Some Results on Unique Relationship between Structure Functions and Credibility Expressions

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ABSTRACT

Premium computation in a Bayesian context requires the use of a prior distribution (structure function) that the risk parameter follows in the heterogeneous portfolio. This paper contributes to the analysis of credibility theory by identifying the unique relationship between the prior distribution and credibility formula. The latter corresponds to a suggestive form of expressing the premium to be charged to a policyholder as a weighted sum of the sample mean and collective premium. Results for net premium principle and Poisson and negative binomial likelihood functions are shown.

Keywords: *Bayes, Credibility Premium, Identification, Negative Binomial, Poisson*

MSC: 62C05, 62C10, 62P05, 91B30

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1 Introduction

Credibility theory provides a tool to compute premiums calculated by combining the sample information together with collateral information by incorporating a prior distribution to the risk parameter. Assuming that the individual risk, X, has a density $f(x|\theta)$, indexed by the risk parameter $\theta \in \Theta$ which has a prior distribution with density $\pi(\theta)$. Let $\mathbf{P}$ be the action space, $L(\theta, P)$ be the loss if the premium $P \in \mathbf{P}$ is chosen, and θ the state of nature. Let, now, $\pi(\theta|k)$ be the posterior density when a sample $\underline{X} = (X_1, \ldots, X_n)$ of size n is observed, $k = \sum_{i=1}^n X_i$, and $\rho(\pi(\theta|k), P)$ the posterior expected loss of P. Then a Bayes premium can be obtained by minimizing $\rho(\pi(\theta|k), P)$. The loss is usually chosen to belong to the family $L(\theta, \mathcal{P}) = h(\theta)\left(w(\theta) - \mathcal{P}\right)^2$, where h and w are functions of θ.

By minimizing the posterior expected loss above, denoting $g(\theta) = h(\theta)w(\theta)$, we get

$$P = \frac{\displaystyle\int_\Theta g(\theta) f(k|\theta)\pi(\theta)d\theta}{\displaystyle\int_\Theta h(\theta) f(k|\theta)\pi(\theta)d\theta}, \tag{1}$$

which is known in the literature as the ratemaking or Bayes premium (P_B).

When we assume that $h(\theta) = 1$ and $w(\theta) = \theta$, we get the net premium principle and for $h(\theta) = e^{\alpha\theta}$, $\alpha > 0$, and $w(\theta) = \theta$ the Esscher premium principle. Other combinations are possible providing, for example, the variance and exponential premium principles. For a more detailed information on premium calculation principles, the reader can consult Calderín et al. (2008), Heilmann (1989), Hürlimann (1994) and Young (2004); among others.

In many ocasions it is possible to write (1) as a weighted sum of the sample mean and the collective premium, the premium to be charged to a group of policyholders in a portfolio. The weighted factor is referred as the credibility factor and, therefore, the premium obtained adopts this suggestive expression:

$$P_B = z_n l(\bar{X}) + (1 - z_n)P_C, \tag{2}$$

for some function of the sample mean $l(\bar{X})$, where $\bar{X}$ is the sample mean, P_C the collective premium and z_n the credibility factor satysfing $z_n \in (0,1)$, $\lim_{n\to 0} z_n = 0$ and $\lim_{n\to\infty} z_n = 1$.

Some historical references on credibility theory are Whitney (1918), Mowbray (1914), Bailey (1945), Bühlmann (1967), Jewell (1974), Kahn (1975), Gerber and Arbor (1980), Eichenauer et al. (1988), Heilmann (1989), Goovaerts et al. (1990) and Herzog (1996). For a recent revision of the credibility theory see Bühlmann and Gisler (2005), Landsmand and Makov (1999), Promislow and Young (2000), Young (2000), Gómez-Déniz (2008) and Gómez-Déniz et al. (2006).

On the other hand, in the statistical literature a basic problem for a mixture model, $f(x) = \int_\Theta f(x|\theta)\pi(\theta)d\theta$, is its identifiability. This problem deals with indentifying in a one to one correspondence between the distribution of the likelihood of the model and the prior, given the posterior mean of the parameter of interest, $\int_\Theta \theta\pi(\theta|k)d\theta$. Some important contributions in this field are Cacoullos and Papageorgiou (1982), Gupta and Wesolowski (1997, 1999), Johnson (1957, 1967), Papageorgiou (1984), Papageorgiou and Wesolowski (1997) and Wesolowski (1995); among others. Nevertheless, in the actuarial setting the posterior magnitude, called the Bayes premium, usually does not coincide with the posterior mean of the parameter. Therefore, we consider that this topic about indentifiability of Bayes premium has never been analyzed in the actuarial literature.

In this paper, the prior distribution under different likelihoods is shown to be completely indentifiable by the form of the Bayes net premium, which results under appropriate likelihood and prior distribution a credibility formula. This will be made through by a one to one correspondence between the likelihood and the prior distribution if the credibility expression is given. Results for net premium principle and Poisson, negative binomial and binomial likelihood functions are shown.

For that reason, we will observe that for a sample $\underline{X} = (X_1, \dots, X_n)$ of size n from model $f(x|\theta)$ and prior distribution $\pi(\theta)$, $\theta \in \Theta$ it is verified

$$\pi(\theta|\underline{X})m(\underline{X}|\pi) = f(\underline{X}|\theta)\pi(\theta), \tag{3}$$

which is obtained directly from Bayes' Theorem. Here, $\pi(\theta|\underline{X})$ represents the posterior distribution of θ given the sample information $\underline{X}$ and $\int_\Theta f(\underline{X}|\theta)\pi(\theta)d\theta$ is the marginal distribution of $\underline{X}$.

Section 2, 3 and 4 includes the results for the Poisson, negative binomial and binomial cases, respectively. Conclusions and extensions are given in the last Section.

2 The Poisson case

Let X be a random variable with the Poisson probability mass function, i.e.

$$f(x|\theta) = \frac{1}{x!}e^{-\theta}\theta^x, \ \theta > 0, \ x = 0, 1, \dots \tag{4}$$

For a sample $\underline{X} = (X_1, \dots, X_n)$ of size n, the likelihood function from model (4) is

$$f(k|\theta) \propto e^{-n\theta}\theta^k,$$

The fact that the regression of X on k is linear, i.e. a credibility formula, was proved by Johnson (1957). In this section, we reproduce the proof in an alternative way.

Theorem 1. *Let us suppose that (X, θ) is a mixture model with pmf given by (4), then the prior distribution of θ is uniquely determined by the posterior mean $\mathbb{E}(\theta|k)$, being $k = \sum_{i=1}^n X_i$.*

Proof. From (3) we have that

$$\pi(\theta|k)\int_0^\infty f(k|\theta)\pi(\theta)d\theta = f(k|\theta)\pi(\theta).$$

Therefore,

$$\int_0^\infty \theta\pi(\theta|k)d\theta \int_0^\infty f(k\,|\theta)\pi(\theta)d\theta = \int_0^\infty \theta f(k|\theta)\pi(\theta)d\theta.$$

Now, replacing $f(k|\theta)$ by the likelihood in (4) we get

$$\int_0^\infty \theta\pi(\theta|k)d\theta \int_0^\infty \theta^k e^{-n\theta}\pi(\theta)d\theta = \int_0^\infty \theta^{k+1}e^{-n\theta}d\theta.$$

Let us define now a prior distribution in the form $\pi_1(\theta) = c\ e^{-n\theta}\pi(\theta)$, where $c = 1/\mathbb{E}_\pi\left(e^{-n\theta}\right)$, is the normalizing constant. Let also U the random variable with probability density function $\pi_1(\theta)$. By putting $m(k) = \int_\Theta \theta\pi(\theta|k)d\theta$, we have that $m(k) = \mathbb{E}U^{k+1}/\mathbb{E}U^k$, from which it is easy to obtain that

$$\mathbb{E}U^k = \prod_{j=0}^{k-1} m(j),$$

Thus, the distribution of U is uniquely determined by the function m and therefore $\pi(\theta)$ is unique. □

It is known (see Heilmann, 1989) that if we assume a Poisson distribution, $\mathcal{P}o(\theta)$, for the risk X and θ has the Pearson Type III distribution, $\mathcal{G}(a,b),\ a>0,\, b>0$, with the density

$$\pi(\theta; a, b) = \frac{b^a}{\Gamma(a)}\theta^{a-1}e^{-b\theta}I_{(0,\infty)}(\theta), \tag{5}$$

being $\Gamma(\cdot)$ the gamma function, the posterior mean (Bayes net premium) of $P(\theta) = \theta$ (the risk net premium) is a credibility formula as in (2), where $l(x) = x$, $z_n = n/(b+n)$ and $P_C = \mathbb{E}\Theta = a/b$.

Therefore we have the following result.

Corollary 1. *If $X \sim \mathcal{P}o(\theta)$ as in (4), the only form of prior probability density function satysfing that the Bayes net premium takes the form (2), is the Pearson Type III distribution (5).*

Proof. It follows directly by applying Theorem 1. □

It is well–known that the mean of the predictive distribution coincides with the posterior mean when $\mathbb{E}(y|\theta) = \theta$, see Herzog (1996) for details. Therefore, we have the following result.

Corollary 2. *Suppose that (X, θ) is a mixture model with pmf given by (4), then the prior distribution of θ is uniquely determined by the mean of the predictive distribution $f(y|k) = \int_\Theta f(y|\theta)\pi(\theta|k)d\theta$.*

Proof. It follows directly by applying Theorem 1 having into account that $\mathbb{E}(y|\theta) = \theta$. □

3 The negative binomial case

Let now X be a random variable with the negative binomial probability mass function, i.e.

$$f(x|\theta) = \binom{r+x-1}{x}\theta^r(1-\theta)^x, \quad 0<\theta<1,\ r>0,\ x=0,1,\dots \tag{6}$$

For a sample $\underline{X} = (X_1,\dots,X_n)$ of size n, the likelihood function from model (6) is

$$f(k|\theta) \propto \theta^{nr}(1-\theta)^k,$$

where $k = n\sum_{i=1}^n X_i \in \mathbb{N}$. The net risk premium under this probability model is given by $P(\theta) = r(1-\theta)/\theta$ (see Heilmann, 1989).

The following Lemma will be used to show that only exists one prior distribution which generates a linear with respect to the data Bayes premium.

Lemma 1. *Suppose that a random variable U satisfies that*

$$r\mathbb{E}U^{x+1} = m(x)\left(\mathbb{E}U^x - \mathbb{E}U^{x+1}\right),$$

for $x \in \mathbb{N}$, $r>0$ and some function $m(x)$, then it is verified that

$$\mathbb{E}U^x = \prod_{j=0}^{x-1}\frac{m(j)}{r+m(j)}.$$

Proof. It is easy to prove by expanding the recursive formula. □

Now, we have the following result.

Theorem 2. *Suppose that (X,θ) is a mixture model with pmf given by (6), then the prior distribution of θ is uniquely determined by the posterior mean $\mathbb{E}\left[r(1-\theta)/\theta|k\right]$, being $k=\sum_{i=1}^n X_i$.*

Proof. From (3) we have that

$$\pi(\theta|k)\int_0^1 f(k|\theta)\pi(\theta)d\theta = f(k|\theta)\pi(\theta).$$

Therefore,

$$\int_0^1 r(1-\theta)/\theta\pi(\theta|k)d\theta \int_0^1 f(k\,|\theta)\pi(\theta)d\theta = \int_0^1 r(1-\theta)/\theta f(k|\theta)\pi(\theta)d\theta.$$

Now, replacing $f(k|\theta)$ by the likelihood in (7) we get

$$\int_0^1 r(1-\theta)/\theta\pi(\theta|k)d\theta \int_0^1 \theta^{nr}(1-\theta)^k\pi(\theta)d\theta = \int_0^1 r\theta^{nr-1}(1-\theta)^{k+1}d\theta.$$

Define now a prior distribution in the form $\pi_1(\theta) = c\,\theta^{nr-1}\,\pi(\theta)$, where $c = 1/\mathbb{E}_\pi\left(\theta^{nr-1}\right)$, is the normalizing constant. Let also W the random variable with probability density function $\pi_1(\theta)$. By putting $m(k) = \int_0^1 r(1-\theta)/\theta\pi(\theta|k)d\theta$, and by using Lemma 1 we have that

$$\mathbb{E}U^k = \prod_{j=0}^{k-1}\frac{m(j)}{r+m(j)},$$

where $U = 1-W$. Therefore, the distribution of U and hence of W are uniquely determined by the function m and $\pi(\theta)$ is unique. □

It is known (see Heilmann, 1989) that if we assume a negative binomial distribution, $\mathcal{NB}(r,\theta)$, for the risk X and θ has the beta distribution of the first kind, $\mathcal{B}e(a,b),\ a > 0,\ b > 0$, with the density

$$\pi(\theta; a, b) = \frac{\theta^{a-1}(1-\theta)^{b-1}}{\mathcal{B}(a,b)} I_{(0,1)}(\theta), \tag{7}$$

being $\mathcal{B}(\cdot,\cdot)$ the beta function, the posterior mean (Bayes net premium) of $P(\theta) = r(1-\theta)/\theta$ (the risk net premium) is a credibility formula as in (2), with $l(x) = x$, $z_n = rn/(a+nr-1)$, $a+nr>1$ and $P_C = \mathbb{E}\left(r(1-\Theta)/\Theta\right) = rb/(a-1)$, $a>1$.

Therefore we have the following result.

Corollary 3. *If $X \sim \mathcal{NB}(r,\theta)$ as in (6), the only form of prior probability density function satysfing that the Bayes net premium takes the form (2), is the beta distribution of the first kind (7).*

Proof. It can be easily proved by using Theorem 2. □

Let us suppose now that X be a random variable with the following negative binomial probability mass function, i.e.

$$f(x|\theta) \;=\; \binom{r+x-1}{x}\left(\frac{r}{r+\theta}\right)^r\left(\frac{\theta}{r+\theta}\right)^x, \quad \theta>0,\ r>0,\ x=0,1,\ldots \tag{8}$$

This parameterization of the negative binomial model in the actuarial context has been considered by Gómez-Déniz and Vázquez (2003) and Meng et al. (1999), among others.

For a sample $\underline{X}=(X_1,\ldots,X_n)$ of size n, the likelihood function from model (8) is

$$f(k|\theta)\propto\left(\frac{r}{r+\theta}\right)^{nr}\left(\frac{\theta}{r+\theta}\right)^k,$$

where $k=n\sum_{i=1}^n X_i\in\mathbb{N}$. In this case, the net risk premium is given by $P(\theta)=\theta$.

Theorem 3. *Suppose that (X,θ) is a mixture model with pmf given by (8), then the prior distribution of θ is uniquely determined by the posterior mean $\mathbb{E}(\theta|k)$, being $k=\sum_{i=1}^n X_i$.*

Proof. From (3) we have that

$$\pi(\theta|k)\int_0^\infty f(k|\theta)\pi(\theta)d\theta=f(k|\theta)\pi(\theta).$$

Therefore,

$$\int_0^\infty\theta\pi(\theta|k)d\theta\int_0^\infty f(k\,|\theta)\pi(\theta)d\theta=\int_0^\infty\theta f(k|\theta)\pi(\theta)d\theta.$$

Now, replacing $f(k|\theta)$ by the likelihood in (13) we get

$$\int_0^\infty\theta\pi(\theta|k)d\theta\int_0^\infty\left(\frac{r}{r+\theta}\right)^{nr}\left(\frac{\theta}{r+\theta}\right)^k\pi(\theta)d\theta=\int_0^\infty\theta\left(\frac{r}{r+\theta}\right)^{nr}\left(\frac{\theta}{r+\theta}\right)^k\pi(\theta)d\theta.$$

Now, if we denote

$$I(k,s)=\int_0^\infty\frac{1}{(r+\theta)^s}\left(\frac{\theta}{\theta+r}\right)^k\pi(\theta)d\theta$$

and writing

$$m(k)=\int_0^\infty\theta\pi(\theta|k)d\theta$$

we have

$$(m(k)+r)I(k,s) = I(k,s-1). \tag{9}$$

On the other hand, from the definition of the function I we get

$$I(k+1,s-1) = I(k,s-1) - rI(k,s), \tag{10}$$

After combining the two equations (9) and (10) it can be verified the following recurrence formula:

$$I(k+1,s-1) = I(k,s-1)\left(\frac{m(k)}{m(k)+r}\right).$$

Let us define a new prior distribution in the form $\pi_1(\theta) = \pi(\theta)/(c(r+\theta)^{s-1})$, where $c = I(0,s-1)$ is the normalizing constant. If Θ is a random variable with prior distribution $\pi(\theta)$ then $U = \Theta/r + \Theta$ is the random variable with probability density function $\pi_1(\theta)$. Then we have that

$$\mathbb{E}U^{k+1} = \mathbb{E}U^k \frac{m(k)}{m(k)+r},$$

and

$$\mathbb{E}U^k = \prod_{j=0}^{k-1} \frac{m(j)}{m(j)+r}.$$

Therefore, the distribution of U is uniquely determined by the function m and, for that reason, $\pi(\theta)$ is unique. $\square$

It is known (see Gómez-Déniz and Vázquez, 2003 and Meng et al., 1999) that if we assume a negative binomial distribution, $\mathcal{NB}(r,\theta)$ as in (8), for the risk X and θ has the generalized Pareto distribution, $\mathcal{GP}(\zeta,r,s)$, $\zeta>0$, $r>0$, $s>0$, with the following density function

$$\pi(\theta;\zeta,r,s) = \frac{\Gamma(s\zeta+sr+1)}{\Gamma(s\zeta)\Gamma(sr+1)} \frac{r^{sr+1}\theta^{s\zeta-1}}{(r+\theta)^{s\zeta+sr+1}} I_{(0,\infty)}(\theta), \tag{11}$$

the posterior mean (Bayes net premium) of $P(\theta)=\theta$ (the risk net premium) is a credibility formula as in (2), where $l(x)=x$, $z_n = n/(s+n)$ and $P_C = \mathbb{E}\Theta = \zeta$.

Thus, we have the following result.

Corollary 4. *If $X \sim \mathcal{NB}(r,\theta)$ as in (8), the only form of prior probability density function satysfing that the Bayes net premium takes the form (2), is the generalized Pareto distribution in (11).*

Proof. The result can be obtained directly by applying Theorem 4. □

4 The binomial case

Finally, let us suppose that X is a random variable with the following binomial probability mass function, i.e

$$f(x|\theta) \quad = \quad \binom{N}{x}\frac{\theta^x}{(1+\theta)^N}, \quad \theta > 0, \ , \ x = 0, 1, \dots, N \tag{12}$$

For a sample $\underline{X} = (X_1, \dots, X_n)$ of size n, the likelihood function from model (12) is

$$f(k|\theta) \propto \frac{\theta^k}{(1+\theta)^{nN}}, \tag{13}$$

where $k = \sum_{i=1}^n X_i \in \mathbb{N}$. In this case, the net risk premium is $P(\theta) = \dfrac{N\theta}{1+\theta}$.

Theorem 4. *Suppose that (X,θ) is a mixture model with pmf given by (12), then the prior distribution of θ is uniquely determined by the posterior mean $\mathbb{E}\left(N\theta/(1+\theta)|k\right)$, being $k = \sum_{i=1}^n X_i$.*

Proof. From (3) we have that

$$\pi(\theta|k)\int_0^\infty f(k|\theta)\pi(\theta)d\theta = f(k|\theta)\pi(\theta).$$

Therefore,

$$\int_0^\infty \frac{N\theta}{1+\theta}\pi(\theta|k)d\theta \int_0^\infty f(k\,|\theta)\pi(\theta)d\theta = \int_0^\infty \frac{N\theta}{1+\theta}f(k|\theta)\pi(\theta)d\theta.$$

Now, replacing $f(k|\theta)$ by the likelihood in (13) we get

$$\int_0^\infty \frac{N\theta}{1+\theta}\pi(\theta|k)d\theta \int_0^\infty \frac{\theta^k}{(1+\theta)^{nN}}\pi(\theta)d\theta = \int_0^\infty \frac{N\theta^{k+1}}{(1+\theta)^{nN+1}}\pi(\theta)d\theta.$$

Now, if we denote

$$I(k,s) = \int_0^\infty \frac{\theta^k}{(1+\theta)^s}\pi(\theta)d\theta$$

and putting

$$m(k) = \int_0^\infty \frac{N\theta}{1+\theta}\pi(\theta|k)d\theta$$

we have

$$m(k)I(k,s) = NI(k+1,s+1). \tag{14}$$

From the definition of the function I, we can derive

$$I(k+1,s+1) = I(k,s) - I(k,s+1), \tag{15}$$

Combining now (14) and (15) it is easy to verify the following recurrence formula:

$$I(k+1,s+1) = I(k,s+1)\left[\frac{m(k)}{N-m(k)}\right].$$

Let us define now a prior distribution in the form $\pi_1(\theta) = \frac{1}{c(1+\theta)^{s+1}}\pi(\theta)$, where $c = I(0,s+1)$ is the normalizing constant. Let also U be a random variable with probability density function $\pi_1(\theta)$. Then we have that

$$\mathbb{E}U^k = \prod_{j=0}^{k-1} \frac{m(j)}{N-m(j)},$$

where $U = \Theta$. Therefore, the distribution of U is uniquely determined by the function m and therefore $\pi(\theta)$ is unique. $\square$

If we assume a binomial distribution, $\mathcal{B}i(N,\theta)$ as in (13), for the risk X and θ has the second kind Beta distribution (inverted Beta distribution), $\mathcal{IB}(\theta,a,b),\ \theta>0,\ a>0,\ b>0$, with the following density function

$$\pi(\theta;a,b) = \frac{1}{\mathcal{B}e(a,b)}\frac{\theta^a}{(1+\theta)^{a+b}}I_{(0,\infty)}(\theta), \tag{16}$$

the posterior mean (Bayes net premium) of $P(\theta) = \dfrac{N\theta}{1+\theta}$ (the risk net premium) is a credibility formula as in (2), where $l(x) = x$, $z_n = Nn/(a+b+Nn)$ and $P_C = \mathbb{E}\left(N\Theta/(1+\Theta)\right) = Na/(a+b)$.

Therefore we have the following result.

Corollary 5. *If $X \sim \mathcal{NB}(r,\theta)$ as in (8), the only form of prior probability density function satysfing that the Bayes net premium takes the form (2), is the generalized Pareto distribution in (16).*

Proof. It follows directly by using Theorem 4. □

5 Conclusions and extensions

The aim of this paper has been to illustrate some basic notions about identifiability. A basic question for credibility formulas is to determine the unique relationship between these expressions and the prior distributions. Usually, it is connected with one to one correspondence between the structure function and the likelihood function if the credibility expression is provided.

One possible aspect to extend this work is based on the search of marginal distributions generated by identifiability method whose posterior expectations satisfy a given format, in the line of the work of Papageorgiou and Wesolowski (1997). This is surely an attractive problem which deserves to be studied deeply in the future.

Acknowledgements

The authors thanks to Ministerio de Educación y Ciencia (project SEJ2006-12685) for partial support of this work.

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