Coherent Distortion Risk Measures in Portfolio Selection (Joint work with Dr Ken Seng Tan)

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Abstract

The theme of this presentation relates to solving portfolio selection problems using linear and fractional programming.
Two key contributions:

- Generalization of the CVaR linear optimization framework (see Rockafellar and Uryasev [3, 4]).
- Equivalences among four formulations of CDRM optimization problems.

Outline

Motivations

- Practical portfolio selection problems
- Good risk measures
- Well-studied programming models

Question

Can we connect this together? We want to solve practical portfolio optimization problems with sophisticated risk measures using a programming model that can be solved efficiently.

We wish to..

- Incorporate a general class of risk measure into a well-studied programming model
- Study equivalences among different formulations of portfolio selection problems
- Solve portfolio selection problems of interest efficiently

Outline

Scenario Generation

Loss Matrix

$$
\begin{array}{cc}
p_1 & \rightarrow \\
p_2 & \rightarrow \\
\vdots & \vdots \\
p_m & \rightarrow
\end{array}
\; \boldsymbol{L} = \begin{bmatrix}
L_{11} & L_{12} & \cdots & L_{1n} \\
L_{21} & L_{22} & \cdots & L_{2n} \\
\vdots & \ldots & \ddots & \vdots \\
L_{m1} & L_{m2} & \cdots & L_{mn}
\end{bmatrix}
\begin{array}{cc}
\rightarrow & l_1 = l(\boldsymbol{x}, p_1) \\
\rightarrow & l_2 = l(\boldsymbol{x}, p_2) \\
\vdots & \vdots \\
\rightarrow & l_m = l(\boldsymbol{x}, p_m)
\end{array}
$$

Let $l_{(1)} \leq \cdots \leq l_{(m)}$ be the ordered losses, $p_{(i)}$, $i = 1, \cdots, m$ be the corresponding probability masses.

Return/Price/Premium/Profit Vector

$$\boldsymbol{c} = [c_1, \cdots, c_m]'$$

CVaR Optimization

Background

Consider the special function

$$F(\boldsymbol{x}, \zeta) = \zeta + \frac{1}{1-\alpha} \sum_{j=1}^{m} p_j (l_j - \zeta)^+$$

Rockafellar and Uryasev [3, 4] showed that

1. $CVaR_\alpha(\boldsymbol{x}) = \min_{\zeta \in \mathbb{R}} F(\boldsymbol{x}, \zeta)$
2. $\min_{\boldsymbol{x} \in \boldsymbol{X}} CVaR_\alpha(\boldsymbol{x}) = \min_{(\boldsymbol{x}, \zeta) \in \boldsymbol{X} \times \mathbb{R}} F(\boldsymbol{x}, \zeta)$

CVaR Optimization

CVaR portfolio selection problems can be formulated as LPs. Suppose $\boldsymbol{X}$ is the set of all feasible portfolios.

CVaR minimization subject to a return constraint

$$
\begin{array}{lrcll}
\text{minimize} & \zeta + \frac{1}{1-\alpha}\sum_{j=1}^{m} p_j z_j & & & \\
\text{subject to} & \boldsymbol{c}'\boldsymbol{x} & \geq & \mu & \\
& l(\boldsymbol{x}, p_j) - \zeta & \leq & z_j & j = 1, \cdots, m \\
& 0 & \leq & z_j & j = 1, \cdots, m \\
& (\boldsymbol{x}, \zeta) & \in & \boldsymbol{X} \times \mathbb{R} &
\end{array}
$$

CVaR Optimization

Return maximization subject to CVaR constraint(s)

$$
\begin{array}{llcll}
\text{maximize} & \boldsymbol{c}'\boldsymbol{x} & & & \\
\text{subject to} & \zeta_i + \frac{1}{1-\alpha}\sum_{j=1}^{m} p_j z_{ij} & \leq & \eta_i & i = 1, \cdots, k \\
 & \boldsymbol{l}(\boldsymbol{x}, p_j) - \zeta_i & \leq & z_{ij} & \forall i, j \\
 & 0 & \leq & z_{ij} & \forall i, j \\
 & (\boldsymbol{x}, \zeta) & \in & \boldsymbol{X} \times \mathbb{R}^k &
\end{array}
$$

Definition and Representation Theorem

Two Equivalent Definitions

A risk measure $\rho(\boldsymbol{x})$ is a CDRM if it is

- A comonotone law-invariant coherent risk measure
- A distortion risk measure with a concave distortion function

Representation Theorem for CDRM

A risk measure $\rho(\boldsymbol{x})$ is a CDRM if and only if there exists a function $w : [0, 1] \mapsto [0, 1]$, satisfying $\int_{\alpha=0}^{1} w_\alpha d\alpha = 1$, such that

$$\rho(\boldsymbol{x}) = \int_{\alpha=0}^{1} CVaR_\alpha(\boldsymbol{x}) w_\alpha d\alpha$$

Representation Theorem in Discrete Case

Finite Generation Theorem for CDRM

Given a concave distortion function g, $\rho(\boldsymbol{x}) = \sum_{i=1}^{m} q_i l_{(i)}$, moreover

$$\rho(\boldsymbol{x}) = \sum_{i=1}^{m} w_i CVaR_{\frac{i-1}{m}}(\boldsymbol{x}), \text{ where}$$

$$w_i = \begin{cases} \frac{q_1}{p_{(1)}} & \text{if} \quad i = 1 \\ (q_i - \frac{p_{(i)}}{p_{(i-1)}} q_{i-1}) \frac{\sum_{j=i}^{m} p_{(j)}}{p_{(i)}} & \text{if} \quad i = 2, \cdots, m \end{cases}$$

CDRM Optimization

CDRM minimization subject to a return constraint

$$
\begin{array}{llcll}
\text{minimize} & \sum_{i=1}^{m} w_i(\zeta_i + \frac{1}{1-\alpha}\sum_{j=1}^{m} p_j z_{ij}) & & & \\
\text{subject to} & \boldsymbol{c}'\boldsymbol{x} & \geq & \mu & \\
 & \boldsymbol{l}(\boldsymbol{x}, p_j) - \zeta_i & \leq & z_{ij} & \forall i, j \\
 & 0 & \leq & z_{ij} & \forall i, j \\
 & (\boldsymbol{x}, \zeta) & \in & \boldsymbol{X} \times \mathbb{R}^m &
\end{array}
$$

CDRM Optimization

Return maximization subject to one CDRM constraint

$$
\begin{array}{llll}
\text{maximize} & \boldsymbol{c}'\boldsymbol{x} & & \\
\text{subject to} & \sum_{i=1}^{m} w_i(\zeta_i + \frac{1}{1-\alpha}\sum_{j=1}^{m} p_j z_{ij}) & \leq \eta & \\
 & \boldsymbol{l}(\boldsymbol{x}, p_j) - \zeta_i & \leq z_{ij} & \forall i, j \\
 & 0 & \leq z_{ij} & \forall i, j \\
 & (\boldsymbol{x}, \zeta) & \in \boldsymbol{X} \times \mathbb{R}^m &
\end{array}
$$

CDRM Optimization

Return-CDRM utility maximization

$$
\begin{array}{llll}
\text{maximize} & \boldsymbol{c}'\boldsymbol{x} - \tau \sum_{i=1}^{m} w_i(\zeta_i + \frac{1}{1-\alpha} \sum_{j=1}^{m} p_j z_{ij}) & & \\
\text{subject to} & \boldsymbol{l}(\boldsymbol{x}, p_j) - \zeta_i \leq z_{ij} & & \forall i, j \\
 & 0 \leq z_{ij} & & \forall i, j \\
 & (\boldsymbol{x}, \zeta) \in \boldsymbol{X} \times \mathbb{R}^m & &
\end{array}
$$

This formulation is very similar to a return maximization problem with m CVaR constraints. Yet we converted m CVaR constraints into the objective function.

CDRM Optimization

CDRM-based Sharpe ratio maximization

$$
\begin{array}{llcll}
\text{maximize} & \dfrac{\mathbf{c}'\mathbf{x}-\nu}{\sum_{i=1}^{m} w_i(\zeta_i+\frac{1}{1-\alpha}\sum_{j=1}^{m} p_j z_{ij})} & & & \\
\text{subject to} & \mathbf{l}(\mathbf{x}, p_j) - \zeta_i & \leq & z_{ij} & \forall i,j \\
 & 0 & \leq & z_{ij} & \forall i,j \\
 & (\mathbf{x}, \zeta) & \in & \mathbf{X} \times \mathbb{R}^m &
\end{array}
$$

This is an LFP, but we can solve it by solving at most two related LPs using a variable transformation method studied by Charnes and Cooper [1].

Formulation Equivalences

Equivalences among four formulations, part 1

Problem	Max-Return	Min-CDRM
Preset Parameter	η	μ
Implied Parameters		
$\eta =$	N/A	$\rho(\boldsymbol{x}^*)$
$\mu =$	$\boldsymbol{c}'\boldsymbol{x}^*$	N/A
$\tau =$	u^1	$\frac{1}{u^2}$
$\nu =$	$\boldsymbol{c}'\boldsymbol{x}^* - u^1\rho(\boldsymbol{x}^*)$	$R(x^*) - \frac{1}{u^2}\rho(\boldsymbol{x}^*)$

If the return and CDRM constraints are binding at respective optimal solutions, the preset parameter for Max-Return equals to the implied parameter for Min-CDRM and vice versa.

Formulation Equivalences

Equivalences among four formulations, part 1

Problem	Max-Utility	Max-Sharpe
Preset Parameter	τ	ν
Implied Parameters		
$\eta =$	$\rho(\boldsymbol{x}^*)$	$\rho(\boldsymbol{x}^*)$
$\mu =$	$\boldsymbol{c}'\boldsymbol{x}^*$	$\boldsymbol{c}'\boldsymbol{x}^*$
$\tau =$	N/A	$\frac{\boldsymbol{c}'\boldsymbol{x}^* - \nu}{\rho(\boldsymbol{x}^*)}$
$\nu =$	$\boldsymbol{c}'\boldsymbol{x}^* - \tau\rho(\boldsymbol{x}^*)$	N/A

We will see that the preset parameter for Max-Return equals to the implied parameter for Min-CDRM and vice versa.

Outline

Case Study 1: Constructing Reinsurance Portfolios

We wish to construct profit-$CVaR_{0.95}(\boldsymbol{L})$ efficient portfolios from the following 10 risk contracts. Simulations are done for 10,000 scenarios.

Contract	Premium	Losses			
		Mean	STD	95%VaR	95%CVaR
1	554271	311388	1377843	2613161	5885442
2	364272	222117	1172497	588329	4338214
3	91763	55953	739026	0	1119065
4	867176	437968	1806626	3845685	7937610
5	798005	438464	2913258	0	8769284
6	107585	43381	263019	0	867624
7	878525	375438	1375166	3160679	5974087
8	3081188	1283828	2199151	5661191	8442634
9	65162	29352	324061	0	587044
10	885897	385173	1047454	1506500	3693435

Case Study 1: Constructing Reinsurance Portfolios

Balanced portfolio consisting of 0.1 unit of each risk.

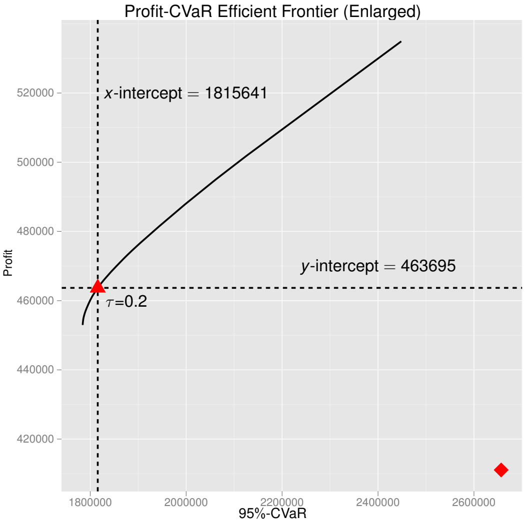
Summary of balanced portfolio

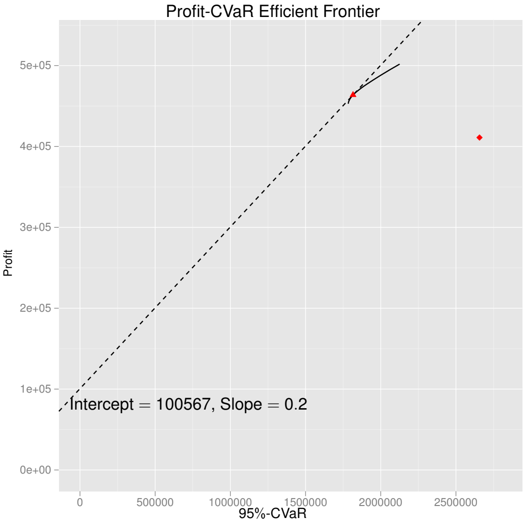
Premium	Losses				Expected Profit
	Mean	STD	95%VaR	95%CVaR	
769384	358306	667647	1716458	2656764	40578

Profit-95%CVaR utility maximization with $\tau = 0.2$

Summary of target portfolio

Premium	Losses				Expected Profit
	Mean	STD	95%VaR	95%CVaR	
769384	305689	492425	1313074	1815641	463695

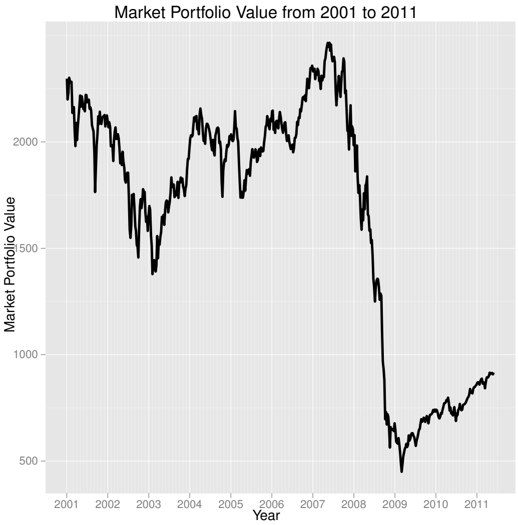

Profit-CVaR Efficient Frontier (Enlarged)
x-intercept = 1815641
y-intercept = 463695
τ=0.2
Profit
520000
500000
480000
460000
440000
420000
1800000
2000000
2200000
2400000
2600000
95%-CVaR


Profit-CVaR Efficient Frontier
5e+05
4e+05
3e+05
2e+05
1e+05
0e+00
Profit
Intercept = 100567, Slope = 0.2
0
500000
1000000
1500000
2000000
2500000
95%-CVaR

Data decriptions

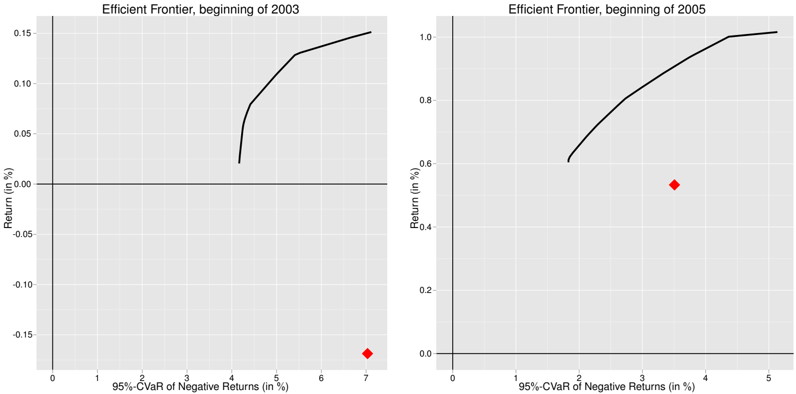
- 2 stocks from each of the 10 sectors defined in Global Industry Classification Standard(GICS).
- Weekly prices from Jan-02-2001 to May-31-2011
- Adjusted closing prices obtained from *finance.yahoo.com*

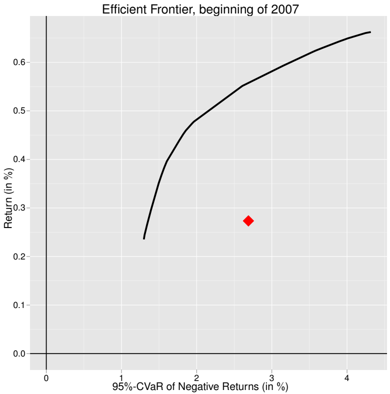
Sum of these 20 stocks' prices can be viewed as the "market"

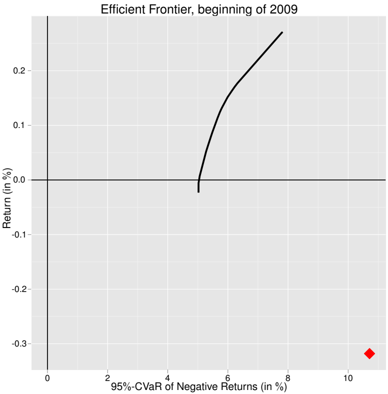


Optimization Settings

- Replace scenario generation by historical data
- Constant "sample" size of 100.
- $\boldsymbol{c}$ = expected sample returns, $\boldsymbol{L}$ = negative returns matrix.
- Weekly rebalancing via CDRM-minimization.
- $\boldsymbol{x} \geq \mathbf{0}$, $\boldsymbol{x} \leq \mathbf{0.2}$, budget constraint, and return constraint.


Efficient Frontier, beginning of 2003
Return (in %)
0.15
0.10
0.05
0.00
-0.05
-0.10
-0.15
0
1
2
3
4
5
6
7
95%-CVaR of Negative Returns (in %)
Efficient Frontier, beginning of 2005
Return (in %)
1.0
0.8
0.6
0.4
0.2
0.0
0
1
2
3
4
5
95%-CVaR of Negative Returns (in %)

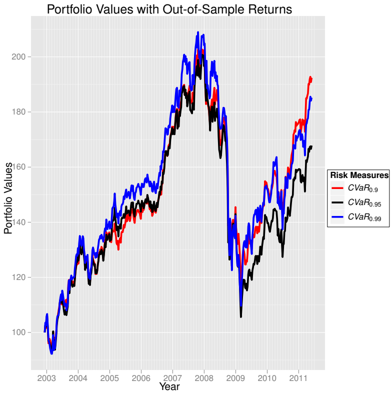

Efficient Frontier, beginning of 2007
Return (in %)
0.0
0.1
0.2
0.3
0.4
0.5
0.6
0
1
2
3
4
95%-CVaR of Negative Returns (in %)

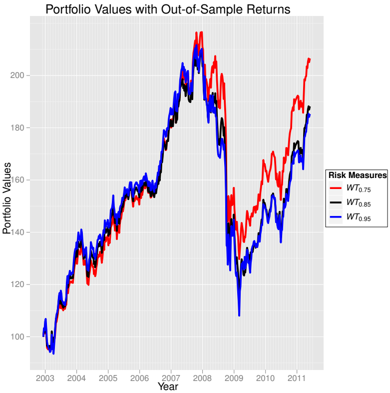

Efficient Frontier, beginning of 2009
Return (in %)
-0.3
-0.2
-0.1
0.0
0.1
0.2
0
2
4
6
8
10
95%-CVaR of Negative Returns (in %)

Portfolio Selection over Different CDRMs

Well-known CDRMs

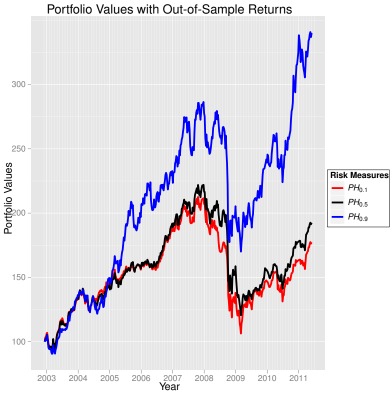
- $CVaR_\alpha$ distortion:
  $g_{CVaR}(x,\alpha) = \min\{\frac{x}{1-\alpha}, 1\}$
- Wang Transform(WT) distortion:
  $g_{WT}(x,\beta) = \Phi[\Phi^{-1}(x) - \Phi^{-1}(\beta)]$
- Proportional hazard(PH) distortion:
  $g_{PH}(x,\gamma) = x^\gamma$ with $\gamma \in (0,1]$
- Lookback(LB) distortion:
  $g_{LB}(x,\delta) = x^\delta(1-\delta\ln x)$ with $\delta \in (0,1]$

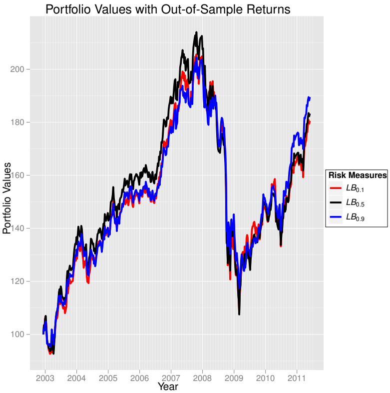

Portfolio Values with Out-of-Sample Returns
Portfolio Values
200
180
160
140
120
100
2003 2004 2005 2006 2007 2008 2009 2010 2011
Year
Risk Measures
$CVaR_{0.9}$
$CVaR_{0.95}$
$CVaR_{0.99}$


Portfolio Values with Out-of-Sample Returns
Portfolio Values
200
180
160
140
120
100
2003 2004 2005 2006 2007 2008 2009 2010 2011
Year
Risk Measures
$WT_{0.75}$
$WT_{0.85}$
$WT_{0.95}$

Summary statistics of optimal out-of-sample returns

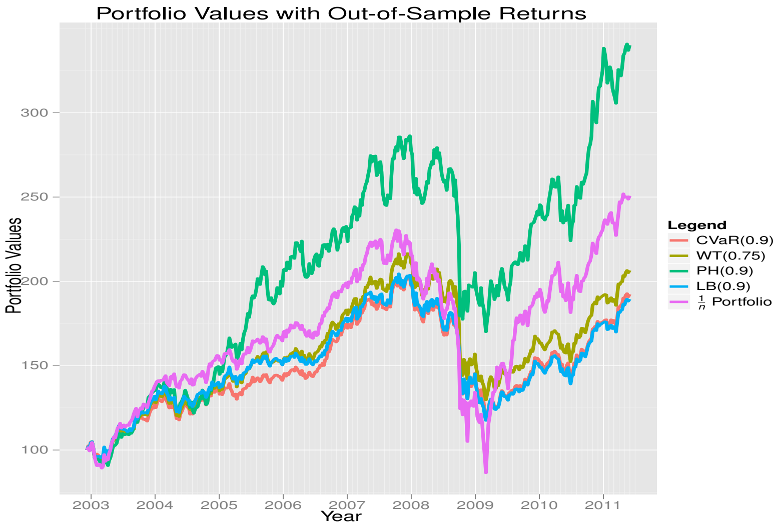
	Mean	STD	Skew	Kurt	Sharpe
$CVaR_{0.9}$	0.00148	0.01891	-0.93697	6.08202	0.07833
$CVaR_{0.95}$	0.00117	0.02050	-0.56738	4.96513	0.05718
$CVaR_{0.99}$	0.00139	0.02243	-0.20805	4.47107	0.06219
$WT_{0.75}$	0.00164	0.01919	-1.00243	7.06069	0.08560
$WT_{0.85}$	0.00261	0.01915	-0.77534	5.88635	0.07477
$WT_{0.95}$	0.00232	0.02107	-0.30517	5.46812	0.06628


Portfolio Values with Out-of-Sample Returns
Portfolio Values
Year
Risk Measures
$PH_{0.1}$
$PH_{0.5}$
$PH_{0.9}$
100
150
200
250
300
2003
2004
2005
2006
2007
2008
2009
2010
2011


Portfolio Values with Out-of-Sample Returns
Portfolio Values
Year
Risk Measures
$LB_{0.1}$
$LB_{0.5}$
$LB_{0.9}$
100
120
140
160
180
200
2003
2004
2005
2006
2007
2008
2009
2010
2011

Summary statistics of optimal out-of-sample returns

	Mean	STD	Skew	Kurt	Sharpe
$PH_{0.1}$	0.00130	0.02218	-0.26156	5.14293	0.05844
$PH_{0.5}$	0.00148	0.02091	-0.83421	8.50931	0.07091
$PH_{0.9}$	0.00277	0.02622	-0.95739	6.78000	0.10574
$LB_{0.1}$	0.00134	0.02230	-0.22880	4.59996	0.05995
$LB_{0.5}$	0.00137	0.02130	-0.34008	5.15387	0.06439
$LB_{0.9}$	0.00145	0.01893	-0.80400	6.04230	0.07645


Portfolio Values with Out-of-Sample Returns
Portfolio Values
300
250
200
150
100
2003
2004
2005
2006
2007
2008
2009
2010
2011
Year
Legend
CVaR(0.9)
WT(0.75)
PH(0.9)
LB(0.9)
$\frac{1}{n}$ Portfolio

Summary statistics of optimal out-of-sample returns

	Mean	STD	Skew	Kurt	Sharpe
$\frac{1}{n}$-portfolio	0.00208	0.03038	0.25175	13.73943	0.06845
$CVaR_{0.9}$	0.00148	0.01891	-0.93697	6.08202	0.07833
$WT_{0.75}$	0.00164	0.01919	-1.00243	7.06069	0.08560
$PH_{0.9}$	0.00277	0.02622	-0.95739	6.78000	0.10574
$LB_{0.9}$	0.00145	0.01893	-0.80400	6.04230	0.07645

Outline

Linear optimization for CDRM portfolio selection

- CDRM portfolio optimization with LPS and LFPs
- CDRM includes CVaR, WT, PH, and LB
- Choose CDRM that suits specific risk appetites
- Four different CDRM formulations are equivalent
- Equivalences are helpful for interpretation of parameters, verification of consistencies, and estimation of implied information

Empirical results

- Simple portfolio construction rules can be very inefficient, active management is important.
- Despite the inefficiency of the $\frac{1}{n}$-portfolio, its terminal wealth (based on out-of sample returns) can be high
- We have found CDRM efficient portfolios with higher Sharpe ratio than the $\frac{1}{n}$-portfolio's

Future Directions

- Apply various decomposition methods to solve CDRM problems more efficiently
- Apply stochastic programming techniques to solve CDRm problems
- Apply CDRM approach in multi-period models
- Explore/identify other members of CDRM (Higher moment coherent risk measure)

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