{\overline{dt}|}+c/r} \Phi_j(ue^{rdt} + c\bar{s}^{(r)}_{\overline{dt}|} - x)dB_{ij}(x)\right.$$
$$\left.+ \int_{ue^{rdt}+c\bar{s}^{(r)}_{\overline{dt}|}+c/r}^{\infty} w(ue^{rdt} + c\bar{s}^{(r)}_{\overline{dt}|}, x - ue^{rdt} - c\bar{s}^{(r)}_{\overline{dt}|})dB_{ij}(x)\right] + o(dt)$$

- Using a change of variable and denoting

$$A_{ij}(u) = \int_{u+c/r}^{\infty} w(u, x-u)dB_{ij}(x),$$

we find

- $$(c+ur)\Phi_i'(u) = \delta\Phi_i(u) - \sum_{j=1}^{m} d_{ij}\Phi_j(u)$$
$$-\sum_{j=1}^{m} D_{ij}\left[\int_{0}^{u+c/r} \Phi_j(u-x)dB_{ij}(x) + A_{ij}(u)\right] \quad (1)$$

- REMARK: In the Poisson case $d_{ij} = -\lambda$, $D_{ij} = \lambda$ and the latter system is reduced to one equation identical to the one in the classical model with interest.

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Initial and boundary conditions

Lemma 1

For a MAP of order n risk process with general claim amounts $B_{ij}(x)$,

$$\lim_{u \to -c/r} \underline{\Phi}(u) = \mathbf{C}^{-1} \underline{a}, \tag{2}$$

where

$$\mathbf{C} = \delta \mathbf{I_n} - \mathbf{D_0}, \qquad a_i = \sum_{j=1}^{n} D_{ij} A_{ij}(-c/r). \tag{3}$$

- We make the natural assumption that the Gerber-Shiu functions vanish at infinity, i.e.,

$$\lim_{u\to\infty} \Phi_i(u) = 0, \quad i = 1, 2, \ldots, n.$$

Lemma 2

The kth derivative of the Gerber-Shiu function satisfies

$$\lim_{u\to\infty} \Phi_i^{(k)}(u) = 0, \quad i = 1, 2, \ldots, n, \quad k = 1, 2, \ldots.$$

Changing premium rates and earning interest on invested capital

Under the multi-layer model, the G-S equations derived for each layer are structurally the same (only the force of interest, the premium rates and initial/boundary conditions being different among the layers).

Erlang interclaims with ME claims

$$
\mathbf{D_0} = \begin{bmatrix} -\lambda_1 & \lambda_1 & 0 & \dots & 0 \\ 0 & -\lambda_2 & \lambda_2 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & -\lambda_n \end{bmatrix},
$$

$$
\mathbf{D_1} = \begin{bmatrix} 0 & 0 & 0 & \dots & 0 \\ 0 & 0 & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ \lambda_n & 0 & 0 & \dots & 0 \end{bmatrix}.
$$

- Assume also that claim sizes are ME distributed:

$$
\tilde{b}_{ij}(s) = \tilde{b}(s) = \frac{p_1 s^{m-1} + p_2 s^{m-2} + \cdots p_m}{q_0 s^m + q_1 s^{m-1} \cdots q_m}, \quad q_0 = 1. \quad (4)
$$

- Consequently,

$$b^{(m)}(\cdot) + q_1 b^{(m-1)}(\cdot) + \cdots q_m b(\cdot) = 0, \tag{5}$$

and

$$\begin{cases} (c+ru)\Phi_1'(u) = (\delta+\lambda_1)\Phi_1(u) - \lambda_1\Phi_2(u), \\ (c+ru)\Phi_2'(u) = (\delta+\lambda_2)\Phi_2(u) - \lambda_2\Phi_3(u), \\ \vdots \\ (c+ru)\Phi_n'(u) = (\delta+\lambda_n)\Phi_n(u) - \lambda_n[N_{\Phi_1}(u) - A(u)]. \end{cases} \tag{6}$$

- If claim sizes satisfy (5) then,

$$\sum_{j=0}^{m} q_j \, N_{\Phi_1}^{(m-j)}(u) = \sum_{j=0}^{m-1} \xi_j \, \Phi_1^{(m-1-j)}(u), \tag{7}$$

where $\xi_j = \sum_{k=0}^{j} q_{j-k} f^{(k)}(0)$.

- For penalty functions that depend only on the deficit, i.e., $w(x,y) = w(y)$, we arrive at

$$\prod_{i=1}^{n} \frac{\delta+\lambda_i}{\lambda_i} \left(\sum_{j=0}^{m} q_{m-j}\, \mathcal{D}_u^{(j)} \right) \left(\prod_{i=1}^{n} \left(1 - \frac{c+ru}{\delta+\lambda_i} \mathcal{D}_u \right) \right) \Phi_1(u)$$
$$= \left(\sum_{j=0}^{m-1} \xi_{m-1-j}\, \mathcal{D}_u^{(j)} \right) \Phi_1(u), \quad \text{where } \mathcal{D}_u^{(0)} = 1. \quad (8)$$

- Let $x = c + ru$ and $\Phi_1(u) = z(x)$.
- We seek the solutions of the form

$$z(x) = z(x,\alpha) = \sum_{k=0}^{\infty} a_k\, x^{k+\alpha},$$

with $a_k = a_k(\alpha)$ and $a_0 = 1$.

Finally, we obtain

$$\sum_{l=0}^{m-1}\left\{\sum_{j=m-l}^{m} r^j\,[Kq_{m-j}\,\gamma_{j-(m-l)}(\alpha) -\xi_{m-j-1}]a_{j-(m-l)}[\alpha+j-(m-l)]_{(j)}\right\}x^{\alpha-(m-l)} + \sum_{k=0}^{\infty}\left\{\sum_{j=0}^{m} r^j\,[Kq_{m-j}\,\gamma_{j+k}(\alpha)-\xi_{m-j-1}]a_{j+k}[\alpha+j+k]_{(j)}\right\}x^{\alpha+k}=0.$$

Since $a_0 = 1$ and $r \neq 0$, coefficient of $x^{-m+\alpha}$ is zero if and only if $K\gamma_0(\alpha)\,(\alpha)_{(m)} = 0$, which yields

$$\left(\prod_{i=1}^{n}\left(1-\frac{r\alpha}{\delta+\lambda_i}\right)\right)\,\alpha\,(\alpha-1)\ldots(\alpha-m+1)=0.$$

Generalized Erlang(2) arrivals with exponential claims

For Gen.Erlang(2)IATs

- $$D_0 = \begin{bmatrix} -\lambda_1 & \lambda_1 \\ 0 & -\lambda_2 \end{bmatrix}, \; D_1 = \begin{bmatrix} 0 & 0 \\ \lambda_2 & 0 \end{bmatrix}$$

  Gerber-Shiu equations:

$$(c+ru)\Phi_1'(u) = (\delta+\lambda_1)\Phi_1(u) - \lambda_1\Phi_2(u)$$
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Probability of ruin for Gen.Erlang(2) arrivals with exp. claims

- The probability of ruin $\psi(u)$ is obtained from $\Phi(u)$ with $\delta = 0,\ w(x_1, x_2) \equiv 1.$
- Initial condition becomes:

$$\psi_1(-c/r) = \psi_2(-c/r) = 1.$$

- Changes of variables:

$$c + ru = x, \quad \psi_1(u) = \zeta(x).$$

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- $$\zeta'''(x) + [\beta_r + (3 - \lambda_{1r} - \lambda_{2r})x^{-1}]\zeta''(x)$$
$$+[\beta_r(1 - \lambda_{1r} - \lambda_{2r})x^{-1} + (1 - \lambda_{1r} - \lambda_{2r} + \lambda_{1r}\lambda_{2r})x^{-2}]\zeta'(x) = 0.$$

- Let $y(x) = \zeta'(x)$. Then
- Since $\zeta(0) = \psi_1(-c/r) = 1$:

$$\psi_1(u) = \zeta(c + ru) = 1 + \int_0^{c+ru} y(x)dx.$$

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- Suppose $\lambda_1 \geq \lambda_2$.
- Let :
  $y(x) = x^{\lambda_{1r}-1}\omega(x),\ \tilde{\omega}(x) = e^{\beta_r x}\omega(x),\ \beta_r x = t,\ \tilde{\omega}(x) = \bar{\omega}(t)$
-
  $$t\bar{\omega}''(t) + (1+\lambda_{1r}-\lambda_{2r}-t)\omega'(t) - (1+\lambda_{1r})\bar{\omega}(t) = 0,$$
  "degenerate hypergeometric equation".
-
  $$\begin{aligned} y(x) &= \kappa_1 x^{\lambda_{1r}-1}e^{-\beta_r x}M(1+\lambda_{1r}, 1+\lambda_{1r}-\lambda_{2r}; \beta_r x) \\ &\quad +\kappa_2 x^{\lambda_{1r}-1}e^{-\beta_r x}U(1+\lambda_{1r}, 1+\lambda_{1r}-\lambda_{2r}; +\beta_r x), \end{aligned}$$

- Suppose $\lambda_1 \geq \lambda_2$.
- Let :
  $y(x) = x^{\lambda_{1r}-1}\omega(x),\ \tilde{\omega}(x) = e^{\beta_r x}\omega(x),\ \beta_r x = t,\ \tilde{\omega}(x) = \bar{\omega}(t)$
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$$M(a,b;x) = 1 + \sum_{n=1}^{\infty} \left[\frac{[a]^{(n)}}{n![b]^{(n)}} \right] x^n,$$

$$[a]^{(n)} = a(a+1)\ldots(a+n-1),$$

$$U(a,b;x) = \frac{\pi}{\sin \pi b} \left\{ \frac{M(a,b;x)}{\Gamma(1+a-b)\Gamma(b)} - x^{1-b} \frac{M(1+a-b,2-b;x)}{\Gamma(a)\Gamma(2-b)} \right\},$$

for b non-integer.

For b integer,

$$U(a,n+1;x) = \frac{(-1)^{n+1}}{n!\Gamma(a-n)} \{M(a,n+1;x)\ln(x) + \sum_{l=0}^{\infty} \frac{[a]^{(l)}}{[n+1]^{(l)}} [\varsigma(a+l) - \varsigma(1+l) - \varsigma(1+n+l)] \frac{x^l}{l!} \} + \frac{(n-1)!}{\Gamma(a)} x^{-n} M(a-n, 1-n, x)_n,$$

for $n = 0, 1, 2, \ldots,$ where the subscript n on the last $M(.)$ function denotes the partial sum of the first n terms. This term is to be interpreted as zero when $n = 0$ and $\varsigma(a) = \frac{\Gamma'(a)}{\Gamma(a)}$. Also,

$$\varsigma(1) = -\gamma, \quad \varsigma(n) = -\gamma + \sum_{k=1}^{n-1} k^{-1},$$

and $\gamma = 0.5772...$ is the Euler constant.

When x approaches infinity (Abramowitz and Stegun)

- $$x^{\lambda_{1r}-1}M(1+\lambda_{1r}, 1+\lambda_{1r}-\lambda_{2r}; \beta_r x) = \infty,$$

- $$x^{\lambda_{1r}-1}e^{-\beta_r x}U(1+\lambda_{1r}, 1+\lambda_{1r}-\lambda_{2r}, +\beta_r x) = 0.$$

- $\lim_{x\to\infty} y(x) = \lim_{u\to\infty} \psi_1'(u) = 0$, so $\kappa_1 = 0$.
- Therefore, $y(x) = \kappa_2 x^{\lambda_{1r}-1} e^{-\beta_r x} U(1+\lambda_{1r}, 1+\lambda_{1r}-\lambda_{2r}; +\beta_r x)$, which yields

$$\zeta(x) = 1 + \kappa_2 \int_0^x v^{\lambda_{1r}-1} e^{-\beta_r v} U(1+\lambda_{1r}, 1+\lambda_{1r}-\lambda_{2r}; \beta_r v) dv.$$

- Recall that $\lim_{u\to\infty} \psi_1(u) = 0$.

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- Recall that $\lim_{u\to\infty} \psi_1(u) = 0$.

- Finally,

$$\kappa_2 = \frac{1}{\int_0^\infty v^{\lambda_{1r}-1} e^{-\beta_r v} U(1+\lambda_{1r}, 1+\lambda_{1r}-\lambda_{2r}; \beta_r v) dv}$$

and

- $$\psi_1(u) = 1 - \frac{\int_0^{ru+c} v^{\lambda_{1r}-1} e^{-\beta_r v} U(1+\lambda_{1r}, 1+\lambda_{1r}-\lambda_{2r}; \beta_r v) dv}{\int_0^\infty v^{\lambda_{1r}-1} e^{-\beta_r v} U(1+\lambda_{1r}, 1+\lambda_{1r}-\lambda_{2r}; \beta_r v) dv}. \quad (9)$$

Examples. Numerical results

- We consider Example 6.1. from Gerber and Yang (2007) (Exp interclaims) versus our results for (Gen)Erlang(2) interclaims.
- We assume the interclaims are gen-Erlang (2), with param. λ_1 and λ_2 under three different scenarios, summarized in the following table.
- The claim size is exponential $\beta = 0.5$ and the premium rate $c = 2$.
- We assume that the interest rate is constant at $r = 0.1$.

Table: Absolute ruin probabilities

u	$\lambda_1 = 1, \lambda_2 = 0.5$	$\lambda_1 = \lambda_2 = 1$	$\lambda_1 = \lambda_2 = 2$	Exp(1)
50	1.6259×10^{-14}	6.4067×10^{-13}	6.4575×10^{-9}	1.821×10^{-7}
10	0.0103×10^{-3}	0.1658×10^{-3}	0.0396	0.0698
5	0.0121×10^{-2}	0.1539×10^{-2}	0.1514	0.2014
1	0.0844×10^{-2}	0.8405×10^{-2}	0.3552	0.3971
0	0.0013	0.0126	0.4238	0.4579
-1	0.0021	0.0188	0.4975	0.5218
-5	0.0137	0.0847	0.7939	0.7764
-10	0.1150	0.3934	0.9835	0.9681
-20	1	1	1	1

Observations

- In all scenarios, as the initial surplus decreases the absolute ruin probability increases.
- Comparing the first three columns, it is clear that the most risky case is the third one, where the process waits less in average for a claim to appear.
- Comparing the last two columns, the latter Erlang(2) and the exponential case are different, although they have the same mean 1.
  - For positive values of the initial surplus, the Erlang case is less likely to lead to ruin than the exponential. However, for sufficient negative values of the surplus, the reverse situation happens.
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Conclusions and possible extensions

- We presented a unifying approach for the determination of the G-S function related to absolute ruin in a single layer Sparre Andersen risk model in the presence of a constant interest rate.
  - These results can be easily extended to the multi-layer case, since the equations are structurally the same.
  - It seems that one can use the same methodology by replacing the Generalized Erlang(n) interclams by a Triangular Phase-type distribution as considered in O'Cinneide (1993).
- We remark that it is very challenging to obtain closed-form solutions for the absolute ruin probability if we move away from the exponential assumption for claim sizes, or if we assume a higher order generalized Erlang interclaim time distributioin. However, one can use out methodology to obtain numerical results for any ME claim sizes.

Thank You!