Consistent Pricing for Equity-Linked Products

Patrice Gaillardetz

X. Sheldon Lin

University of Toronto
e-mail: patrice@utstat.utoronto.ca

ARC Iowa, 2004

I. Objectives and Outline

- **Objectives**:
  - Develop pricing methods for equity-linked products.
  - Reproduce the current market prices of standard insurances and annuities.
  - If insurance products were tradable, their prices would not admit arbitrage (Harrison and Pliska (1981)).

- **Applications** of our model:
  - Variable annuities
  - Segregated funds
  - Unit-linked insurances
  - Universal life
  - Equity-indexed annuities

- **Outline** of the talk:
  - II. Binomial financial and insurance models
  - III. Martingale probability measures:
    - (a) for insurances and annuities
    - (b) for endowment insurances
    - (c) for equity-linked products
  - IV. Equity-indexed annuity valuation
  - V. Numerical examples
  - VI. Concluding remarks
  - VII. References

II. Binomial Financial and Insurance Models

N : number of trading dates per year ($\Delta = 1/N$)

Assume that the short-term rate is deterministic: $r(0), r(1), r(2), \ldots$ are known at time 0 (can be extended to stochastic interest rates).

Let $S(k)$ be the "index level" described by a modified Cox, Ross and Rubinstein (1979) model, where $k = 0, \Delta, 2\Delta, \ldots$.

$S(k)$ can either increase to $S(k)u(t^* + 1)$ or decrease to $S(k)d(t^* + 1)$ at time $k + \Delta$, where $t^* = \lfloor k \rfloor$ and $u(t)$ and $d(t)$ $(t = 1, 2, \ldots)$ are known at time 0.

Under the martingale measure Q, define

$$\begin{aligned}\widetilde{\pi}(k) &= \widetilde{\Pr}[S(k+\Delta) = S(k)u(t^*+1) | S(k)] \\ &= \frac{(1 + r(t^*))^{\Delta} - d(t^*+1)}{u(t^*+1) - d(t^*+1)}.\end{aligned}$$

Under the CRR model $\frac{S(t+1)}{S(t)}$ can take one of the following $N + 1$ possible values

$$\gamma(t+1, i) = u(t+1)^i d(t+1)^{N-i},$$

with corresponding martingale probability

$$\widetilde{\Pr}\left[\frac{S(t+1)}{S(t)} = \gamma(t+1, i)\right] = \binom{N}{i} \widetilde{\pi}(t)^i (1 - \widetilde{\pi}(t))^{N-i},$$

for $i = 0, \ldots, N$.

$K(x)$: curtate-future-lifetime of (x).

$V^{(1)}(x, t, n)$: market price at time t of an n-year **term life insurance** issued to (x) at time 0.

$V^{(1)}(x, 0, n)$ are the **current market prices** (single premium) and are given exogenously $\forall n$.

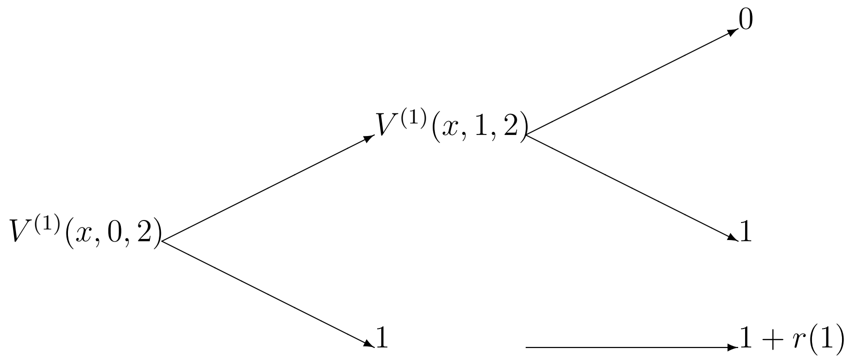
II. Binomial Financial and Insurance Models

Let $W^{(1)}(x,t,n)$ be the stochastic processes generated by the n-year **term life insurance**

$$W^{(1)}(x,t,n) = \begin{cases} \frac{B(t)}{B(K(x)+1)}, & K(x) < t \\ V^{(1)}(x,t,n), & K(x) \geq t \end{cases},$$

where $V^{(1)}(x,n,n) = 0$ and $B(t) = \prod_{i=0}^{t-1}(1+r(i))$, (the money market account).

Stochastic Evolution of $W^{(1)}(x,t,2)$



$V^{(2)}(x,t,n)$: market price at time t of an n-year **pure endowment insurance** issued to (x) at time 0.

$V^{(2)}(x,0,n)$ are the **current market prices** and are given exogenously $\forall n$.

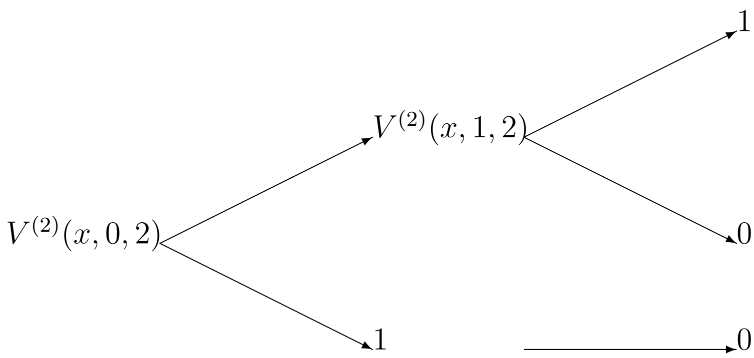
II. Binomial Financial and Insurance Models

Let $W^{(2)}(x,t,n)$ be the stochastic processes generated by the n-year **pure endowment insurance**

$$W^{(2)}(x,t,n) = \begin{cases} 0, & K(x) < t \\ V^{(2)}(x,t,n), & K(x) \geq t \end{cases},$$

where $V^{(2)}(x,n,n) = 1$.

Stochastic Evolution of $W^{(2)}(x,t,2)$



III.(a) Martingale Probability Measures for Insurances and Annuities

Method A:

- We suppose a separation of the insurance and the annuity markets.
- The price of any product containing death and accumulation benefits is then the sum of the two insurances priced separately.

Martingale measures $Q_x^{(j)}$ $(j = 1, 2)$ are any probability measure, equivalent to the insurance market measure, such that $\frac{W^{(j)}(x,t,n)}{B(t)}$ are martingales.

Let $\widetilde{q}_x^{(j)}(t)$ $(j = 1, 2)$ be the probability under $Q_x^{(j)}$ that (x) **dies** before age $x + t + 1$ given that the insured is alive at age $x + t$

$$\widetilde{q}_x^{(j)}(t) = \widetilde{\Pr}^{(j)} \left[K(x) = t | K(x) \geq t\right].$$

Define the probability (under $Q_x^{(j)}$) that (x) survives at least 1 year given that the insured is **alive** at age $x + t$ by

$$\widetilde{p}_x^{(j)}(t) = \widetilde{\Pr}^{(j)} \left[K(x) > t | K(x) \geq t\right] = 1 - \widetilde{q}_x^{(j)}(t).$$

The goal is to define $\widetilde{q}_x^{(j)}$'s and $\widetilde{p}_x^{(j)}$'s for $j = 1, 2$ using the martingale properties.

For given $V^{(j)}(x, 0, n)$ $(j = 1, 2$ and $n = 1, 2, ...)$ the age-dependent, mortality risk-adjusted probabilities are

$$\widetilde{q}_x^{(1)}(n-1) = \frac{\left(V^{(1)}(x,0,n) - V^{(1)}(x,0,n-1)\right) \prod_{i=0}^{n-1}(1 + r(i))}{\prod_{i=0}^{n-2} \widetilde{p}_x^{(1)}(i)},$$

and

$$\widetilde{p}_x^{(2)}(n) = \frac{V^{(2)}(x,0,n+1)}{V^{(2)}(x,0,n)} \left(1 + r(n)\right).$$

III.(b) Martingale Probability Measures for Endowment Insurances

Method B:

- Any product containing death and accumulation benefits is now priced by unifying the underlying contingent claims.

$V^{(3)}(x,t,n)$: market price at time t of an n-year **endowment insurance** issued to (x) at time 0.

$V^{(3)}(x,0,n)$ are the **current market prices** and are given exogenously $\forall n$.

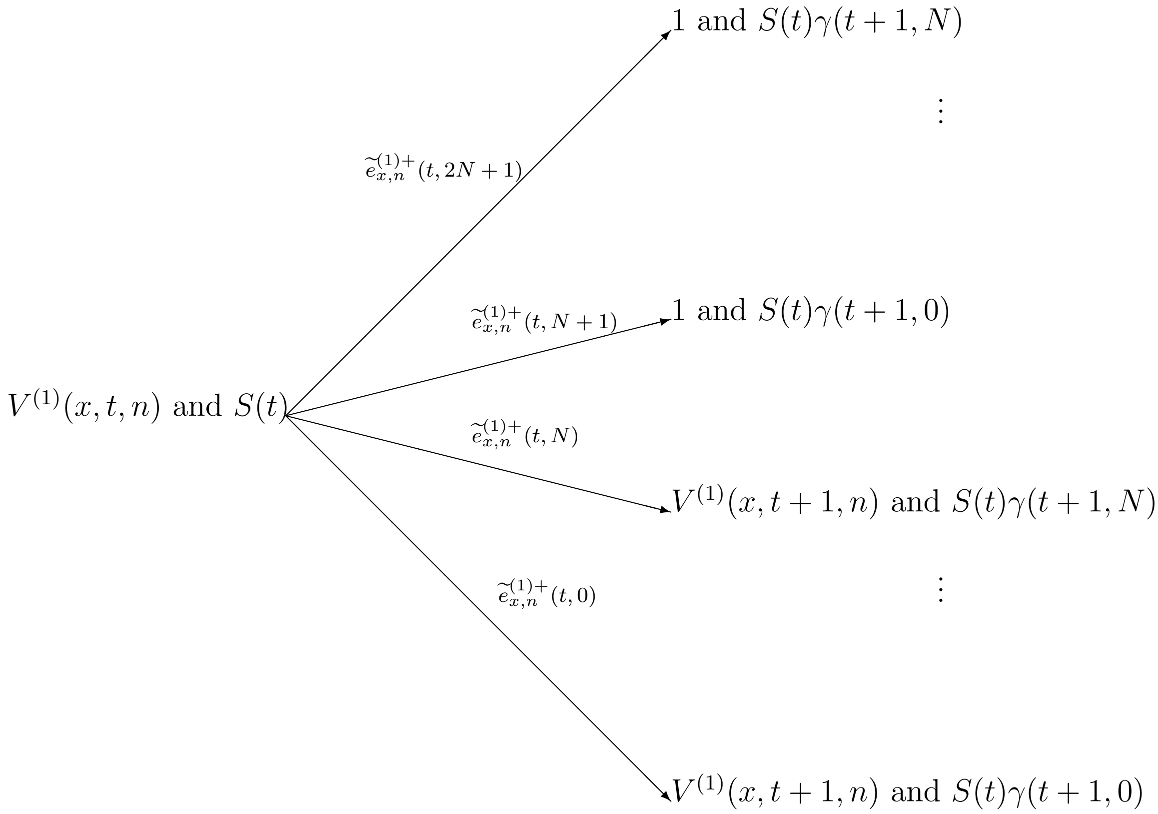
For given $V^{(3)}(x,0,n)$ $(n=1,2,...)$ the age-dependent, mortality risk-adjusted probabilities are

$$\prod_{i=0}^{n-1} \tilde{p}_x^{(3)}(i) = \frac{V^{(3)}(x,0,n+1) - V^{(3)}(x,0,n)}{\prod_{i=0}^{n}(1+r(i))^{-1} - \prod_{i=0}^{n-1}(1+r(i))^{-1}}.$$

III.(c) Martingale Probability Measures for Equity-Linked Products

- Equity-linked products provide death and accumulation benefits.
- Those benefits are based on the performance of the underlying index.
- The equity-linked products market is composed by the financial and the insurance markets.
- The goal is to determine the martingale measure for the combined market.
- The combined martingale measure reproduces the insurance and the financial martingale measures.

Stochastic Evolution of the Combined Insurance and Financial Markets Between t and $t+1$



For a fixed term n, define

$$\widetilde{e}_{x,n}^{(1)+}(t,i) = \widetilde{\Pr}^{(1)+}\left[\frac{S(t+1)}{S(t)} = \gamma(t+1,i), \right.$$
$$\left. W^{(1)}(x,t+1,n) = V^{(1)}(x,t+1,n) | S(t), K(x) \geq t\right], i = 0, ..., N$$

and

$$\widetilde{e}_{x,n}^{(1)+}(t,i) = \widetilde{\Pr}^{(1)+}\left[\frac{S(t+1)}{S(t)} = \gamma(t+1,i-N-1), \right.$$
$$\left. W^{(1)}(x,t+1,n) = 1 | S(t), K(x) \geq t\right], i = N+1, ..., 2N+1$$

where $\widetilde{\Pr}^{(1)+}[.]$ represents the probability under $Q_{x,n}^{(1)+}$.

III.(c) Martingale Probability Measures for Equity-Linked Products

Martingale measures $Q^{(1)+}_{x,n}$ are any probability measure, equivalent to the combined market physical measure, such that $\frac{W^{(1)}(x,t,n)}{B(t)}$ and $\frac{S(t)}{B(t)}$ are martingales.

Define the joint c.d.f. of $S(t+1)$ and $W^{(1)}(x,t+1,n)$ between t and $t+1$ by

$$G^{(1)+}(y_1,y_2) \;=\; \widetilde{\Pr}^{(1)+}\left[S(t+1)\leq y_1, W^{(1)}(x,t+1,n)\leq y_2 | S(t), K(x)\geq t\right].$$

From the martingale property of $\frac{W^{(1)}(x,t,n)}{B(t)}$

$$G^{(1)+}(\infty,y_2) \;=\; \widetilde{\Pr}^{(1)}\left[W^{(1)}(x,t+1,n)\leq y_2 | K(x)\geq t\right].$$

From the martingale property of $\frac{S(t)}{B(t)}$

$$G^{(1)+}(y_1,\infty) \;=\; \widetilde{\Pr}\left[S(t+1)\leq y_1 | S(t)\right].$$

Based on the **copulas approach**, the joint c.d.f. is defined by

$$G^{(1)+}(y_1,y_2) \;=\; C\left(G^{(1)+}(y_1,\infty), G^{(1)+}(\infty,y_2); \kappa(t)\right),$$

where $C:[0,1]^2\rightarrow[0,1]$ and $\kappa(t)$ is the copula's free parameter indicating the level of dependence between the index and the insured's life.

The following three non-parametric copulas are used to determine the dependence relation between the index and the insured's life.

Independent copula: $G^{(1)+}(y_1,y_2) = G^{(1)+}(y_1,\infty)G^{(1)+}(\infty,y_2)$

Upper Frechet bound: $G^{(1)+}(y_1,y_2) = \min\left(G^{(1)+}(y_1,\infty), G^{(1)+}(\infty,y_2)\right)$

Lower Frechet bound: $G^{(1)+}(y_1,y_2) = \max\left(G^{(1)+}(y_1,\infty) + G^{(1)+}(\infty,y_2) - 1, 0\right)$

Using Cossette, Gaillardetz, Marceau and Rioux (2002) the $\tilde{e}^{(1)+}_{x,n}$'s can be extracted from $G^{(1)+}$.

IV. Equity-Indexed Annuity Valuation

The payoff of the **Total Return** EIA class can be expressed by

$$D(t) = \max\left[1 + \alpha R(t), \beta(1+g)^t\right].$$

α: participation level.
β and g: minimum guarantee.

Point-to-Point: $R(t) = \frac{S(t)}{S(0)} - 1$

High-Water Mark: $R(t) = \max_{k\in\{0,\Delta,2\Delta,\ldots,t\}} \frac{S(k)}{S(0)} - 1$

$C_{x,n}$: price of the EIA.

Method A: $C_{x,n} = C_{x,n}^{(1)} + C_{x,n}^{(2)}$

Method B: $C_{x,n} = C_{x,n}^{(3)}$

Leading to

$$C_{x,n}^{(1)} = \widetilde{E}^{(1)+}\left[\sum_{k=1}^{n} \frac{D(k)I(K(x)=k-1)}{B(k)}\right],\ C_{x,n}^{(2)} = \widetilde{E}^{(2)+}\left[\frac{D(n)I(K(x)\geq n)}{B(n)}\right],$$

and

$$C_{x,n}^{(3)} = \widetilde{E}^{(3)+}\left[\sum_{k=1}^{n-1} \frac{D(k)I(K(x)=k-1)}{B(k)} + \frac{D(n)I(K(x)\geq n-1)}{B(n)}\right],$$

where $\widetilde{E}^{(j)+}[.]$ represents the expected value under $Q_{x,n}^{(j)+}$.

V. Numerical Examples

For a life age (55) and term of 5 years, the following data are observed (Bowers et al. (1997)):

t	$r(t-1)$	q_{55+t-1}	$V^{(1)}(55,0,t)$	$V^{(2)}(55,0,t)$	$V^{(3)}(55,0,t)$
1	4.50%	8.9605	13.1464	957.4531	961.5385
2	5.00%	9.7538	23.8093	904.8839	916.3769
3	5.50%	10.6230	34.1888	850.0821	869.8067
4	5.75%	11.5257	44.4799	795.8125	824.3219
5	6.00%	12.6181	54.7425	742.4034	780.1222

Probabilities and prices are multiplied by 1,000.

For illustration purposes, assume that the market prices are determined using the standard deviation premium principle with a factor of 5.00%.

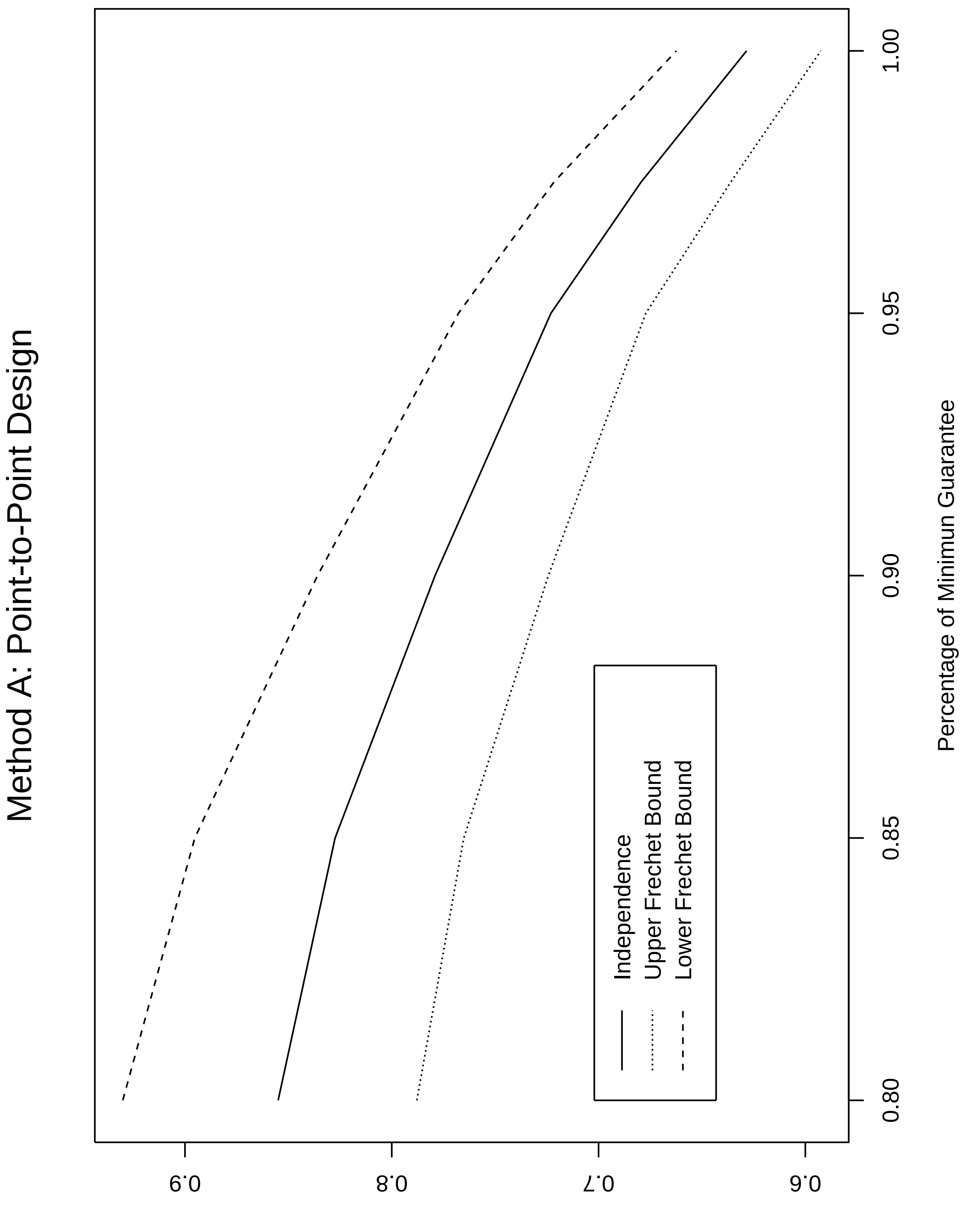
For the financial model, assume $N = 3$ $(\Delta = 1/3)$, $u(t) = e^{0.15\sqrt{\Delta}} = 1.0905$ and $d(t) = e^{-0.1\sqrt{\Delta}} = 0.9439$ for $t = 1, ..., 5$.

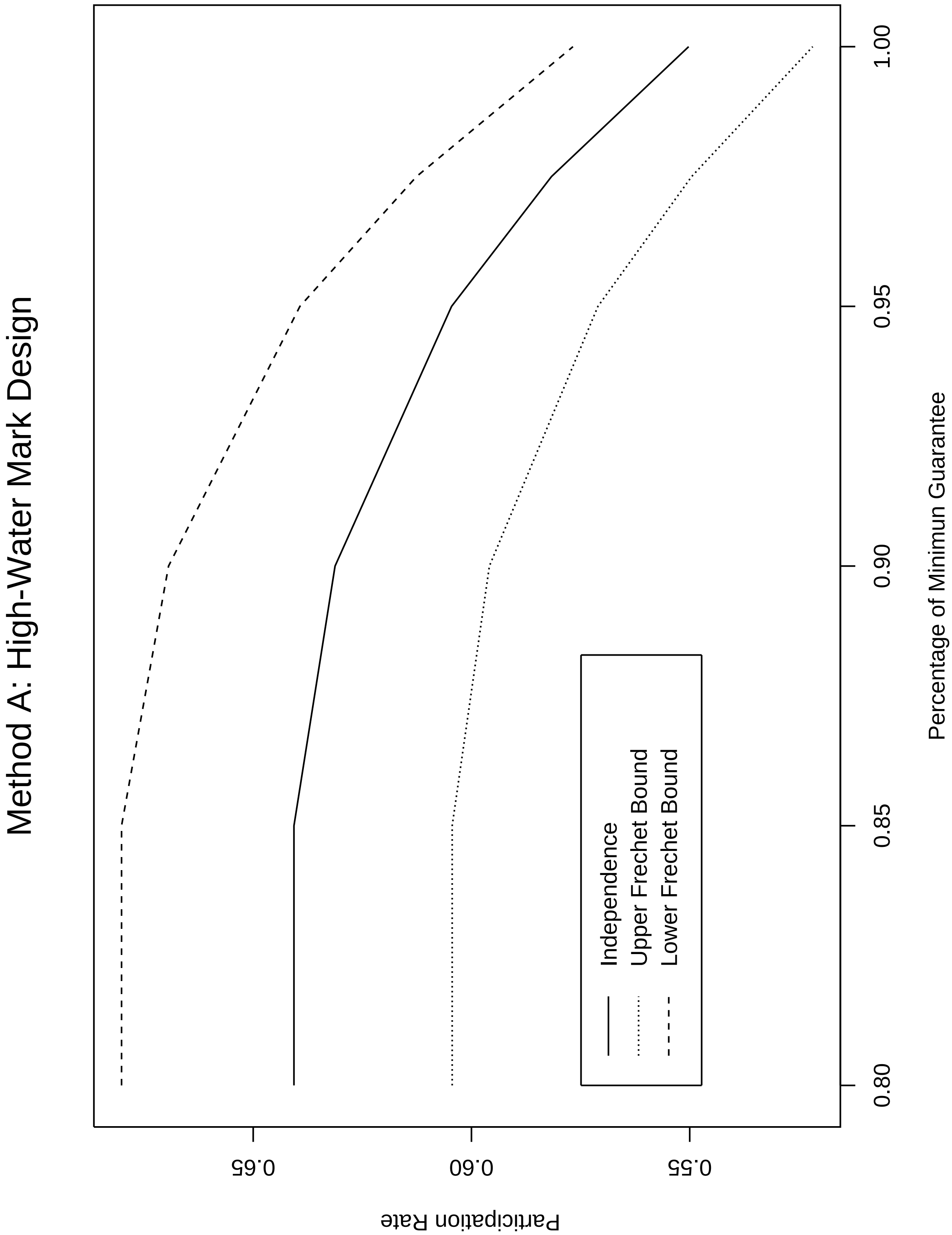
The martingale probabilities are:

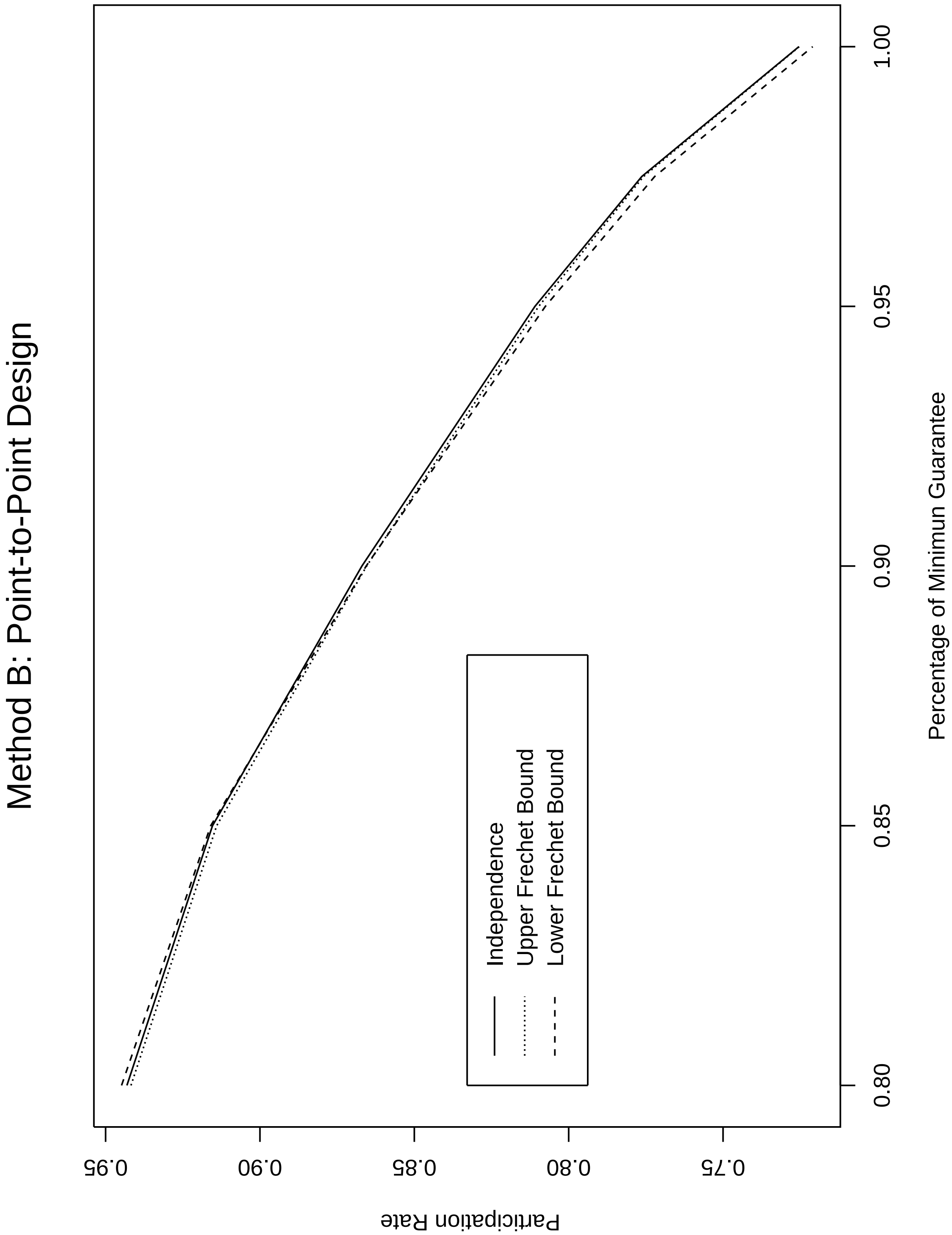
t	$\widetilde{q}^{(1)}_{55}(t)$	Loading	$\widetilde{q}^{(2)}_{55}(t)$	Loading	$\widetilde{q}^{(3)}_{55}(t)$	Loading	$\widetilde{\pi}(t)$
0	13.6722	4.7117	4.2488	-4.7117	13.6722	4.7117	472.6
1	11.8053	2.0515	7.6505	-2.1033	10.9922	1.2384	494.6
2	12.2683	1.6453	8.8932	-1.7298	12.0553	1.4322	505.6
3	13.0232	1.4480	10.0113	-1.5639	12.8658	1.2906	511.1
4	13.9480	1.3298	11.1394	-1.4787	?	?	516.6

Probabilities and loadings are multiplied by 1,000.

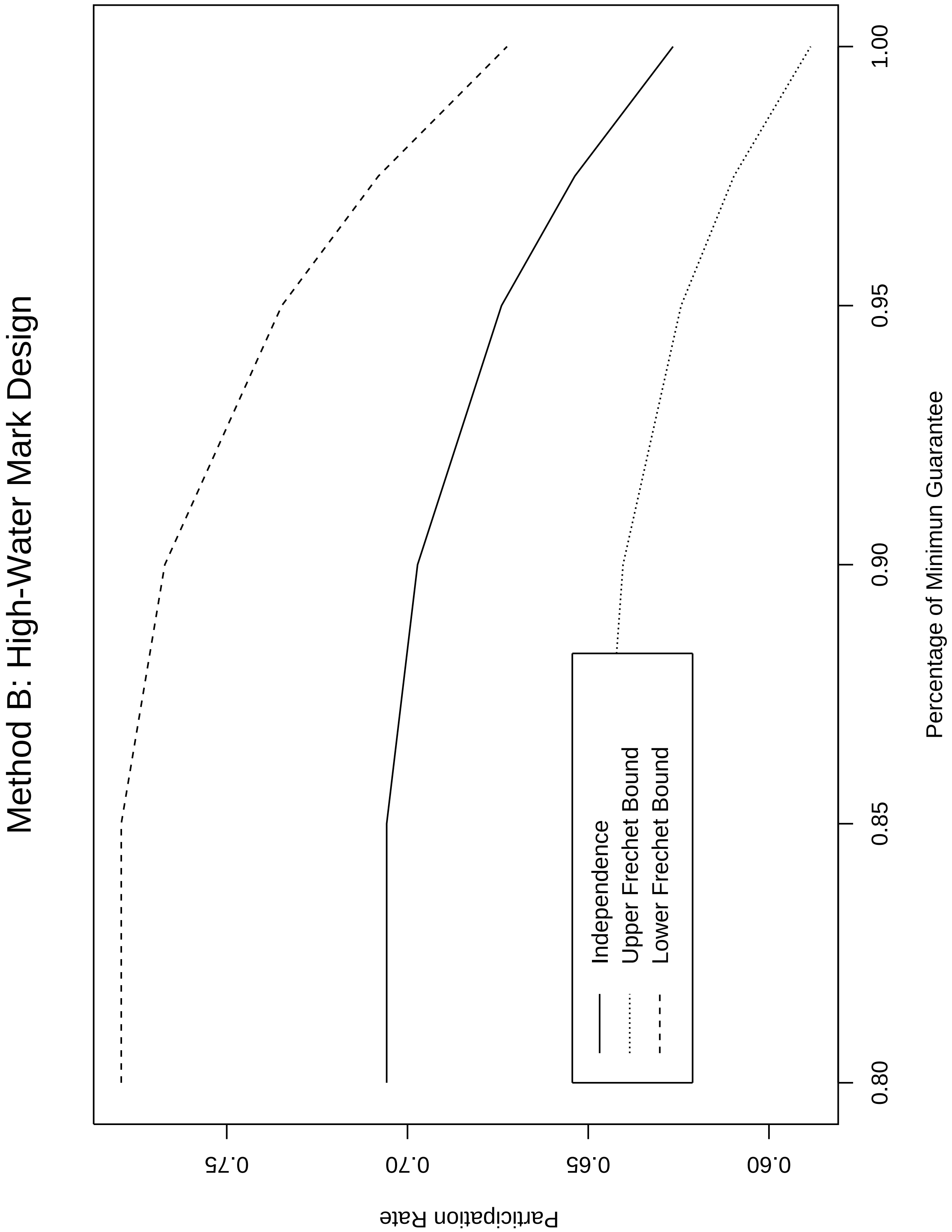
Determine α by numerical methods after setting $g = 3.00\%$ and β.


Method A: Point-to-Point Design
Independence
Upper Frechet Bound
Lower Frechet Bound
0.80
0.85
0.90
0.95
1.00
Percentage of Minimun Guarantee
0.6
0.7
0.8
0.9
Participation Rate


Method A: High-Water Mark Design
Independence
Upper Frechet Bound
Lower Frechet Bound
0.80
0.85
0.90
0.95
1.00
Percentage of Minimun Guarantee
0.55
0.60
0.65
Participation Rate



Method B: Point-to-Point Design
Independence
Upper Frechet Bound
Lower Frechet Bound
Participation Rate
0.95
0.90
0.85
0.80
0.75
Percentage of Minimun Guarantee
0.80
0.85
0.90
0.95
1.00


Method B: High-Water Mark Design
Independence
Upper Frechet Bound
Lower Frechet Bound
Percentage of Minimun Guarantee
0.80
0.85
0.90
0.95
1.00
Participation Rate
0.60
0.65
0.70
0.75

VI. Concluding Remarks

- We derived an age-dependent, mortality risk-adjusted martingale probability measure for each market.
- We combined the information from the insurance and the financial markets and derived martingale measures.
- We introduced two pricing methods for equity-linked products:
  Method A: split the benefits and use the insurance and annuity markets;
  Method B: unify the contingent claims and use the endowment market.
- Difficulty to find current market prices.
- Use other parametric copulas (free-parameters can fit the equity-linked product prices).
- Extend to surrender charges and stochastic interest rates.

VII. References

- Bacinello, A. R. (2003). "Fair Valuation of a Guaranteed Life Insurance Participating Contract Embedding a Surrender Option". *The Journal of Risk and Insurance*, 70, 461-487.
- Black, F., Derman, E. and Toy, W. (1990). "A One-Factor Model of Interest Rates and its Application to treasury Bond Options". *Financial Analysts Journal*, 46, 33-39.
- Cossette, H., Gaillardetz, P., Marceau, E. and Rioux, J. (2002). "On Two Dependent Individual Risk Models". *Insurance: Mathematics and Economics*, 30, 153-166.
- Frees E.W. and Valdez E.A. (1998). "Understanding Relationships Using Copulas ". *North American Actuarial Journal*, 2, 1-25.
- Jarrow, R. and Turnbull, S. (1995). "Pricing Options on Financial Securities Subject to Default Risk". *Journal of Finance*, 50, 53-86.
- Lin, S.X. and Tan, K.S. (2003). "Valuation of Equity-Indexed Annuities under Stochastic Interest Rate". *North American Actuarial Journal*, 7, 72-91.
- Moller, T. (1998). "Risk-Minimizing Hedging Strategies for Unit-Linked Life Insurance Contracts". *Astin Bulletin*, 28, 17-47.
- Wang, S. (1998). "Aggregation of Correlated Risk Portfolios: Models and Algorithms". 1998 *CAS Proceedings*, 848-939.