Implementation of Intensity Model Approach to Constant Maturity Credit Default Swap Pricing

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Abstract: Constant maturity credit default swaps (CMCDS) are useful as hedging tools. In intensity model approach, the default time is defined as the first arrival time of the Poisson process. From the market quotes of CDS forward rates and bonds, we are able to numerically compute the default probabilities. Approximating CMCDS price depends largely on CDS forward rates' volatilities and their correlations. We implement the price algorithm based on Brigo's work (2006). Starting with current market data such as CDS forward rates and non-defaultable bond prices, we describe steps involved to obtain the price CMCDS. We demonstrate the impact of convexity on the CMCDS price structure.

Introduction

A Constant Maturity Credit Default Swap (CMCDS) is a combination of a Constant Maturity Swap and Credit Default Swap. The valuation of a CMCDS is implemented in Excel VBA and derives important quantities relating to CMCDS valuation based on the work of D. Brigo[1]

1. Credit Default Swap (CDS)

A CDS contract insures protection against default. If a third company C (Reference credit) defaults at the time τ_c with $T_a < \tau_c \leq T_b$, then B (Protection Seller) pays to A (Protection buyer) a certain cash amount $L_{GD.}$ In turn, A pays to B a rate R at time $T_{a+1}, \cdots T_b$ or until default τ_c.

2. Constant Maturity Credit Default Swap (CMCDS)

Consider a contract protecting in $[T_a, T_b]$ against default of a reference credit C. If default occurs in $[T_a, T_b]$, a protection payment L_{GD} is made from the protection seller B to the protection buyer A at the first T_j following the default time, called protection leg. A pays to B at each T_j before default a C+1 – long CMCDS rate $R_{i-1,i+c}(T_{i-1})$

$$\mathbf{B} \rightarrow \text{protection } L_{GD} \text{ at default } \tau_c \text{ if } T_a < \tau_c \leq T_b \rightarrow \mathbf{A}$$
$$\leftarrow R_{i-1,i+c}(T_{i-1}) \text{ at } T_i = T_{a+1}, \ \ldots \ , T_b \text{ or until } \tau_c \leftarrow$$

The value of the CMCDS to "B" is the value of the premium leg minus the value of the protection leg.

3. Constant Maturity Swap (CMS)

A swap contract is when, on specified payment dates, party 1 agrees to pay the floating LIBOR rate of a notional amount to party 2 and in return party 2 agrees to pay a fixed swap rate to party 1 of the same notional amount. A constant maturity swap differs in that neither group will be paying a fixed rate. In a constant maturity swap scenario, party 1 agrees to pay the floating LIBOR rate on a notional amount to party 2 and party 2 will agree to pay a floating c-period swap-rate (CMS rate) on the same notional amount to party 1. The CMS rate is not constant and is calculated at the reset dates stated on the contract. The period of time 'c' used to calculate the CMS rate will be stated in the contract. In essence both parties will be paying a fluctuating rate. A CMS can be used to hedge against short term changes in interest rates.

4. CDS Payoff

Instead of considering the exact default time τ , the protection payments L_{GD} is postponed to the first time T_i following default and the premium payment R is paid at T_i as long as the default occurs after T_i. Consequently the CDS payoff as seen from B can be expressed as

$$\Pi_{PRCDS_{a,b}}(t) := \sum_{i=a+1}^{b} D(t,T_i)\alpha_i R 1_{\{\tau \geq T_i\}} - \sum_{i=a+1}^{b} 1_{\{T_{i-1} < \tau \leq T_i\}} D(t,T_i) L_{GD}$$

where α_i is the year fraction between T_i and T_{i-1} and $D(t,T_i)$ is a stochastic discount factor at time t for maturity T_i.

5. CDS Value at time t

The CDS price with respect to the risk neutral valuation can be written as

$$PRCDS_{a,b}(t,R,L_{GD}) = E\{\Pi_{PRCDS_{a,b}}(t) \mid G_t\}$$
$$= \frac{1_{\{\tau>t\}}}{\Pr(\tau > t \mid F_t)} E[\Pi_{PRCDS_{a,b}}(t) \mid F_t]$$

where the filtration G_t includes default information and default free market information F_t. That is, $G_t = F_t \vee \sigma(\{\tau < u\}, u \le t)$

Intensity Models

We describe the default by means of Poisson's first arrival process. We assume the default is independent of all the default free market information. We denote default time by τ. We consider time in-homogeneous Poisson process and let $\lambda(t)$ be intensity (hazard) rate function. Having not defaulted before t, the risk neutral probability of defaulting in the next dt instants is

$$\Pr(\tau \in [t, t+dt) \big| \tau > t, \text{ Market information up to } t) = \lambda(t)dt$$

We define the cumulative intensity function by

$$\Gamma(t) = \int_0^t \lambda(u)du$$

It is called hazard function. Assume $\lambda(t)$ is a deterministic function. We define

$$\xi := \Gamma(\tau) = \int_0^\tau \lambda(u)du \,.$$

It is known that ξ is a standard exponential random variable. That is, $\Pr(\xi \le x) = 1 - e^{-x}$. We can show that

$$\Pr(\tau > t) = \Pr(\Gamma(\tau) > \Gamma(t)) = \Pr(\xi > \Gamma(t)) = e^{-\int_0^t \lambda(u)du}$$

If $\lambda(u)$ is a stochastic process, then

$$\Pr(\tau > t) = \Pr(\Gamma(\tau) > \Gamma(t)) = \Pr(\xi > \Gamma(t)) = E[e^{-\int_0^t \lambda(u)du}]$$

$$\Pr\left(s < \tau \le t\right) = e^{-\Gamma(s)} - e^{-\Gamma(t)} \approx e^{\int_s^t \lambda(u)du}$$

The defaultable bond price $\bar{P}(t,T)$ is defined as:

$$1_{\{\tau>t\}}\bar{P}(t,T) = E[D(t,T)1_{\{\tau>T\}}\left|G_t\right.] = E[e^{-\int_t^T (r(u)+\lambda(u))du}]$$

Thus, the survival probability looks like the price of a zero coupon bond in an interest rate model with short rate r replaced by $\lambda(t)$ and $\lambda(t)$ is interpreted as instantaneous credit spread. In particular,

$$\bar{P}(0,T) = E[D(0,T)1_{\{\tau>T\}}\left|G_t\right.] = E[e^{-\int_0^T (r(u)+\lambda(u))du}]$$

The Filtration Switching Formula

It can be shown that the following switching formula is valid:

$$E[1_{\{\tau>T\}} Payoff \left|G_t\right.] = \frac{1_{\{\tau>t\}}}{Q\{\tau > t\left|F_t\right.\}} E[1_{\{\tau>T\}} Payoff \left|F_t\right.]$$

where Q represents probability.

The proof of this result is in the reference [1]. Switching from G_t to F_t is useful because most times it is easy to compute the F_t conditional expectation.

CDS Forward Rates

The CDS forward rates can be defined as that rate R that makes the CDS value equal to zero at time t. That is, $CDS_{a,b}(t, R, L_{GD}) = E\{\Pi_{PRCDS_{a,b}}(t) \,|\, G_t\} = 0$. where Π represents CDS payoff at time t.

A defaultable zero coupon bond (DZCB), $\bar{P}(t,T)$ is defined as

$$1_{\{\tau>t\}}\bar{P}(t,T) := E[D(t,T)1_{\{\tau>T\}} \mid G_t]$$

Using the filtration switching formula, the following expression can be obtained

$$E[D(t,T)1_{\{\tau>T\}} \mid F_t] = Q(\tau > t \mid F_t)\bar{P}(t,T)$$

$$CDS_{a,b}(t,R,L_{GD}) = E[\Pi_{PRCDS_{a,b}}(t)\big|G_t]$$
$$= \frac{1_{\{\tau>t\}}}{\Pr(\tau>t\big|F_t)}E[\Pi_{PRCDS_{a,b}}(t)\big|F_t]$$

$$= \frac{1_{\{\tau>t\}}}{Q(\tau>t\big|F_t)}\left\{\sum_{i=a+1}^{b}\alpha_i E\left[D(t,T_i)R1_{\{\tau>T_i\}}\big|F_t\right] - L_{GD}\sum_{i=a+1}^{b}E\left[D(t,T_i)1_{\{T_{i-1}<\tau\le T_i\}}\big|F_t\right]\right\}$$

$$= \frac{1_{\{\tau>t\}}}{Q(\tau>t\big|F_t)}\left\{\sum_{i=a+1}^{b}\alpha_i RQ\left(\tau>t\big|F_t\right)\bar{P}(t,T_i) - L_{GD}\sum_{i=a+1}^{b}E\left[D(t,T_i)1_{\{T_{i-1}<\tau\le T_i\}}\big|F_t\right]\right\}$$

Hence, we have this expression,

$$R^{PR}_{a,b}(t) = \frac{L_{GD}\sum_{i=a+1}^{b}E[D(t,T_i)1_{\{T_{i-1}<\tau<T_i\}} \mid F_t]}{\sum_{i=a+1}^{b}\alpha_i E[D(t,T_i)1_{\{\tau>T_i\}}\big|F_t]} = \frac{L_{GD}\sum_{i=a+1}^{b}E[D(t,T_i)1_{\{T_{i-1}<\tau<T_i\}} \mid F_t]}{\sum_{i=a+1}^{b}\alpha_i \Pr(\tau>t\big|F_t)\bar{P}(t,T_i)}$$

CDS Options

Consider the option for a protection buyer to enter a CDS at a future time $T_a > 0$, $T_a < T_b$, paying a fixed premium rate K at times $T_{a+1},...,T_b$ or in exchange for a protection payment L_{GD} until default happens in $[T_a,T_b]$. This option expires at T_a. The discounted CDS option payoff at time t is:

$$\Pi_{callPRCDS_{a,b}}(t;K) = D(t,T_a)\left[CDS_{a,b}(T_a,R_{a,b}(T_a),L_{GD}) - CDS_{a,b}(T_a,K,L_{GD})\right]^+$$
$$= D(t,T_a)\left[-CDS_{a,b}(T_a,K,L_{GD})\right]^+$$

which can be also written:

$$\Pi_{callPRCDS_{a,b}}(t;K)=\frac{1_{\{\tau>T_a\}}}{Q(\tau>T_a|F_{T_a})}D(t,T_a)\left[\sum_{i=a+1}^{b}\alpha_i Q(\tau>T_a|F_{Ta})\bar{P}(T_a,T_i)\right](R_{a,b}(T_a)-K)^+$$

Thus, we obtain the market formula for CDS option:

$$\begin{aligned}
&CallCDS_{a,b}(t,K,L_{GD})=E\left[\Pi_{CallCDS_{a,b}}(t,K;L_{GD})\middle|G_t\right]\\
&=E\left\{1_{\{\tau>T_a\}}D(t,T_a)\bar{C}_{a,b}(T_a)(R_{a,b}(T_a)-K)^+\middle|G_t\right\}\\
&=1_{\{\tau>t\}}\bar{C}_{a,b}(t)\left[R_{a,b}(t)N(d_1(t))-KN(d_2(t))\right]
\end{aligned}$$

where $d_{1,2}=\dfrac{(\frac{\ln(R_{a,b}(t)}{K})\pm\frac{(T_a-t)\sigma_{a,b}^2}{2})}{\sigma_{a,b}\sqrt{T_a-t}}$

One period CDS forward rate $R_j(t)$

$R_{a,b}(t)$ can be expressed as a linear combination of one period CDS forward rates $R_j(t)$ as the swap rate, $S_{a,b}(t)$ is expressed as a linear combination of forward rates $F_j(t)=F(t,T_{j-1},T_j)$. The one period CDS rates is defined as

$$R_j(t)=R_{j-1,j}(t):=\frac{L_{GD}E\left[D(t,T_j)\ \ 1_{\{T_{j-1}<\tau<T_j\}}\middle|F_t\right]}{\alpha_j Q(\tau>t|F_t)\ \ \bar{P}(t,T_j)}$$

Let $p(t,T_i)$ be a zero-coupon bond, and $F_i(t)=F(t,T_{i-1},T_i)$ be a forward rate maturing at T_i. The swap rate is defined as:

$$S_{a,b}(t)=\frac{\sum_{i=a+1}^{b}\alpha_i P(t,T_i)F_i(t)}{\sum_{i=a+1}^{b}\alpha_i P(t,T_i)}$$

Likewise, we define the credit default swap rate as:

$$R_{a,b}(t) = \frac{\sum_{i=a+1}^{b} \alpha_i R_i(t) \bar{P}(t,T_i)}{\sum_{i=a+1}^{b} \alpha_i \bar{P}(t,T_i)}$$

$$= \sum_{i=a+1}^{b} \bar{W}_i(t) R_i(t) \approx \sum_{i=a+1}^{b} \bar{W}_i(0) R_i(t)$$

One period CDS rate $R_j(t)$ is approximated by $\tilde{R}_j(t)$. $\tilde{R}_j(t)$ is defined as:

$$\frac{L_{GD}[\mathrm{E}[D(t,T_{j-1})1_{\{\tau > T_{j-1}\}} \frac{P(t,T_j)}{P(t,T_{j-1})} | F_t] - \mathrm{E}[D(t,T_j)1_{\{\tau > T_j\}} | F_t]}{\alpha_j Q(\tau > t | F_t) \bar{P}(t,T_j)}$$

$$= L_{GD} \frac{\bar{P}(t,T_{j-1}) \frac{P(t,T_j)}{P(t,T_{j-1})} - \bar{P}(t,T_j)}{\alpha_j \bar{P}(t,T_j)}$$

$$= \frac{L_{GD}}{\alpha_j} \left(\frac{\bar{P}(t,T_{j-1})}{(1+\alpha_j F_j(t)) \bar{P}(t,T_j)} - 1 \right)$$

$$\approx \frac{L_{GD}}{\alpha_j} \left(\frac{\bar{P}(t,T_{j-1})}{(1+\alpha_j F_j(0)) \bar{P}(t,T_j)} - 1 \right)$$

Let $\hat{C}_{j-1,j}(t) = \alpha_j Q(\tau > t | F_t) \bar{P}(t,T_j)$. Then, $\tilde{R}_j(t)$ is a martingale under the probability measure $\hat{Q}_{j-1,j}$ associated with numeraire $\hat{C}_{j-1,j}$.

In other words, we have this expression

$$\frac{\bar{P}(t,T_{j-1})}{\bar{P}(t,T_j)} = \left(\frac{\alpha_j}{L_{GD}} \tilde{R}_j + 1 \right) \left(1 + \alpha_j F_j(0) \right) > 1$$

This implies $\bar{P}(t,T_j)$ is completely determined by $\tilde{R}_i$ and $\bar{P}(t,T_{j-1})$. The dynamics of $\tilde{R}_i$ is needed to compute CMCDS values.

One period CDS rate approximation and dynamics

First, $\tilde{R}_i$ is a Martingale under $\hat{Q}^i$ measure as long as $\tilde{R}_i$ remains positive. $d\tilde{R}_i = \sigma_i \tilde{R}_i dZ_i^i$ under $\hat{Q}^i$ measure

Second, we need to change a probability measure from $\hat{Q}^i$ to $\hat{Q}^j$ for all $i \geq j$. The following result is obtained and the proof is in the reference [1].

$d\tilde{R}_i = \sigma_i \tilde{R}_i dZ_i^i$ under $\hat{Q}^i$ measure

$$= \sigma_i \tilde{R}_i \left(\sum_{h=i+1}^{j} \rho_{i,h} \frac{\sigma_h \tilde{R}_h}{\tilde{R}_h + \frac{L_{GD}}{\alpha_h}} dt + dZ_i^j \right)$$

$$= \tilde{R}_i [\mu_i^j (\tilde{R}) dt + \sigma_i Z_i^j] \text{ under } \hat{Q}^j \text{ measure}$$

where $\rho_{i,h}$ is a correlation of $\tilde{R}_i$ and $\tilde{R}_h$

Hence, Monte Carlo simulation is possible given $\tilde{R}(0)$, the volatilities and correlations.

Furthermore, the expected value of $\tilde{R}_i(T_{j-1})$ under $\hat{Q}^j$ measure is computed.

We assume the volatility σ_i is piecewise constant.

Let $\tilde{\mu}_i^j = \left(\sum_{h=j+1}^{i} \rho_{i,h} \frac{\sigma_h \tilde{R}_h(0)}{\tilde{R}_h(0) + \frac{L_{GD}}{\alpha_h}} \right) \sigma_i$. Then we have

$$\hat{E}^{j-1,j} \left[\tilde{R}_i(T_{j-1}) \right] = \tilde{R}_i(0) \exp\left\{ \int_0^{T_{j-1}} \tilde{\mu}_i^j (\tilde{R}(0)) du \right\} \approx \tilde{R}_i(0) \exp\left[T_{j-1} \sigma_i \left(\sum_{h=j+1}^{i} \rho_{ih} \frac{\sigma_h \tilde{R}_h(0)}{\tilde{R}_h(0) + \frac{L_{GD}}{\alpha_h}} \right) \right]$$

Calibration Technique

We use the intensity models to obtain implied default probabilities from market quotes. We assume the intensity rate function $\gamma(t)$ to be deterministic and piecewise function.

$$\Gamma(t) = \int_0^t \gamma(s)ds = \sum_{i=1}^{\beta(t)-1} (T_{i+1} - T_i)\gamma_i + (t - T_{\beta(t)-1})\gamma_{\beta(t)}$$

$$\Gamma_j := \int_0^{T_j} \gamma(s)ds = \sum_{i=1}^{j} (T_i - T_{i-1})\gamma_i$$

We have the following expression for the Protection leg at time 0:

$$\begin{aligned}
& L_{GD} \cdot E[\sum_{i=a+1}^{b} D(0,T_i) 1_{\{T_{i-1}<\tau<T_i\}} \mid F_0] \\
&= L_{GD} \int_0^{\infty} \sum_{i=a+1}^{b} E[D(0,T_i) 1_{\{T_{i-1}<\tau<T_i\}}] \Pr(\tau \in [u, u+du)) \\
&= L_{GD} \sum_{i=a+1}^{b} \int_{T_{i-1}}^{T_i} E[D(0,T_i)] \Pr(\tau \in [u, u+du)) \\
&= L_{GD} \sum_{i=a+1}^{b} \int_{T_{i-1}}^{T_i} P(0,T_i)\gamma(u) \exp(-\int_0^u \gamma(s)ds) du \\
&= L_{GD} \sum_{i=a+1}^{b} \gamma_i \int_{T_{i-1}}^{T_i} \exp(-\Gamma_{i-1} - \gamma_i (u - T_{i-1})) P(0,T_i) du
\end{aligned}$$

We also have the expression for the Premium leg at time 0

$$\begin{aligned}
& \sum_{i=a+1}^{b} E[D(0,T_i)\alpha_i R 1_{\{\tau \ge T_i\}}] \\
&= \sum_{i=a+1}^{b} E[D(0,T_i)]\alpha_i R E[1_{\{\tau \ge T_i\}}] \\
&= \sum_{i=a+1}^{b} P(0,T_i)\alpha_i R \Pr(\tau \ge T_i) \\
&= R \sum_{i=a+1}^{b} P(0,T_i)\alpha_i \Pr(\tau \ge T_i)
\end{aligned}$$

It can be shown that

$$CDS_{a,b}(t,R,L_{GD};\Gamma(\cdot))$$
$$=\sum_{i=a+1}^{b}P(t,T_i)R\alpha_i e^{\Gamma(t)-\Gamma(T_i)}+L_{GD}\sum_{i=a+1}^{b}\int_{T_{i-1}}^{T_i}P(t,T_i)d_u(e^{-\Gamma(u)-\Gamma(t)})]$$

In particular, under the piecewise assumption on $\gamma(t)=\gamma_i,\ \ \gamma_i\in[T_{i-1},T_i]$, we have,

$$CDS_{a,b}(t,R,L_{GD};\Gamma(\cdot))$$
$$=\sum_{i=a+1}^{b}P(t,T_i)R\alpha_i e^{-(\Gamma(T_i)-\Gamma(t))}-L_{GD}\sum_{i=a+1}^{b}\gamma_i\int_{T_{i-1}}^{T_i}\exp(-\Gamma_{i-1}-\gamma_i(u-T_{i-1}))P(0,T_i)du$$

Here we use the discrete form of the CDS value at time 0 to implement in computation scheme to estimate the piecewise constant intensity.

$$CDS_{a,b}(0,R,L_{GD,}\Gamma)$$
$$=R\sum_{i=a+1}^{b}P(0,T_i)\alpha_i\exp(-\Gamma(T_i))$$
$$-L_{GD}\sum_{i=a+1}^{b}P(0,T_i)\gamma(T_i)\exp(-\Gamma_{i-1}-\gamma_i(T_i-T_{i-1}))(T_i-T_{i-1})$$
$$=R\sum_{i=a+1}^{b}P(0,T_i)\alpha_i\exp(-\Gamma(T_i))-L_{GD}\sum_{i=a+1}^{b}\gamma_i P(0,T_i)\exp(-\Gamma_i)\alpha_i$$

where $\alpha_i=T_i-T_{i-1}$ and $\gamma(t)=\gamma_i,\ \ \gamma_i\in[T_{i-1},T_i]$

In the market $T_a=0$ and we have R quotes for $T_b=1,2,3,....,10$ years, by setting T_i quarterly, we can solve the

$$CDS_{0,1}(0,R_{0,1}^{M},L_{GD};\gamma_1=\gamma_2=\gamma_3=\gamma_4;\gamma_1)=0$$
$$CDS_{0,2}(0,R_{0,2}^{M},L_{GD};\gamma_1;\gamma_5=\gamma_6=\gamma_7=\gamma_8;\gamma_2)=0$$

by Matlab Software. See the code in the Appendix.

Numerical Example

At this point, we present some numerical examples, based on IBM Company CDS data on 28th, October, 2008.

Recovery Rate= 40%

Maturity Tb(yr)	Maturity (date)	R(0,Tb)
0.5	2009-4-28	39.1
1	2009-10-28	47.327
2	2010-10-28	54.669
3	2011-10-28	63.894
4	2012-10-28	72.652
5	2013-10-28	77.16
7	2015-10-28	77.472
10	2018-10-28	79.439

Table 1 Maturity dates and corresponding CDS quotes in bps for T_0 =28th, October, 2008.

Date	Intensity	Survival Probability
2009-4-28	0.0065167	99.675%
2009-10-28	0.009276	99.213%
2010-10-28	0.010365	98.190%
2011-10-28	0.013868	96.838%
2012-10-28	0.016849	95.220%
2013-10-28	0.016254	93.685%
2015-10-28	0.013067	91.268%
2018-10-28	0.014322	87.430%

Table 2 Calibration with piecewise linear intensity on 28th, October, 2008.

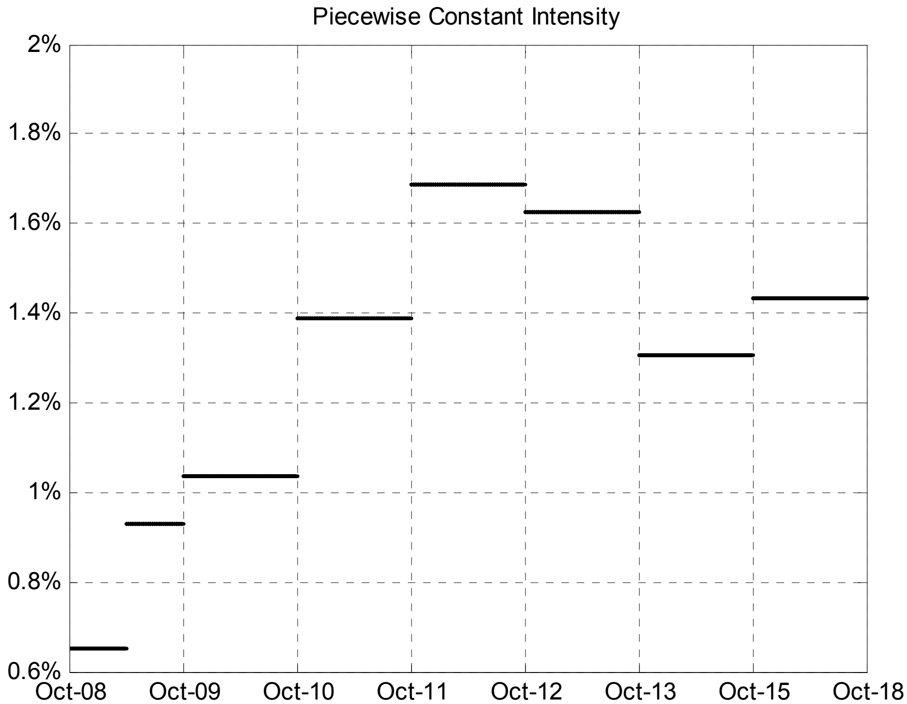

Figure 1 Piecewise constant intensity γ calibrated on CDS quotes on October 28th 2008.

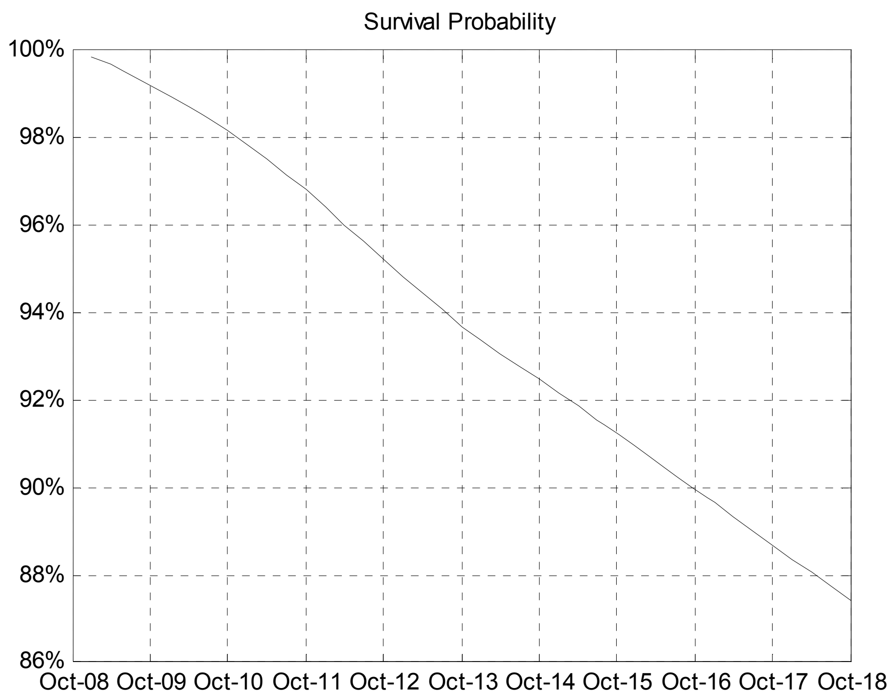

Figure 2 Survival Probability $\exp(-\Gamma)$ resulting from calibration on CDS quotes on October 28th, 2009

An Approximation to Valuation of CMCDS

Constant Maturity Credit Default Swap (CMCDS)

Consider a contract protecting in $[T_a, T_b]$ against default of a reference credit C. If default occurs in $[T_a, T_b]$, a protection payment L_{GD} is made from the protection seller B to the protection buyer A at the first T_j following the default time, called protection leg. The value of the CMCDS to 'B' is the value of the premium leg minus the value of the protection leg. The protection leg valuation in $[T_a, T_b]$ is expressed as

$$\mathrm{R}_{a.b}(0)\sum_{j=a+1}^{b}\alpha_j \bar{P}(0,T_j) = \sum_{j=a+1}^{b}\alpha_j R_j(0)\bar{P}(0,T_j)$$

The value of the premium leg at time $t = 0$ is expressed as,

$$\sum_{j=a+1}^{b}\alpha_j E\left[\ D(0,T_j)1_{\{\tau>T_j\}} R_{j-1,j+c}(T_{j-1})\right]$$

A $C+1$-long CMCDS rate is defined as

$$R_{j-1,j+c}(T_{j-1}) = \sum_{i=j}^{j+c}\bar{W}_i(T_{j-1})R_i(T_{j-1})$$

where $\bar{W}_i(T_{j-1}) = \dfrac{\alpha_i \bar{P}(T_{j-1},T_i)}{\sum_{h=j}^{j+c}\alpha_h \bar{P}(T_{j-1},T_h)}$

The first approximation of $R_{j-1,j+c}(T_{j-1}) \approx \sum_{i=j}^{j+c}\bar{W}_i(0)R_i(T_{j-1})$

where $\bar{W}_i(0) = \dfrac{\alpha_i \bar{P}(0,T_i)}{\sum_{h=j}^{j+c}\alpha_h \bar{P}(0,T_h)}$

$$\sum_{j=a+1}^{b} \alpha_j E\left[D(0,T_j) 1_{\{\tau > T_j\}} R_{j-1,j+c}(T_{j-1}) \right]$$

$$\cong \sum_{j=a+1}^{b} \sum_{i=j}^{j+c} \alpha_j \bar{W}_i^j(0) E\left[D(0,T_j) 1_{\{\tau > T_j\}} R_{j-1,j+c}(T_{j-1}) \right]$$

$$= \sum_{j=a+1}^{b} \sum_{i=j}^{j+c} \alpha_j \bar{W}_i^j(0) \hat{C}_{j-1,j}(0) \hat{E}^{j-1,j}\left[R_i(T_{j-1}) \right]$$

$$= \sum_{j=a+1}^{b} \sum_{i=j}^{j+c} \alpha_j \bar{W}_i^j(0) \bar{P}(0,T_j) \hat{E}^{j-1,j}\left[R_i(T_{j-1}) \right]$$

We need to compute $\hat{E}^{j-1,j}\left[R_i(T_{j-1}) \right]$. We approximate the expectation by $\hat{E}^{j-1,j}\left[\tilde{R}_i(T_{j-1}) \right]$.

$$\hat{E}^{j-1,j}\left[\tilde{R}_i(T_{j-1}) \right] \cong \tilde{R}_i(0) \exp\left\{ \int_0^{T_{j-1}} \tilde{\mu}_i^j(\tilde{R}(0)) du \right\}$$

$$= \tilde{R}_i(0) \exp\left\{ \sum_{k=j+1}^{i} \frac{\tilde{R}_K(0)}{\tilde{R}_K(0) + \frac{L_{GD}}{\alpha_K}} \quad \rho_{i,k} \int_0^{T_{j-1}} \sigma_i(u) \sigma_k(u) du \right\}$$

Assume σ_i is piecewise constant

$$\hat{E}^{j-1,j}\left[\tilde{R}_i(T_{j-1}) \right] \cong \tilde{R}_i(0) \exp\left\{ \int_0^{T_{j-1}} \tilde{\mu}_i^j(\tilde{R}(0)) du \right\}$$

$$\approx \tilde{R}_i(0) \exp\left\{ T_{j-1} \sigma_i \left(\sum_{k=j+1}^{i} \rho_{i,k} \frac{\sigma_k \tilde{R}_k(0)}{\tilde{R}_K(0) + \frac{L_{GD}}{\alpha_k}} \right) \right\}$$

The value of CMCDS at $t = 0$

$$CDS_{CMa,b,c}(0, L_{GD}) = \sum_{j=a+1}^{b} \alpha_j \bar{P}(0,T_j) \left\{ \sum_{i=j}^{j+c} \frac{\alpha_i \bar{P}(0,T_i)}{\sum_{h=j}^{j+c} \alpha_h \bar{P}(0,T_h)} \tilde{R}_i(0) \exp\left[T_{j-1} \sigma_i \left(\sum_{k=j+1}^{i} \rho_{ik} \frac{\sigma_k \tilde{R}_k(0)}{\tilde{R}_k(0) + \frac{L_{GD}}{\alpha_k}} \right) \right] - R_j(0) \right\}$$

The detailed proof of this result is in the reference [1].

Important Quantities for Comparisons

The following quantities are worthy of consideration for comparison. We define L_i as $\frac{R_{i-1,i+c}(0)}{R_{0,b}(0)}$. This measures how the CMCDS differs from a standard CDS at the premium rate at each period. We define M_i as follows,

$$
\begin{aligned}
A &= E\left[D(0,T_i)1_{\{\tau > T_i\}} R_{i-1,i+c}(T_{i-1}) \right] \\
&= \sum_{j=i}^{i+c} \overline{W}_j(0)\overline{P}(0,T_i) * \hat{E}^{i-1,i}\left[\tilde{R}_j(T_{i-1}) \right]
\end{aligned}
$$

$$
\hat{E}^{i-1,i}\left[\tilde{R}_j(T_{i-1}) \right] \approx \tilde{R}_j(0)\exp\left\{ T_{i-1}\sigma_j \left(\sum_{k=i+1}^{j} \rho_{j,k} \frac{\sigma_k \tilde{R}_k(0)}{\tilde{R}_K(0) + \frac{L_{GD}}{\alpha_k}} \right) \right\}
$$

$$
M_i = \frac{A}{\overline{P}(0,T_i)R_{0,b}(0)}
$$

The quantity M_i is the same as L_i except taking into consideration the expression of the random values and correlations. We define N_i as follows,

$$
N_i = \frac{\hat{E}_0^{i-1,i}[D(0,T_i)1_{\{\tau > T_i\}} R_{i-1,i+c}(T_{i-1})]}{\overline{P}(0,T_i)R_{i-1,i+c}(0)}
$$

This is a measure of the impact of the convexity at each period in the premium leg.

We define X_i as follows,

$$
X_i = \frac{\text{"}premium\ leg\ CDS\text{"}}{\text{"}premium\ leg\ CMCDS\text{"}} = \frac{\sum_{j=1}^{i} \alpha_j \overline{P}(0,T_j)R_{0,i}(0)}{\sum_{j=1}^{i} \alpha_j \overline{P}(0,T_j)R_{j-1,j+c}(0)}
$$

We define Y_i as follows,

$$\mathrm{Y}_i = \frac{\text{"}\,premium\ leg\ CDS\,\text{"}}{\text{"}\,premium\ leg\ CMCDS\ with\ convexity\,\text{"}}$$

$$= \frac{\sum_{j=1}^{i} \alpha_j \bar{P}(0,T_j) R_{0,i}(0)}{\sum_{j=1}^{i} \alpha_j \hat{E}_0^{j-1,j}[D(0,T_j) 1_{\{\tau > T_j\}} R_{j-1,j+c}(T_{j-1})}$$

In this case the convexity due to correlation and volatilities are taken into consideration.

We make a note of the difference between two quantities:

$$premiumlegCDS = \sum_{j=1}^{N} \alpha_j \bar{P}(0,T_j) R_{0,N}(0)$$

$$premiumlegCMCDS = \sum_{j=1}^{N} \alpha_j E\left[D(0,T_j) 1_{\{\tau > T_j\}} R_{j-1,j-1+c}(T_{j-1}) \right]$$

Numerical Implementation and Procedure

This will be implemented in Excel VBA. The CMCDS value at $t = 0$ is based on

$$CDS_{CMa,b,c}(0, L_{GD}) = \sum_{j=a+1}^{b} \alpha_j \bar{P}(0,T_j) \left\{ \sum_{i=j}^{j+c} \frac{\alpha_i \bar{P}(0,T_i)}{\sum_{h=j}^{j+c} \alpha_h \bar{P}(0,T_h)} \tilde{R}_i(0) \exp\left[T_{j-1}\sigma_i \left(\sum_{k=j+1}^{i} \rho_{ik} \frac{\sigma_k \tilde{R}_k(0)}{\tilde{R}_k(0) + \frac{L_{GD}}{\alpha_K}} \right) \right] - R_j(0) \right\} \quad \mathbf{(1)}$$

The list of inputs is as follows,

Inputs
ρ_{ij}
σ_i
$p(0,T_i)$
L_{GD}
a,b,c
T_i
$\alpha_i = T_i - T_{i-1}$
$R_{0,b}(0)$

ρ_{ij} represents the instantaneous correlation between R_i and R_j, and σ_i represents the volatility of $R_i(t)$. $p(0,T_i)$ and the market value $R_{0,b}(0)$ are available in the appendix.

The list of intermediate inputs is as follows,

Intermediate Inputs
$\lambda(t)$
$Q(\tau > T_i)$
$\bar{p}(0,T_i) = p(0,T_i)*Q(\tau > T_i)$

We use the intensity model to obtain implied default probability from market quotes under the assumption that there is independence between interest rates and the default time. We need a calibration process to extract implied hazard rates and $Q(\tau > T_i)$. We have the following equation,

$$R_{a,b}(t) = \frac{\sum_{i=a+1}^{b} \alpha_i \; R_i(t)\bar{P}(t,T_i)}{\sum_{i=a+1}^{b} \alpha_i \bar{P}(t,T_i)} \qquad \textbf{(2)}$$

$$= \sum_{i=a+1}^{b} \bar{W}_i(t) R_i(t) \approx \sum_{i=a+1}^{b} \bar{W}_i(0) R_i(t)$$

Then, $R_i(t)$ is approximated by $\tilde{R}_i(t)$.

We make use of the expression,

$$R_{j-1,j+c}(T_{j-1}) = \sum_{i=j}^{j+c} \bar{W}_i(T_{j-1}) R_i(T_{j-1})$$

where $\bar{W}_i(T_{j-1}) = \dfrac{\alpha_i \bar{P}(T_{j-1},T_i)}{\sum_{h=j}^{j+c} \alpha_h \bar{P}(T_{j-1},T_h)}$

$$R_{j-1,j+c}(T_{j-1}) \approx \sum_{i=j}^{j+c} \bar{W}_i(0) R_i(T_{j-1}) \tag{3}$$

where $\bar{W}_i(0) = \dfrac{\alpha_i \bar{P}(0,T_i)}{\sum_{h=j}^{j+c} \alpha_h \bar{P}(0,T_h)}$

$\tilde{R}_K(0)$ is an approximation of $R_K(0)$

$$\tilde{R}_K(0) = L_{GD} \frac{\bar{P}(0,T_{K-1}) \dfrac{P(0,T_K)}{P(0,T_{K-1})} - \bar{P}(0,T_K)}{\alpha_K \bar{P}(0,T_K)} \tag{4}$$

The price of the premium leg at $t = 0$

$$\begin{aligned} &\sum_{j=a+1}^{b} \alpha_j E\left[D(0,T_j) 1_{\{\tau > T_j\}} R_{j-1,j+c}(T_{j-1}) \right] \\ &= \sum_{j=a+1}^{b} \sum_{i=j}^{j+c} \alpha_j \bar{W}_i^{\,j}(0) \bar{P}(0,T_j) \hat{E}^{\,j-1,j}\left[R_i(T_{j-1}) \right] \end{aligned} \tag{5}$$

$$\begin{aligned} &\hat{E}^{\,j-1,j}\left[\tilde{R}_i(T_{j-1}) \right] \cong \tilde{R}_i(0) \exp\left\{ \int_0^{T_{j-1}} \tilde{\mu}_i^j(\tilde{R}(0)) du \right\} \\ &\approx \tilde{R}_i(0) \exp\left\{ T_{j-1} \sigma_i \left(\sum_{k=j+1}^{i} \rho_{j,k} \frac{\sigma_k \tilde{R}_k(0)}{\tilde{R}_K(0) + \dfrac{L_{GD}}{\alpha_k}} \right) \right\} \end{aligned} \tag{6}$$

The list of outputs is as follows,

Outputs
$CMCDS_{a,b,c}(0, L_{GD}, \sigma, \rho)$
$CMCDS_{a,b,c}(0, L_{GD}, \rho = 0)$
$conv(\sigma, \rho) = CMCDS_{a,b,c}(0, L_{GD}, \sigma, \rho) - CMCDS_{a,b,c}(0, L_{GD}, \rho = 0)$
L_i
M_i
N_i
X_i
Y_i

The value of CMCDS at $t = 0$ is denoted by $CMCDS_{a,b,c}(0, L_{GD}, \sigma, \rho)$ when $\rho \neq 0$, otherwise the CMCDS at $t = 0$ is denoted by $CMCDS_{a,b,c}(0, L_{GD}, \rho = 0)$.

$conv(\sigma, \rho)$ measures the convexity difference. The equations (2), (3), (4), (5) and (6) are sufficient to compute the expression (1). For L_i, the equations (3) and (4) are utilized. For M_i, the equations (2), (5), and (6) are utilized. For N_i, the equations (3), (5), and (6) are utilized. For X_i, the equations (2) and (3) are utilized. For Y_i, the equations (2), (5), and (6).

Numerical Samples and Results:

- Ford Company on July 1st, 2008

Inputs:

LGD=0.6

Maturity Tb(yr)	Maturity (date)	R(0,Tb)
0.5	2009-1-1	700
1	2009-7-1	1188.89
2	2010-7-1	1664.141
3	2011-7-1	1936.845
4	2012-7-1	2010.546
5	2013-7-1	2043.658
7	2015-7-1	1980.742
10	2018-7-1	1922.815

Table 3 Maturity dates and corresponding CDS quotes in bps for T_0 = July 1st,2008

	a=0
	b=20
	c=10
assume constant volatility for all R(0)	$\sigma = 0.4$
assume constant correlation ρ	$\rho = 0.9$

Table 4 Constant volatility and correlation

$\alpha(i)$	T_i	$P(0,T_i)$	$Q(\tau > T_i)$	$\bar{P}(0,T_i)$		$\alpha(i)$	T_i	$P(0,T_i)$	$Q(\tau > T_i)$	$\bar{P}(0,T_i)$
0.25	0.25	0.99541	97.13%	0.966802		0.25	5.25	0.83575	14.23%	0.118927
0.25	0.5	0.98917	94.33%	0.933124		0.25	5.5	0.82607	13.53%	0.111751
0.25	0.75	0.98329	92.97%	0.914184		0.25	5.75	0.81633	12.86%	0.10498
0.25	1	0.9771	86.44%	0.844595		0.25	6	0.80653	12.23%	0.098606
0.25	1.25	0.97062	78.38%	0.760762		0.25	6.25	0.79671	11.62%	0.092602
0.25	1.5	0.96384	71.07%	0.685001		0.25	6.5	0.78685	11.05%	0.086939
0.25	1.75	0.95679	64.44%	0.616584		0.25	6.75	0.77698	10.50%	0.081614
0.25	2	0.94946	58.43%	0.554807		0.25	7	0.76712	9.99%	0.076603
0.25	2.25	0.94188	51.52%	0.485247		0.25	7.25	0.75725	9.58%	0.072514
0.25	2.5	0.93406	45.42%	0.424259		0.25	7.5	0.74741	9.18%	0.068634
0.25	2.75	0.926	40.05%	0.370826		0.25	7.75	0.7376	8.81%	0.064953
0.25	3	0.91773	35.31%	0.324014		0.25	8	0.72782	8.44%	0.061461
0.25	3.25	0.90925	31.66%	0.287859		0.25	8.25	0.7181	8.10%	0.058152
0.25	3.5	0.90057	28.39%	0.255663		0.25	8.5	0.70842	7.77%	0.055013
0.25	3.75	0.89173	25.46%	0.227008		0.25	8.75	0.69882	7.45%	0.05204
0.25	4	0.88272	22.83%	0.201498		0.25	9	0.68928	7.14%	0.049224
0.25	4.25	0.87356	20.54%	0.179447		0.25	9.25	0.67981	6.85%	0.046555
0.25	4.5	0.86427	18.49%	0.15976		0.25	9.5	0.67043	6.57%	0.044028
0.25	4.75	0.85486	16.63%	0.142197		0.25	9.75	0.66115	6.30%	0.041637
0.25	5	0.84535	14.97%	0.126532		0.25	10	0.65195	6.04%	0.039372

Table 5 Intermediate input of survival probability and defaultable bond price on July, 1st, 2008

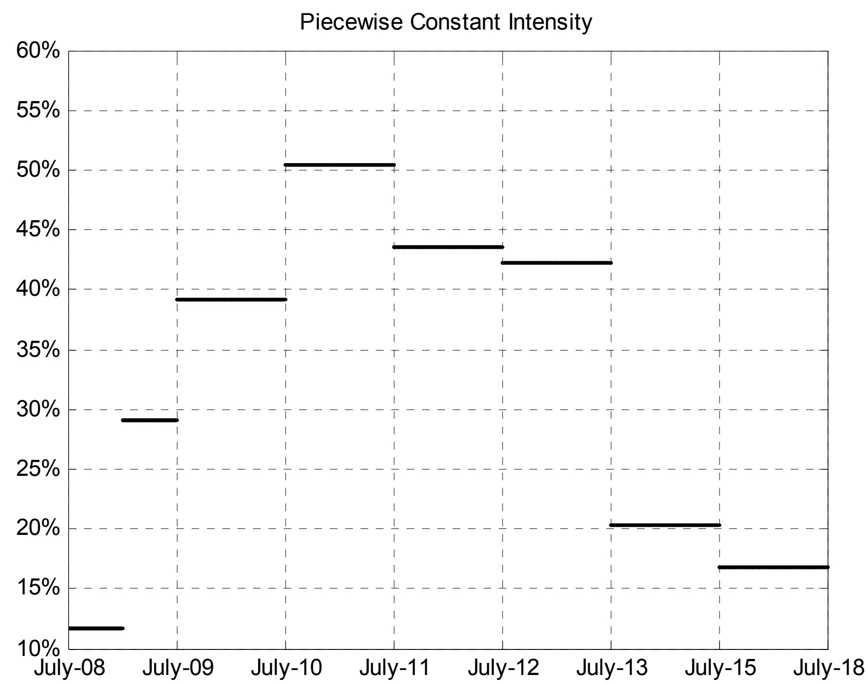

Figure 3 Piecewise constant intensity, calibrated on CDS quotes on July,01,2008

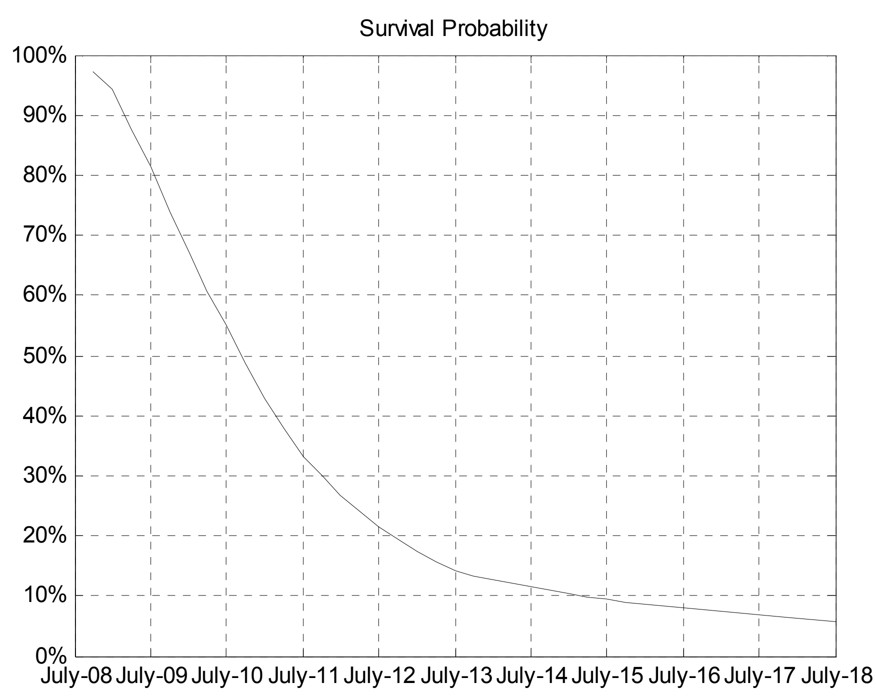

Figure 4 Survival probability resulting from calibration on CDS quotes on July,01, 2008

Outputs:

Case 1: Constant volatilities

CMCDS(0, LGD, σ, ρ)				
ρ	0.7	0.8	0.9	0.99
σ				
0.1	0.547515	0.547917	0.548319	0.548682
0.2	0.556096	0.557766	0.559448	0.560971
0.4	0.593735	0.601566	0.609628	0.61709
0.6	0.670855	0.694274	0.719441	0.743726

CMCDS(0, LGD, ρ=0)	
	0.544722

Table 6 Value of CMCDS at time 0

	Convexity Difference CDSCM(0, L_{GD}, σ, ρ) - CDSCM(0, L_{GD}, ρ=0)			
	ρ			
σ	0.7	0.8	0.9	0.99
0.1	0.002793005	0.003195	0.003597	0.0039598
0.2	0.011374291	0.013044	0.014725	0.0162485
0.4	0.049013263	0.056844	0.064906	0.0723673
0.6	0.126132793	0.149552	0.174719	0.199004

Table 7 Convexity difference of CMCDS valuation

Li	Mi	Ni	Yi
0.995243	0.995243	1	0.352163
1.061809	1.087031	1.023753	0.337659
1.139777	1.196512	1.049777	0.269031
1.257166	1.345613	1.070355	0.376253
1.296076	1.414308	1.091223	0.475277
1.295823	1.443666	1.114092	0.53359
1.295206	1.472389	1.136799	0.570925
1.293714	1.499578	1.159126	0.596125
1.289702	1.511423	1.171917	0.634408
1.234561	1.467193	1.188433	0.664402
1.173101	1.411179	1.202948	0.688881
1.106895	1.345212	1.215303	0.709768
1.034926	1.276013	1.23295	0.722199
0.983813	1.22356	1.243692	0.733409
0.929173	1.163573	1.252268	0.743716
0.871339	1.096527	1.25844	0.753378
0.810055	1.023984	1.264092	0.761845
0.750642	0.950634	1.266428	0.769829
0.690044	0.873227	1.265466	0.777613
0.627188	0.792213	1.263119	0.784956

Table 8 Outputs for a range of terminal dates $T_b = T_i$ spanning five years at quarterly intervals

Case 2: Piecewise constant volatilities

$\rho_{ij} = 0.7$ when $i \neq j$, σ_i is piecewise constant in the time interval linearly changing from 0.1 to 0.9 on the time axis.

CMCDS(0, LGD, σ, ρ) 0.7648041
CMCDS(0, LGD, ρ=0) 0.5447222

Table 9 Value of CMCDS at time 0

Convexity Difference CDSCM(0, L_{GD}, σ, ρ) - CDSCM(0, L_{GD}, ρ=0)
0.220082

Table 10 Convexity difference of CMCDS valuation

Li	**Mi**	**Ni**	**Yi**
0.995243	**0.995243**	**1**	**0.352163**
1.061809	**1.097971**	**1.034056**	**0.335924**
1.139777	**1.225954**	**1.075609**	**0.265817**
1.257166	**1.407595**	**1.119658**	**0.368418**
1.296076	**1.525115**	**1.176717**	**0.460141**
1.295823	**1.619376**	**1.249689**	**0.509686**
1.295206	**1.729348**	**1.335192**	**0.53707**
1.293714	**1.852642**	**1.432034**	**0.551488**
1.289702	**1.960995**	**1.520503**	**0.577209**
1.234561	**2.008776**	**1.627118**	**0.594787**
1.173101	**2.02983**	**1.730312**	**0.607455**
1.106895	**2.015768**	**1.821102**	**0.617496**
1.034926	**1.985049**	**1.918058**	**0.620776**
0.983813	**1.937182**	**1.969056**	**0.624011**
0.929173	**1.844003**	**1.984564**	**0.627655**
0.871339	**1.706165**	**1.958096**	**0.632004**
0.810055	**1.537101**	**1.897528**	**0.63651**
0.750642	**1.345477**	**1.792435**	**0.641676**
0.690044	**1.137539**	**1.648503**	**0.647587**
0.627188	**0.930152**	**1.483052**	**0.653823**

Table 11 Outputs for a range of terminal dates $T_b = T_i$ spanning five years at quarterly intervals

- IBM Company on July 1st, 2008

Inputs:

LGD=0.6

Maturity Tb(yr)	Maturity (date)	R(0,Tb)
0.5	2009-1-1	23.125
1	2009-7-1	30.003
2	2010-7-1	40.829
3	2011-7-1	50.527
4	2012-7-1	55.659
5	2013-7-1	60.561
7	2015-7-1	63.858
10	2018-7-1	67.17

Table 12 Maturity dates and corresponding CDS quotes in bps for T_0 = July 1st, 2008

	a=0
	b=20
	c=10
assume constant volatility for all R(0)	$\sigma = 0.4$
assume constant correlation ρ	$\rho = 0.9$

Table 13 Constant volatility and correlation

$\alpha(i)$	T_i	$P(0,T_i)$	$Q(\tau > T_i)$	$\bar{P}(0,T_i)$		$\alpha(i)$	T_i	$P(0,T_i)$	$Q(\tau > T_i)$	$\bar{P}(0,T_i)$
0.25	0.25	0.99541	99.90%	0.994454		0.25	5.25	0.83575	94.70%	0.791439
0.25	0.5	0.98917	99.81%	0.987271		0.25	5.5	0.82607	94.41%	0.779868
0.25	0.75	0.98329	99.65%	0.979888		0.25	5.75	0.81633	94.12%	0.768305
0.25	1	0.9771	99.50%	0.972215		0.25	6	0.80653	93.83%	0.756751
0.25	1.25	0.97062	99.29%	0.96368		0.25	6.25	0.79671	93.54%	0.745243
0.25	1.5	0.96384	99.07%	0.954876		0.25	6.5	0.78685	93.25%	0.733761
0.25	1.75	0.95679	98.86%	0.945835		0.25	6.75	0.77698	92.97%	0.722335
0.25	2	0.94946	98.64%	0.936557		0.25	7	0.76712	92.68%	0.710982
0.25	2.25	0.94188	98.35%	0.92633		0.25	7.25	0.75725	92.38%	0.699563
0.25	2.5	0.93406	98.06%	0.915921		0.25	7.5	0.74741	92.08%	0.688245
0.25	2.75	0.926	97.77%	0.905332		0.25	7.75	0.7376	91.79%	0.677021
0.25	3	0.91773	97.48%	0.894594		0.25	8	0.72782	91.49%	0.66589
0.25	3.25	0.90925	97.19%	0.883655		0.25	8.25	0.7181	91.20%	0.654871
0.25	3.5	0.90057	96.89%	0.87258		0.25	8.5	0.70842	90.90%	0.643961
0.25	3.75	0.89173	96.60%	0.861402		0.25	8.75	0.69882	90.61%	0.633187
0.25	4	0.88272	96.31%	0.85013		0.25	9	0.68928	90.32%	0.622523
0.25	4.25	0.87356	95.98%	0.838417		0.25	9.25	0.67981	90.02%	0.611992
0.25	4.5	0.86427	95.65%	0.82664		0.25	9.5	0.67043	89.73%	0.601597
0.25	4.75	0.85486	95.32%	0.814827		0.25	9.75	0.66115	89.44%	0.591352
0.25	5	0.84535	94.99%	0.80299		0.25	10	0.65195	89.16%	0.581246

Table 14 Intermediate input of survival probability and defaultable bond price on July, 1st, 2008

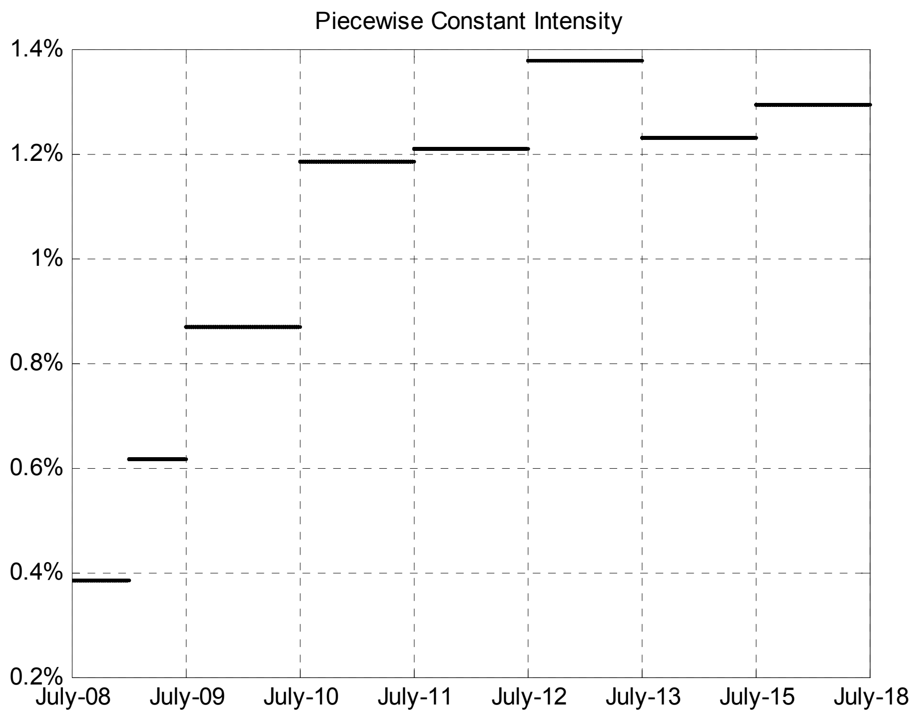

Figure 5 Piecewise constant intensity, calibrated on CDS quotes on July,01,2008

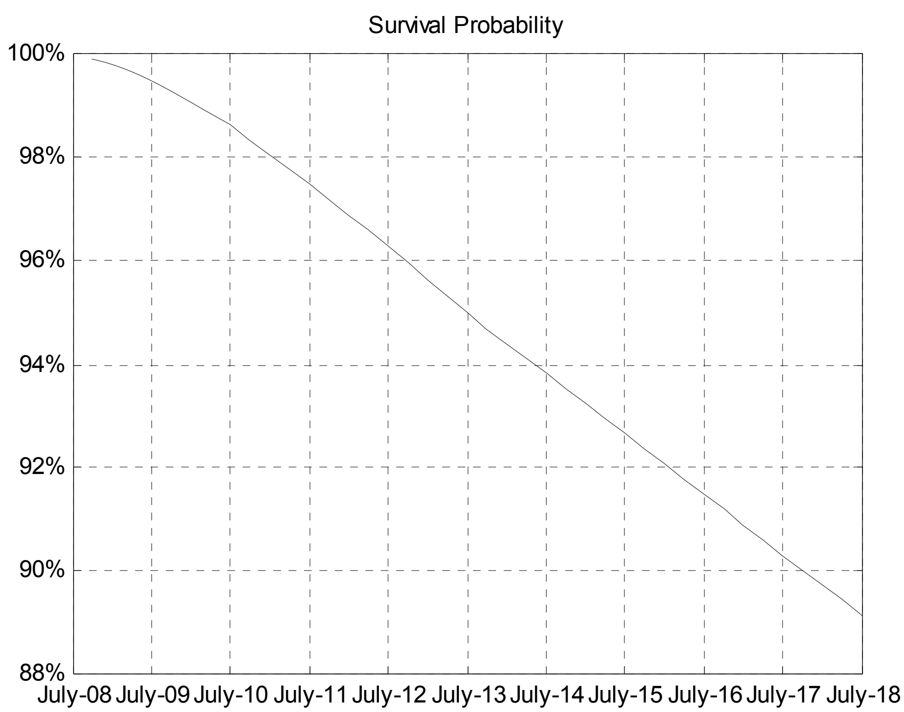

Figure 6 Survival probability resulting from calibration on CDS quotes on July,01, 2008

Outputs:

Case 1: Constant volatility

CMCDS(0, LGD, σ, ρ)				
ρ	0.7	0.8	0.9	0.99
σ				
0.1	0.033044	0.033046	0.033049	0.033051
0.2	0.033093	0.033102	0.033112	0.03312
0.4	0.033291	0.033329	0.033367	0.033401
0.6	0.033626	0.033713	0.0338	0.033879

CMCDS(0, LGD, ρ=0)	
	0.033028

Table 15 Value of CMCDS at time 0

	Convexity Difference CDSCM(0, L_{GD}, σ, ρ) - CDSCM(0, L_{GD}, ρ=0)			
	ρ			
σ	0.7	0.8	0.9	0.99
0.1	1.63371E-05	1.87E-05	2.1E-05	2.311E-05
0.2	6.54416E-05	7.48E-05	8.42E-05	9.263E-05
0.4	0.000263269	0.000301	0.000339	0.0003735
0.6	0.000598062	0.000685	0.000773	0.0008519

Table 16 Convexity difference of CMCDS valuation

Li	Mi	Ni	Yi
1.009037	1.009037	1	0.376851
1.051253	1.052299	1.000994	0.369147
1.094056	1.096265	1.00202	0.434181
1.125118	1.128616	1.003109	0.462432
1.156606	1.16142	1.004162	0.520229
1.175351	1.181558	1.005281	0.556304
1.194281	1.20196	1.00643	0.580204
1.213406	1.222643	1.007612	0.596178
1.235774	1.246421	1.008616	0.636591
1.240948	1.253028	1.009735	0.667827
1.246226	1.259763	1.010862	0.692528
1.251612	1.266632	1.012	0.712411
1.257288	1.273787	1.013123	0.730131
1.261545	1.27954	1.014264	0.744846
1.265891	1.285395	1.015408	0.757397
1.270277	1.291325	1.01657	0.767811
1.274809	1.297099	1.017485	0.784212
1.270108	1.293618	1.01851	0.798941
1.265047	1.289765	1.019539	0.811973
1.259906	1.285804	1.020556	0.823737

Table 17 Outputs for a range of terminal dates $T_b = T_i$ spanning five years at quarterly intervals

Case 2: Piecewise constant volatilities

$\rho_{ij} = 0.7$ when $i \neq j$, σ_i is piecewise constant in the time interval linearly changing from 0.1 to 0.9 on the time axis.

CMCDS(0, LGD, σ, ρ) 0.0338529
CMCDS(0, LGD, ρ=0) 0.0330276

Table 18 Value of CMCDS at time 0

Convexity Difference CDSCM(0, L_{GD}, σ, ρ) - CDSCM(0, L_{GD}, ρ=0)
0.000825

Table 19 Convexity difference of CMCDS valuation

Li	Mi	Ni	Yi
1.009037	1.009037	1	0.376851
1.051253	1.053286	1.001933	0.368971
1.094056	1.098661	1.00421	0.433719
1.125118	1.132819	1.006845	0.461621
1.156606	1.167804	1.009681	0.518909
1.175351	1.190415	1.012816	0.554414
1.194281	1.213503	1.016096	0.5777
1.213406	1.236997	1.019441	0.593032
1.235774	1.263513	1.022447	0.632615
1.240948	1.272684	1.025574	0.663012
1.246226	1.281683	1.028451	0.686888
1.251612	1.290389	1.030981	0.705981
1.257288	1.298796	1.033015	0.722953
1.261545	1.30513	1.034549	0.736996
1.265891	1.310789	1.035468	0.748967
1.270277	1.315682	1.035744	0.758913
1.274809	1.319015	1.034676	0.774895
1.270108	1.312094	1.033057	0.789354
1.265047	1.303859	1.03068	0.802278
1.259906	1.294637	1.027566	0.814098

Table 20 Outputs for a range of terminal dates $T_b = T_i$ spanning five years at quarterly intervals

Appendix

Simulation data

1. Default free zero coupon bond price of different maturities for 3 month to 10 years on July 1st, 2008 and October 28th, 2008

Maturity (yr)	0.25	0.5	0.75	1	1.25	1.5	1.75	2
2008-7-10	1.64	2.0955	2.1445	2.1956	2.2488	2.3036	2.3599	2.4173
2008-10-28	0.76	1.6734	1.5522	1.4729	1.4329	1.4241	1.4444	1.4874
Maturity (yr)	**2.25**	**2.5**	**2.75**	**3**	**3.25**	**3.5**	**3.75**	**4**
2008-7-10	2.4758	2.5351	2.5949	2.6553	2.7159	2.7766	2.8373	2.8979
2008-10-28	1.5512	1.631	1.7251	1.8298	1.9436	2.0639	2.1893	2.318
Maturity (yr)	**4.25**	**4.5**	**4.75**	**5**	**5.25**	**5.5**	**5.75**	**6**
2008-7-10	2.9582	3.0182	3.0777	3.1366	3.1949	3.2524	3.3092	3.3651
2008-10-28	2.4488	2.5804	2.7117	2.842	2.9703	3.0963	3.2191	3.3386
Maturity (yr)	**6.25**	**6.5**	**6.75**	**7**	**7.25**	**7.5**	**7.75**	**8**
2008-7-10	3.4201	3.4741	3.527	3.579	3.6298	3.6795	3.728	3.7753
2008-10-28	3.454	3.5655	3.6725	3.7751	3.8728	3.9659	4.0541	4.1377
Maturity (yr)	**8.25**	**8.5**	**8.75**	**9**	**9.25**	**9.5**	**9.75**	**10**
2008-7-10	3.8214	3.8664	3.91	3.9524	3.9935	4.0334	4.072	4.1093
2008-10-28	4.2163	4.2904	4.3596	4.4243	4.4844	4.5403	4.5918	4.6392

2. Maturity dates and corresponding CDS quotes

   a. IBM Company on October 28th, 2008

Maturity Tb(yr)	Maturity (date)	R(0,Tb)
0.5	2009-4-28	39.1
1	2009-10-28	47.327
2	2010-10-28	54.669
3	2011-10-28	63.894
4	2012-10-28	72.652
5	2013-10-28	77.16
7	2015-10-28	77.472
10	2018-10-28	79.439

b. IBM Company on July 1st, 2008

LGD=0.6

Maturity Tb(yr)	Maturity (date)	R(0,Tb)
0.5	2009-1-1	23.125
1	2009-7-1	30.003
2	2010-7-1	40.829
3	2011-7-1	50.527
4	2012-7-1	55.659
5	2013-7-1	60.561
7	2015-7-1	63.858
10	2018-7-1	67.17

c. Ford Company on July 1st, 2008

LGD=0.6

Maturity Tb(yr)	Maturity (date)	R(0,Tb)
0.5	2009-1-1	700
1	2009-7-1	1188.89
2	2010-7-1	1664.141
3	2011-7-1	1936.845
4	2012-7-1	2010.546
5	2013-7-1	2043.658
7	2015-7-1	1980.742
10	2018-7-1	1922.815

3. A sample of VBA Code

Code

```
Sub FindingXi()
Const D = 41
Dim Ri(D)
Dim DefaultBondPrice(D)
Dim alphai(D)
Dim Ti(D)
sigma = 0
Rho = 0
Lgd = 0
```

```
'Taking input data from sheet1
For i = 0 To D
   Ri(i) = Sheet1.Cells(15 + i, 9) 'The values of Ri(0)
   DefaultBondPrice(i) = Sheet1.Cells(15 + i, 7)
   alphai(i) = Sheet1.Cells(15 + i, 2)
   Ti(i) = Sheet1.Cells(15 + i, 3)
   Next i

sigma = Sheet1.Cells(59, 6)
Rho = Sheet1.Cells(60, 6)
Lgd = Sheet1.Cells(5, 7)
a = Sheet1.Cells(6, 7)
b = Sheet1.Cells(7, 7)
c = Sheet1.Cells(8, 7)
'Finish taking input data from sheet1


Dim ConstantMaturityRate(20)
StandardRate = 0

'Finding the Standard Rate Ro,b Using Equation (2) in the paper
numerator = 0
denominator = 0
For j = a + 1 To b 'Corresponds to b=20
   numerator = numerator + alphai(j) * Ri(j) * DefaultBondPrice(j)
   denominator = denominator + alphai(j) * DefaultBondPrice(j)
Next j
StandardRate = numerator / denominator
Sheet3.Cells(4, 4) = StandardRate 'Output to Intermediate2 Sheet
'Finish finding the Standard Rate


'Finding Constant Maturity Rate and Calculating Li using Equation (3) and (4) in the paper
numerator = 0
denominator = 0
For i = a + 1 To b
   For j = i To i + c
      numerator = numerator + alphai(j) * Ri(j) * DefaultBondPrice(j)
      denominator = denominator + alphai(j) * DefaultBondPrice(j)
   Next j
   ConstantMaturityRate(i) = numerator / denominator 'Found the Constant Maturity Rate
   numerator = 0
   denominator = 0
   Sheet3.Cells(i + 6, 4) = ConstantMaturityRate(i) 'Output to Intermediate2 Sheet
```

```
    Sheet2.Cells(i + 6, 8) = ConstantMaturityRate(i) / StandardRate
    Next i
```

'Finish Finding Constant Maturity Rate and Calculating Li

'Calculating Mi Using Equation (2), (5) and (6) in the Paper

```
Dim YiNumerator(20)
winumerator = 0
widenominator = 0
wi = 0
exponential = 0
For j = a + 1 To b
    For i = j To j + c
        'calculating wi
        For h = j To j + c
            widenominator = widenominator + alphai(h) * DefaultBondPrice(h)
        Next h
    winumerator = alphai(i) * DefaultBondPrice(i)
    wi = winumerator / widenominator
    winumerator = 0
    widenominator = 0
    'Done calculating wi
    'Calculating Expected value of Ri(Tj-1)
    'Calculating exponential
    For k = j + 1 To i
        exponential = exponential + Rho * sigma * Ri(k) / (Ri(k) + Lgd / alphai(k))
    Next k
    exponential = Exp(exponential * Ti(j - 1) * sigma)
    'done calculating exponential
    YiNumerator(j) = YiNumerator(j) + wi * DefaultBondPrice(j) * exponential * Ri(i) 'Expected
value of Ri(Tj-1)
    exponential = 0
    Next i
    Sheet3.Cells(5 + j, 7) = YiNumerator(j) 'Output to Intermediate2 Sheet
    Sheet3.Cells(5 + j, 12) = DefaultBondPrice(j) * StandardRate 'Output to Intermediate2 Sheet
    Sheet2.Cells(6 + j, 9) = YiNumerator(j) / (DefaultBondPrice(j) * StandardRate)
Next j
```

'Already have all information for Ni Using Equation (3), (5) and (6)

```
For j = a + 1 To b
    Sheet2.Cells(6 + j, 10) = YiNumerator(j) / (DefaultBondPrice(j) * ConstantMaturityRate(j))
    Sheet3.Cells(31 + j, 3) = DefaultBondPrice(j) * ConstantMaturityRate(j) 'Output to
Intermediate2 Sheet
    Next j
```

'Finishing the output Ni

'Finding Xi Using Equation (2) and (3)

```
CDS = 0
For j = a + 1 To b
   CDS = CDS + alphai(j) * DefaultBondPrice(j) * Ri(j) 'The values of Ri(0) from the Input sheet rather then Rab formula
   DiscCMRate = DiscCMRate + alphai(j) * DefaultBondPrice(j) * ConstantMaturityRate(j)
   Sheet3.Cells(30 + j, 7) = CDS    'Output to Intermediate2 Sheet
   Sheet3.Cells(30 + j, 12) = DiscCMRate

   Sheet2.Cells(6 + j, 11) = CDS / DiscCMRate 'Psi is outputted
   Next j
```

'Finish Calculating Xi

'Finding Yi by Equation (2), (5) and (6)

```
CDS = 0
CMCDSprice = 0
holdsum2 = 0
exponential = 0
denominator = 0

For j = 1 To b
   CDS = CDS + alphai(j) * DefaultBondPrice(j) * Ri(j)

   For i = j To j + c
      For k = j + 1 To i
         exponential = exponential + Rho * sigma * Ri(k) / (Ri(k) + Lgd / alphai(k))
      Next k
      exponential = Exp(Ti(j - 1) * sigma * exponential)

      For h = j To j + c
         denominator = denominator + alphai(h) * DefaultBondPrice(h)
      Next h

      holdsum2 = holdsum2 + alphai(i) * DefaultBondPrice(i) / denominator * Ri(i) * exponential
      exponential = 0
      denominator = 0
   Next i
CMCDSprice = CMCDSprice + alphai(j) * DefaultBondPrice(j) * holdsum2
holdsum2 = 0

ratio = CDS / CMCDSprice
Sheet3.Cells(55 + j, 3) = CMCDSprice 'Output to Intermediate2 Sheet
```

```
Sheet2.Cells(6 + j, 12) = ratio 'Yi is outputted
Next j
'Finish computing Yi
End Sub
```

4. A sample of Matlab Code calculation of the intensity and survival probability for maturity of 0.5 year

Code

```
clear all;
global x0 LGD T alpha Y P R
%Date 2008-10-28 for IBM
x0=0;
LGD=0.6;
% CDS rate
R=[0.00391 0.0047327 0.0054669 0.0063894 0.0072652 0.007716 0.0077472 0.0079439];
% Time
T=[0.25 0.5 0.75 1 1.25 1.5 1.75 2 2.25 2.5 2.75 3 3.25 3.5 3.75 4 4.25 4.5 4.75 5 ...
  5.25 5.5 5.75 6 6.25 6.5 6.75 7 7.25 7.5 7.75 8 8.25 8.5 8.75 9 9.25 9.5 9.75 10];
% Time interval
alpha=0.25;
% zero coupon bond yield
Y=[0.0076 0.0167340004 0.0155219996 0.0147290003 0.0143289995 0.0142410004
0.0144439995 0.0148740005 0.0155120003 0.0163100004 0.0172510004 0.0182980001
0.0194360006 0.0206389999 0.0218930006 0.0231800008 0.0244880009 0.0258039999
0.0271169996 0.0284200001 0.0297029996 0.0309629989 0.032191 0.0333859992 0.03454
0.0356550002 0.036724999 0.0377509999 0.0387280011 0.0396589994 0.0405410004
0.0413770008 0.0421630001 0.0429040003 0.0435960007 0.0442430019 0.044843998
0.0454029989 0.0459180021 0.0463920021];
% zero coupon bond price
P=exp(-Y.*T);
% intensity
g=zeros(1,40);
% survival probability
gamma=zeros(1,40);
g(1:2)=fsolve(@IBMyear0,x0);
temp=0;
for i=1: 2
   temp=temp+g(i)*alpha;
   gamma(i)=exp(-temp);
end
```

Function IBMyear0

```
function F = IBMyear0(x)
global LGD T alpha Y P R
```

```
F=R(1)*P(1)*alpha*exp(-x*T(1))+R(1)*P(2)*alpha*exp(-x*T(2))-LGD*x*(exp(-x*T(1))*P(1))*alpha-LGD*x*(exp(-x*T(2))*P(2))*alpha;
```

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