DISCOUNTED CLAIMS IN A RENEWAL RISK MODEL

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Abstract

In the classical risk theoretic model with Poisson claim arrival and exponential claims, consider the discounted value of claims nos. 3, 6, 9, and so on. (To risk theory specialists, considering every third claim in a Poisson arrival process is the same as assuming waiting times are Erlang(3) in a Sparre-Andersen model.) The distribution of the discounted value of future claims can be found. It turns out that this is the same as the distribution of the product of two independent variables, one having a gamma distribution, the other a "complex-parameter beta product distribution". The latter will be defined, it involves extending the usual beta distribution.

1. An example

Suppose i.i.d. claims $\{C_n\}$ occur at times $\{T_n\}$, and that one wishes to find the distribution of the discounted value of all future claims. If the discount rate is $r > 0$, then this is

$$X \;=\; \sum_{n=1}^{\infty} e^{-rT_n} C_n.$$

Let the waiting times

$$W_1 = T_1, \qquad W_n = T_n - T_{n-1}, \qquad n \geq 2,$$

be i.i.d., making $\{T_n\}$ a renewal process, and assume moreover that $\{T_n\}$ and $\{C_n\}$ are independent. Then the above sum may be rewritten as

$$X \;=\; \sum_{n=1}^{\infty} B_1 \cdots B_n C_n \qquad \text{if} \qquad B_n = e^{-rW_n}.$$

Such sums of products of random variables occur in a variety of applications and have been studied for some time.

It is well known that X is finite w.p.1 if: (a) $\mathsf{P}(W_1 > 0) > 0$ and (b) $\mathsf{E} \log |C_1| < \infty$, and that it is then the unique solution of the identity in distribution

$$X \stackrel{\mathrm{d}}{=} B_1(X + C_1).$$

("$\stackrel{\mathrm{d}}{=}$" means "has the same distribution as".)

There is yet no general method to find the distribution of X given arbitrary B_1, C_1, but a number of explicit cases are known. The case derived here is not in the literature, to this author's knowledge.

The assumptions are:

— claims $\{C_n\}$ have an exponential distribution with mean 1;

— waiting times $\{W_n\}$ have a $\mathbf{Gamma}(3, \lambda)$ distribution.

(This could be realized as follows: if claims arrive according to a Poisson process with intensity λ, but if only claims number 3,6,9, and so on are discounted, then the waiting time between two consecutive discounted claims does have a $\mathbf{Gamma}(3, \lambda)$ distribution.)

The identity $X \stackrel{\mathrm{d}}{=} B_1(X + C_1)$ then holds, X has finite moments of all order, and the moments satisfy:

$$\mathsf{E}\, X^n \;=\; (\mathsf{E} B^n) \sum_{j=0}^{n} \binom{n}{j} (\mathsf{E}\, X^j)(n-j)! \qquad n \geq 0.$$

This recursive equation uniquely determines the moments of X. It can be shown that

$$\mathsf{E}\, X^n \;=\; \frac{[(\beta)_n]^3}{(\gamma)_n (\overline{\gamma})_n},$$

where

$$\beta \;=\; \frac{\lambda}{r}, \qquad \gamma \;=\; 1 + \frac{\beta}{2}(3 + i\sqrt{3}).$$

(Here $(a)_n = a(a+1)\cdots(a+n-1)$ is the "rising factorial".)

It is easy to see that the distribution of X is determined by its moments.

Question: what is the distribution of X?

(Background: it is already known that:

— if waiting times $\{W_n\}$ have an $\mathbf{Exp}(\lambda) = \mathbf{Gamma}(1, \lambda)$ distribution, then X has a gamma distribution;

— if waiting times $\{W_n\}$ have a $\mathbf{Gamma}(2, \lambda)$ distribution, then the distribution of X is that of the product of a beta variable and an independent gamma variable.)

2. The beta product distribution with complex parameters

Let us introduce the product convolution: if independent variables $X, Y > 0$ have densities h_X, h_Y, then the density of $U = XY$ is

$$h_U(u) \;=\; h_X \odot h_Y(u) \;=\; \int_0^\infty \frac{dx}{x} h_X(x) h_Y\left(\frac{u}{x}\right).$$

The function

$$f_{a,b}(u) \;=\; \frac{1}{B(a,b)}\, u^{a-1}(1-u)^{b-1} \mathbf{1}_{\{0<u<1\}}$$

is a probability density function if $a, b > 0$, but it is not a probability density function otherwise.

Nevertheless, it can be shown that there are choices of complex numbers a, b, c, d, not all real, such that the product convolution

$$f_{a,b} \odot f_{c,d}(u) \;=\; \int_0^\infty \frac{dx}{x} f_{a,b}(x) f_{c,d}\left(\frac{u}{x}\right).$$

is a true probability density function.

Summarizing:

— for $a, b, c, d > 0$ the product convolution $f_{a,b} \odot f_{c,d}$ is always the density of a product of independent $\mathbf{Beta}(a, b)$ and $\mathbf{Beta}(c, d)$ variables;

— there are cases where $(a, b, c, d) \in \mathbb{C}^4 - \mathbb{R}_+^4$ and $f_{a,b} \odot f_{c,d}$ is a true probability density function.

It can be verified that for any a, b, c, d with positive real parts and $0 < u < 1$,

$$f_{a,b} \odot f_{c,d}(u) \;=\; \frac{\Gamma(a+b)\Gamma(c+d)}{\Gamma(a)\Gamma(c)\Gamma(b+d)}\, u^{a-1}(1-u)^{b+d-1}\, {}_2F_1(a+b-c, d; b+d; 1-u).$$

(Here

$$ {}_2F_1(a,b;c;z) \;=\; \sum_{n=0}^{\infty} \frac{(a)_n(b)_n}{(c)_n}\frac{z^n}{n!}, \qquad c \neq 0,-1,-2,\ldots, \quad |z|<1. $$

is the Gauss hypergeometric function.) We thus define

$$ g_{a,b,c,d}(u) \;=\; \frac{\Gamma(a+b)\Gamma(c+d)}{\Gamma(a)\Gamma(c)\Gamma(b+d)}\, u^{a-1}(1-u)^{b+d-1}\, {}_2F_1(a+b-c,d;b+d;1-u)\mathbf{1}_{\{0<u<1\}}. $$

We look for values of a,b,c,d (besides $a,b,c,d>0$) for which the expression given for $g_{a,b,c,d}$ is a probability density function.

It is always true that $\int g_{a,b,c,d} = 1$ (in all cases where $g_{a,b,c,d}$ is integrable). The theorem below gives sufficient conditions for $g_{a,b,c,d}$ to be non-negative, thus proving that it is a probability density function.

Theorem 1. *The function $g_{a,b,c,d}$ is a probability density function if $a, c, b+d$ are real and positive, $\mathsf{Re}(a+b), \mathsf{Re}(c+d)>0$, and either*

(a) (real case) all parameters are real and $\min(a,c) < \min(a+b,c+d)$; or

(b) (complex case) $\mathrm{Im}(b) = -\mathrm{Im}(d) \neq 0$ and $a+b=\overline{c+d}$.

Definition. *The probability distribution with density $g_{a,b,c,d}$ will be called **beta product**, denoted "$\mathbf{BetaP}(a,b,c,d)$."*

(Comment: This is one of the rare families of distribution that "naturally" has complex parameters.)

3. Characterization of the BetaP distribution

Theorem 1 leaves open the question of whether there might be other parameters (a,b,c,d) that lead to a true probability density function $g_{a,b,c,d}$.

Consider the case

$$ a = 3+i/10,\;\; b=2+i,\;\; c=3-i/10,\;\; d=2-i $$

(see Figure 1). This case is not covered by Theorem 1. Here the moments

$$ \int_0^1 du\, u^n g_{a,b,c,d}(u) \;=\; \frac{|(a)_n|^2}{|(a+b)_n|^2}, \qquad n=0,1,2,\ldots, $$

are all positive, and $g_{a,b,c,d}$ integrates to 1. By all appearances the function in Figure 1 is a probability density function.

This leads to a more detailed analysis of the problem, and to the following result.

Theorem 2. *Let $a,b,c,d \in \mathbb{C}$, and suppose a random variable U has the same moments as the product convolution $f_{a,b} \odot f_{c,d} = g_{a,b,c,d}$, that is, assume*

$$ \mathsf{E}\,U^n \;=\; \int du\, u^n g_{a,b,c,d}(u), \qquad n=0,1\ldots $$

Then the distribution of U is either:

— an ordinary beta distribution, or

— the $\mathbf{BetaP}(a,b,c,d)$ distribution, with a, c and $b+d$ positive, $\mathsf{Re}(a+b), \mathsf{Re}(c+d)>0$, and either

(a) (real case) all parameters are real and $\min(a,c)<\min(a+b,c+d)$; or

(b) (complex case) $\mathrm{Im}(b) = -\mathrm{Im}(d) \neq 0$ *and* $a + b = \overline{c + d}$.

This means that the cases listed in Theorem 1 are the only ones where $g_{a,b,c,d}$ is a probability density function, apart from the "degenerate" cases where one of a or c equals $a + b$ or $c + d$, which correspond to an ordinary beta distribution.

Theorem 2 apparently contradicts my initial guess that the case given above,

$$a = 3 + i/10,\ \ b = 2 + i,\ \ c = 3 - i/10,\ \ d = 2 - i,$$

corresponds to a true probability distribution. The explanation is in Figure 2.

4. Return to the example

Recall that

$$X \;=\; \sum_{n=1}^{\infty} e^{-rT_n} C_n,$$

where the waiting times $W_n = T_n - T_{n-1}$ are $\mathbf{Gamma}(3, \lambda)$ variables, and $C_n \sim \mathbf{Exp}(1)$ are independent.

We have seen that the distribution of X is determined by its moments

$$\mathsf{E}\, X^n \;=\; \frac{[(\beta)_n]^3}{(\gamma)_n(\overline{\gamma})_n},$$

where

$$\beta \;=\; \frac{\lambda}{r}, \qquad \gamma \;=\; 1 + \frac{\beta}{2}(3 + i\sqrt{3}).$$

It can be checked that

$$\mathsf{E}\, X^n \;=\; \mathsf{E}\, U^n G^n, \qquad n = 0, 1, \ldots,$$

where $U \sim \mathbf{BetaP}(\beta, \gamma - \beta, \beta, \gamma - \beta)$ and $G \sim \mathbf{Gamma}(\beta)$ are independent. The distributions of all those variables are determined by their moments, and we conclude that

$$X \overset{\mathrm{d}}{=} UG,$$

which we write as $X \sim \mathbf{BetaP}(\beta, \gamma - \beta, \beta, \gamma - \beta) \odot \mathbf{Exp}(\beta)$.

The density of X is thus

$$f_X(x) \;=\; \int_0^1 \frac{du}{u} g_{\beta,\gamma-\beta,\beta,\gamma-\beta}(u) \left(\frac{x}{u}\right)^{\beta-1} e^{-x/u}\, \mathbf{1}_{\{x>0\}}.$$

More details are given in Dufresne (2007).

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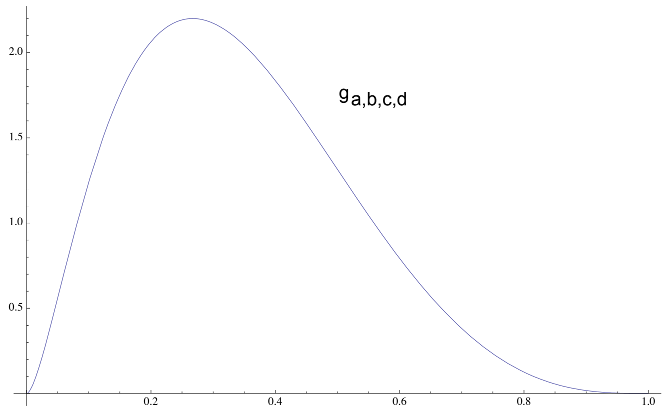

Figure 1

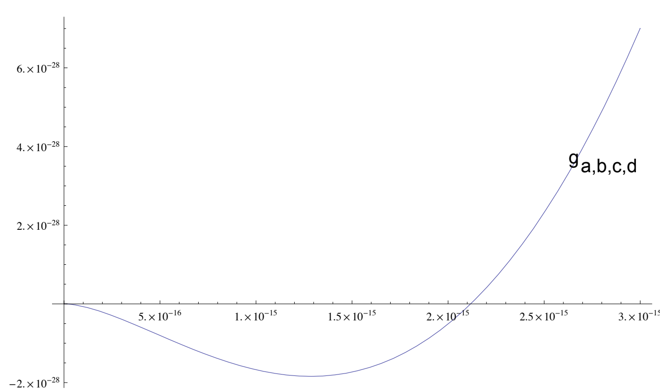

Figure 2