The $1/n$ pension investment puzzle

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Heath Windcliff and Phelim Boyle[1] , University of Waterloo

Abstract

This paper examines the so called $1/n$ investment puzzle which has been observed in defined contribution plans whereby some participants divide their contributions equally among the available asset classes. It has been argued that this is a very naive strategy since it contradicts the fundamental tenets of modern portfolio theory. We use simple arguments to show that this behavior is perhaps less naive than it at first appears. It is well known that the optimal portfolio weights in a mean-variance setting are extremely sensitive to estimation errors, especially those in the expected returns. We show that when we account for estimation error the $1/n$ rule has some advantages in terms of robustness and we demonstrate this with numerical experiments.

1 Introduction

There is evidence (see Benartzi and Thaler [1]) that many participants in defined contribution plans use simple heuristic diversification rules in allocating their contributions among the available asset classes. One popular diversification heuristic is often referred to as the $1/n$ rule. Under this rule the investor divides her holdings equally among the available assets and we refer to this portfolio as an *equally-weighted portfolio*. This strategy has drawn some criticism since it is not an optimal portfolio, in the sense that, in general, it does not lie on the efficient frontier. In fact, some researchers have suggested that pension plans should offer less flexibility in order to avert some of the poor investment decisions made by uninformed individuals.

In this paper we demonstrate that the $1/n$ portfolio is consistent with the Markowitz efficient portfolios, given a limited set of information. We then argue that, even with all of the available historical information available to investment professionals, in light of the parameter estimation risk the performance of the $1/n$ heuristic can be quite satisfactory, assuming an appropriate set of investment choices in the plan. As a result, we feel that it is difficult to justify limiting the flexibility

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of these plans based on the argument that people using the $1/n$ heuristic are not optimally allocating their pension contributions.

There is an extensive literature on the significance of estimation risk in the context of the portfolio optimization problem. Over twenty years ago Jobson and Korkie [12, 13] in a series of papers analyzed the problem and proposed some remedies. Since then there have been several contributions to this topic. In the present paper we use methods that have been described in the finance literature. We emphasize that we make no claims that the techniques used in this paper are original and almost all the ideas presented here can be found in the literature. However we feel that the current paper serves a useful purpose. First, we present a short self contained treatment which should be of interest to an actuarial audience unfamiliar with the finance literature. Second, we provide an alternative[2] explanation of the $1/n$ puzzle in the context of the asset allocation decision in a defined contribution plan. Third, the $1/n$ rule is sometimes viewed as naive and we suggest that this judgement may be unduly harsh.

We begin with a brief overview of the classical mean-variance portfolio theory. To start with, we assume that the true expected return vector and the variance-covariance matrix are known. Under these assumptions we provide an example illustrating that the equally-weighted portfolio underperforms the set of optimal portfolios generated by mean-variance optimization. If the investor makes particularly simple specifications of the expected returns and covariances then we show in section 4 that the equally-weighted portfolio is optimal in a mean-variance sense. This leads us to question how naive this simple specification is for the input parameters of the mean-variance analysis. We find that when we take into account the estimation risk when calibrating these parameters to historical data, this simple specification of the mean-variance input parameters can be viewed as being quite reasonable.

2 Mean-Variance Portfolio Theory

The portfolio optimization methods introduced by Markowitz [15] in the 1950s, termed *mean-variance* optimization methods, signified the birth of modern quantitative finance. The key insight provided by this approach is that an investor can use information describing the relationships between assets to construct a portfolio with better risk-return characteristics than if she considers the assets individually. As the name implies the information that mean-variance portfolio theory uses to describe the relationships between the available assets are the expected returns of the assets, their variances and correlations. We now summarize[3] the basic approach

Given a market of N available assets, we define a portfolio to be a vector:

$$\mathbf{x} = (x_1, x_2, \ldots, x_N)^T \qquad \text{subject to:} \quad \mathbf{e}^T\mathbf{x} = 1, \tag{1}$$

where x_i represents the fraction of wealth held in asset i and where $\mathbf{e} = (1, 1, \ldots, 1)^T$ is the N by one vector consisting of N ones. The constraint $\mathbf{e}^T\mathbf{x} = 1$ represents the budget constraint.

Given a risk tolerance[4] level, $\tau \in [0, \infty)$, the optimal portfolio can be found by solving the

[2]Benartzi and Thaler [1] in their influential paper discuss behavioral explanations of this puzzle.
[3]See Panjer et al [17] for more details.
[4]See Panjer et al [17] for a definition and interpretation of risk tolerance.

parametric quadratic program :

$$\max_{\mathbf{x}} \left(2\tau \boldsymbol{\mu}^T \mathbf{x} - \mathbf{x}^T \boldsymbol{\Sigma} \mathbf{x} \right) \tag{2}$$

$$\text{subject to: } \mathbf{e}^T \mathbf{x} = 1, \tag{3}$$

where:

$$\boldsymbol{\mu} = (\mu_1, \mu_2, \ldots, \mu_N)^T \ , \tag{4}$$

and μ_i is the expected return on asset i, and:

$$\boldsymbol{\Sigma} = \left[\rho_{ij} \sigma_i \sigma_j \right] , \qquad i, j = 1, \ldots, N \ , \tag{5}$$

where ρ_{ij} is the correlation between assets i and j, and σ_i is the volatility of asset i. We assume that the matrix $\boldsymbol{\Sigma}$ is positive definite.

In this paper we will assume that all of the assets are risky. First we consider the case where there are no restrictions on the portfolio weights. In other words short selling is permitted. In this case we can solve (2)-(3) analytically [17] to find the optimal portfolio:

$$\mathbf{x}^*(\tau) = \mathbf{x}_{min} + \tau \Delta \mathbf{x}_{risk} \ , \tag{6}$$

where:

$$\mathbf{x}_{min} = \frac{1}{\mathbf{e}^T \boldsymbol{\Sigma}^{-1} \mathbf{e}} \boldsymbol{\Sigma}^{-1} \mathbf{e} \ , \tag{7}$$

$$\Delta \mathbf{x}_{risk} = \boldsymbol{\Sigma}^{-1} \boldsymbol{\mu} - \frac{\mathbf{e}^T \boldsymbol{\Sigma}^{-1} \boldsymbol{\mu}}{\mathbf{e}^T \boldsymbol{\Sigma}^{-1} \mathbf{e}} \ \boldsymbol{\Sigma}^{-1} \mathbf{e} \ . \tag{8}$$

The portfolio $\mathbf{x}_{min}$ represents the minimum-variance portfolio in this market while $\Delta \mathbf{x}_{risk}$ represents a self-financing[5] portfolio adjustment that optimally trades off risk versus reward in this market. Notice that $\mathbf{x}_{min}$ does not depend on the expected returns, while $\Delta \mathbf{x}_{risk}$ depends on both the expected returns and on the covariances of the assets.

The expected return on the optimal portfolio is given by:

$$\mu[\tau] = \boldsymbol{\mu}^T \mathbf{x}^*(\tau) \ , \tag{9}$$

and the variance of the return on the optimal portfolio is given by:

$$\left(\sigma[\tau] \right)^2 = \left(\mathbf{x}^*(\tau) \right)^T \ \boldsymbol{\Sigma} \ \mathbf{x}^*(\tau) \ , \tag{10}$$

where the optimal weights are obtained from (6). As the risk tolerance parameter τ varies the points $(\sigma[\tau]^2, \ \mu[\tau])$ trace out the top half of a parabola in mean-variance space. This provides a nice geometric interpretation of the optimal portfolios and the top half of the parabola is known as the *efficient frontier*. An investor who cares only about expected return and variance will want to hold a portfolio that is on this efficient frontier. The precise location will depend on how she trades off expected return and variance, and this will be determined by her risk tolerance, τ.

Since we will be analyzing the investment decisions of employees in defined contribution pension plans where short sales are not permitted we impose the condition:

$$\mathbf{x} \geq \mathbf{0} \ . \tag{11}$$

[5] Self financing means that the weights sum to zero. It is readily checked that $\mathbf{e}^T \Delta \mathbf{x}_{risk} = 0$.

In this case we do not obtain an analytical solution for the optimal portfolio weights. However we can solve (2)-(3) using numerical quadratic programming methods. Details are given by Best and Grauer [3]. When the asset weights are restricted to be non negative we rule out some of the solutions which were feasible for the unrestricted case. Thus the efficient portfolio in the case when there is no short selling lies inside[6] (to the south-east of) the efficient portfolio for the case when there is short selling.

In practice the decision maker will not know the true values of the expected return vector $\boldsymbol{\mu}$ and the variance-covariance matrix $\boldsymbol{\Sigma}$. Markowitz [15] suggested that $\boldsymbol{\mu}$ and $\boldsymbol{\Sigma}$ should be estimated using forward looking projections, but usually these parameters are estimated from historical data. The historical estimates for $\boldsymbol{\Sigma}$ tend to be somewhat robust, but that the estimates for $\boldsymbol{\mu}$ are very noisy, even in a stationary market as illustrated by Broadie [9]. This estimation risk has important consequences for the optimal portfolio selection problem. Black and Litterman [6] suggested using reverse optimization and implied expected returns in lieu of the historically estimated returns.

3 Mean-Variance Efficient Portfolios for a Five Asset Universe

We use a five asset universe to illustrate some of these ideas. We assume that an investor wishes to select an efficient portfolio based on five different asset classes. This mirrors that of a hypothetical pension plan participant where the plan sponsor allows participants to allocate their holdings between five asset classes. To be specific we assume that the returns on these asset classes correspond to portfolios that track the following indices:

- The S&P 500 Large Cap (Large Cap),
- The S&P Mid Cap 400 (Mid Cap),
- The Russell 2000 Small Cap (Small Cap),
- The Morgan Stanley World Equity excluding US (World Index),
- The Lehman Brothers long-term government bond index (Long Bond Index).

Holden's [11] descriptive study of pension plan holdings indicates that these asset choices are broadly representative of those offered to many pension plan participants within their 401k plan.

We assume that the annualized expected returns, volatilities and correlations of the five assets in the pension plan are given in Table 1. These descriptive statistics were estimated using 15 years of historical data from February 1981 through September 1997. We then adjusted the expected returns of each asset downwards by 5% per annum to account for current market conditions. Since this study is only meant to be representative of the conditions that pension plan participants face, we assume that the market parameters given in these tables describe the true distribution of future returns. In other words the expected returns, variances and covariances reflected in Table 1 are assumed to *correspond to the true population parameters* and thus there is no estimation risk. In section 5 we will investigate the impact of parameter estimation risk on the mean-variance optimal portfolios.

[6] The efficient portfolio with positive weights may sometimes coincide with the efficient portfolio when short selling is permitted over a certain segment. Best and Grauer [5] analyze this .

Asset Name	μ	σ			ρ		
Large Cap	.1165	.1449	1.0000	0.9180	0.8341	0.4822	0.3873
Mid Cap	.1288	.1549	0.9180	1.0000	0.9386	0.4562	0.3319
Small Cap	.0968	.1792	0.8341	0.9386	1.0000	0.4206	0.2002
World Index	.0921	.1715	0.4822	0.4562	0.4206	1.0000	0.2278
Long Bond Index	.0547	.0558	0.3873	0.3319	0.2002	0.2278	1.0000

TABLE 1: *The expected returns, volatilities and correlations for the distribution of returns of the assets available in the pension plan described in this section.*

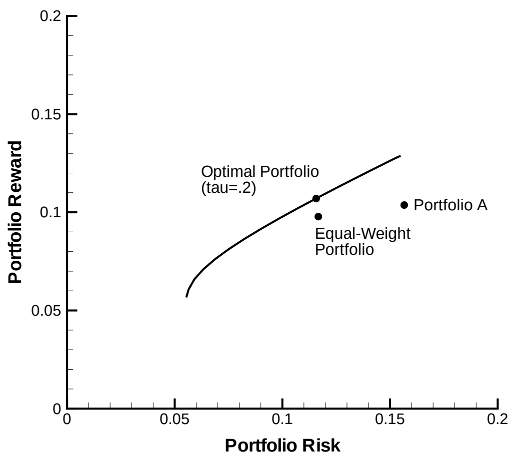


FIGURE 1: *The risk-reward tradeoff, measured in units of standard deviations of portfolio returns and expected returns respectively, between portfolios on the efficient frontier and the equally-weighted portfolio. The descriptive statistics of the assets available in the pension plan are given in Table 1. The point labeled as Portfolio A will be described in a later section.*

The efficient frontier for portfolios comprised of these assets, assuming that short selling is not permitted, is plotted in Figure 1. This figure shows the optimal tradeoff between reward, measured in units of expected portfolio return, and risk, measured in units of standard deviations. Note that the lowest point on extreme south west corner of the efficient frontier corresponds to the case when the portfolio is entirely invested in the Long Bond so that the expected return is 5.47%, and the standard deviation is 5.58%. On the other hand the highest point on the extreme north east corner of the efficient frontier corresponds to the case when the portfolio is entirely invested in the Mid Cap fund so that the expected return is 12.88%, and the standard deviation is 15.49%. In this figure we also show the risk-reward tradeoff for the equally-weighted portfolio. Since the risk-reward tradeoff for the equally-weighted portfolio lies below the efficient frontier it is not optimal in the sense that there exist preferred portfolios that simultaneously have greater expected return and less risk; for example the frontier portfolio with $\tau = .2$ that is labeled in Figure 1.

4 When is the Equally-Weighted Portfolio Optimal?

Since there is empirical evidence that some investors use the equally weighted portfolio, it is of interest to find a set of simple assumptions that justify the equally-weighted portfolio. The motivation here is find the assumptions that would lead to investors selecting the equally weighted portfolio. We show that the equally-weighted portfolio is optimal in a very simple market where the assets are indistinguishable and uncorrelated.

Consider the simple market described above where the asset returns are indistinguishable and uncorrelated. Using a zero subscript to denote this special market we can write the expected return vector and variance-covariance matrix as:

$$\boldsymbol{\mu}_0 = \bar{\mu}\mathbf{e} \ , \tag{12}$$

$$\boldsymbol{\Sigma}_0 = \bar{\sigma}^2\mathbf{I} \ , \tag{13}$$

where $\bar{\mu}$ and $\bar{\sigma}$ are the representative levels of the expected returns and asset volatilities respectively. For any choice of $\bar{\mu}$, and for any non-zero choice[7] of $\bar{\sigma}$, the optimal portfolios, obtained using equations (7)-(8), are given by:

$$\mathbf{x}_{min,0} = \frac{1}{N}\,\mathbf{e} \ , \tag{14}$$

$$\Delta\mathbf{x}_{risk,0} = \mathbf{0} \ . \tag{15}$$

In other words, investors who share this particularly simple view of the market should select the equally-weighted portfolio for any risk-tolerance level, τ. It is also easy to see that this portfolio is also optimal when short selling is not allowed.

It is well known that the this simple description of the market is not accurate. Chan et al [10] find that for a sample of 500 US stocks the average correlation is 28%. The lowest correlation was minus 37% and the highest 92%. Chan et al also demonstrate that correlation between two stocks is higher when they belong to the same industry than when they belong to different industries. Hence the simple distributional assumptions that lead to the equally weighted portfolio being optimal are not supported by the empirical evidence.

We also saw in section 3 that the equally-weighted portfolio does not lie on the efficient frontier, and hence is suboptimal. However this analysis assumed that the true population parameters are known. In the following section we look at the impact of parameter estimation on the mean-variance optimal portfolios.

5 An Introduction to Parameter Estimation Risk

The efficient frontier computed in section 3 assumed that we knew the both the expected return vector, $\boldsymbol{\mu}$ and the variance-covariance matrix, $\boldsymbol{\Sigma}$. Although Markowitz in his original paper suggested using a discount dividend model to specify the expected returns, many practitioners estimate the moments of the future asset return distributions using historical data. A number of papers [16, 9, 4, 8, 2] have explored the impact of parameter estimation risk on mean-variance optimal portfolios.

[7] This is consistent with our assumption that all of the assets available in the pension plan are risky.

We denote the mean-variance optimal portfolio based on the true population parameters $\boldsymbol{\mu}$ and $\boldsymbol{\Sigma}$ by $\mathbf{x_{true}}(\tau) = \mathbf{x_{true}}$. We drop the τ just to simplify the notation, with the understanding that $\mathbf{x_{true}}$ is still a function of τ. Suppose that the estimates of the expected return and variance-covariance matrix based on sample of M historical observations are denoted by $\hat{\boldsymbol{\mu}}$ and $\hat{\boldsymbol{\Sigma}}$. Denote the mean-variance optimal portfolio based on these estimated parameters by $\mathbf{x_{est}}(\tau) = \mathbf{x_{est}}$. Following Jobson and Korkie [13] and Broadie [9], we can distinguish three frontiers:

- *The true frontier.* In this case the expected return and variance are computed as follows:

$$\mathbf{x_{true}}^{\mathbf{T}}\, \boldsymbol{\mu}, \qquad \mathbf{x_{true}}^{\mathbf{T}}\, \boldsymbol{\Sigma}\, \mathbf{x_{true}}\ . \tag{16}$$

  The true frontier is unattainable since in practice we do not know the true parameters, and hence cannot realistically compute $\mathbf{x_{true}}$.

- *The estimated frontier.* In this case the expected return and variance are computed as follows:

$$\mathbf{x_{est}}^{\mathbf{T}}\, \hat{\boldsymbol{\mu}}, \qquad \mathbf{x_{est}}^{\mathbf{T}}\, \hat{\boldsymbol{\Sigma}}\, \mathbf{x_{est}}\ . \tag{17}$$

  The estimated frontier is based on the portfolio $\mathbf{x_{est}}$, which is derived using the estimated weights, and hence can be calculated. However it is does not provide a sensible comparison since in reality the future returns will be drawn from the true distribution and not the distribution given by the sample estimates, $\hat{\boldsymbol{\mu}}$ and $\hat{\boldsymbol{\Sigma}}$.

- *The actual frontier.* In this case the expected return and variance are computed as follows:

$$\mathbf{x_{est}}^{\mathbf{T}}\, \boldsymbol{\mu}, \qquad \mathbf{x_{est}}^{\mathbf{T}}\, \boldsymbol{\Sigma}\, \mathbf{x_{est}}\ . \tag{18}$$

  The actual frontier depicts the performance of the portfolios constructed using the sample estimates, $\mathbf{x_{est}}$, under the true distribution of returns, $\boldsymbol{\mu}$ and $\boldsymbol{\Sigma}$.

The relationships between the true, estimated and actual frontiers are illustrated in Figure 2. In summary the true portfolio is the unattainable ideal, the estimated portfolio is illusory, and the actual portfolio is the most realistic one for many comparison purposes.

The simplest way to illustrate this is using a numerical example. Consider the following example. Suppose the asset return distributions given in Table 1 follow a joint multivariate normal distribution. Also suppose that the parameters values given in Table 1 are the true population parameters. Hence the means, volatilities and correlations given in Table 1 are the population values. Of course an investor who is attempting to construct the mean-variance optimal portfolio is not able to directly observe the true descriptive statistics of the return distributions. Instead, an investor can observe a sample of returns generated by the assets. For example, suppose that the investor takes a sample of five years of monthly return data (60 observations), and uses the sample estimates as proxies to describe the future asset return distributions. Representative sample estimates for a typical numerical scenario are provided in Table 2.

Comparing the results provided in Table 2 with the true parameters from Table 1, we see that although the volatilities and correlations can be estimated with reasonable accuracy, the estimates of the expected returns are very poor. Specifically, we note that the sample returns seriously overestimate the mean for the Small Cap Index and underestimates the mean for the World Index. To see the effect that this will have on the portfolios generated using mean-variance optimization,

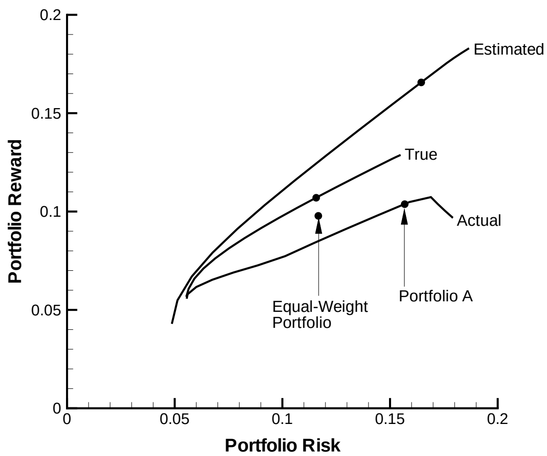


FIGURE 2: *Typical relationships between the true, estimated and actual frontiers. The points on the frontiers correspond to a risk tolerance level of $\tau = .2$. The estimates of $\boldsymbol{\mu}$ and $\boldsymbol{\Sigma}$ are given in Table 2.*

	$\hat{\mu}$	$\hat{\sigma}$			$\hat{\rho}$		
Large Cap	.0991	.1468	1.0000	0.9208	0.8266	0.4017	0.5018
Mid Cap	.1602	.1620	0.9208	1.0000	0.9413	0.4188	0.4641
Small Cap	.1830	.1867	0.8266	0.9413	1.0000	0.4386	0.2925
World Index	.0042	.1747	0.4017	0.4188	0.4386	1.0000	0.1863
Long Bond Index	.0440	.0489	0.5018	0.4641	0.2925	0.1863	1.0000

TABLE 2: *Estimates of the expected returns, volatilities and correlations for the asset return distributions obtained from one scenario using a sample of 60 monthly observations.*

Portfolio	True Market μ	σ	Asset Holdings Large Cap	Mid Cap	Small Cap	World Index	Bond Index
$\mathbf{x_{true}}$	.1070	.1156	.0000	.6796	.0000	.0529	.2675
$\mathbf{x_{est}}$	.1048	.1590	.0000	.3311	.6081	.0000	.0608
Equal-weight	.0978	.1167	.2000	.2000	.2000	.2000	.2000

TABLE 3: *The expected returns, volatilities and holdings of the true optimal portfolio for a risk tolerance of $\tau = .2$, the actual portfolio obtained by mean-variance optimization with $\tau = .2$ using the estimated parameters reported in Table 2, and the equally-weighted portfolio. The expected returns and volatilities given in the second and third columns are computed the true population parameters of the asset return distributions specified in Table 1.*

in Table 3 we compare the true optimal portfolio and the reportedly optimal portfolio constructed using the estimated parameters, both using a risk tolerance level of $\tau = .2$. We find that the portfolio constructed using the estimated parameters has over-weighted the Small Cap Index and under-weighted the World Index. This demonstrates the error maximizing property (discussed by Broadie [9]) of the mean-variance optimization procedure. Notice that assets whose sample returns have been optimistically biased are over-weighted in the recommended portfolio.

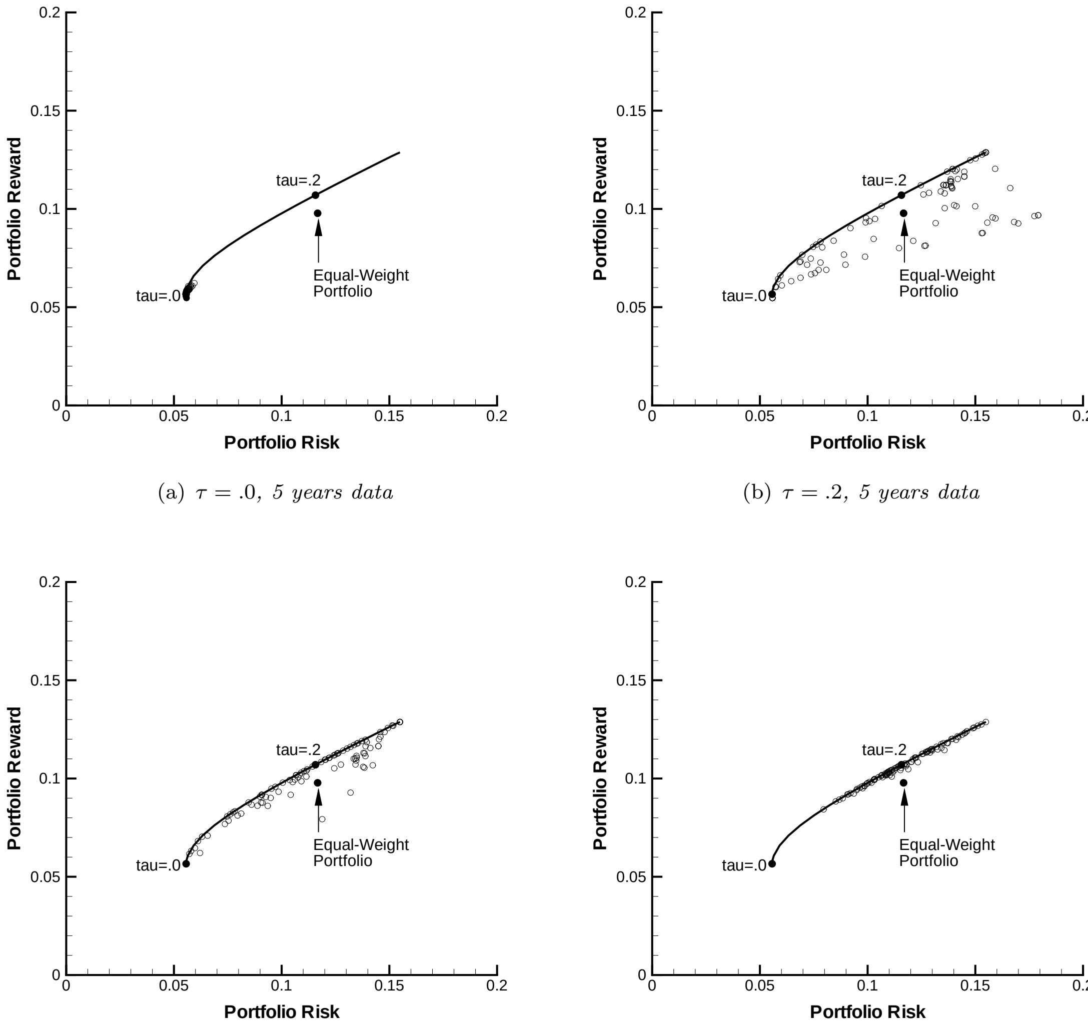
We say that a portfolio dominates another if it has both a higher return and a lower standard deviation. In Table 3 we see that if the investor knows the true market distributions, then the optimal portfolio for $\tau = .2$ dominates the equally-weighted portfolio. However, it is not clear whether the portfolio obtained from mean-variance optimization with $\tau = .2$ using the estimated parameters is better than, or worse than, the equally-weighted portfolio. This portfolio is labeled as Portfolio A in Figure 1 and Figure 2. We see that it offers a somewhat higher expected return but takes on substantially more risk than the equally weighted portfolio.

This numerical example illustrates some of the important issues raised by parameter estimation error. Although the implications of parameter estimation on optimal portfolio selection are discussed in the literature, we feel that these issues have not been addressed properly in some textbooks. In the next section we will explore the effect of parameter estimation error on the performance of the mean-variance optimized portfolios.

6 Parameter Estimation Error

In the preceding section we saw that portfolios generated using mean-variance optimization can lie quite far away from the efficient frontier when there is parameter estimation error. In this section we provide a statistical comparison of the performances of mean-variance optimal portfolios with parameter estimation error and the equally-weighted portfolio. Assuming that the true market parameters are as given in Table 1 we generate scenarios of market data from which we obtain estimates of the market parameters. Using these estimated parameters we generate mean-variance optimal portfolios and then study their performance under the true market parameters.

In Figure 3 we investigate the mean-variance optimal portfolios computed using estimated parameters for risk tolerances of $\tau = 0$ and $\tau = .2$. In the first panel when $\tau = 0$, we see that risk-reward tradeoffs of the estimated portfolios, $\mathbf{x_{est}}$, are all close to the true global minimum-variance portfolio. This illustrates that the global minimum-variance portfolio can be approximated quite well with the estimated parameters using 5 years of data (60 monthly samples).



(a) $\tau = .0$, *5 years data*

(b) $\tau = .2$, *5 years data*

(c) $\tau = .2$, *20 years data*

(d) $\tau = .2$, *5 years data, Expected returns known*

FIGURE 3: *The actual risk-reward tradeoff under the true market parameters, measured in units of standard deviations and expected returns respectively, of a sample of 100 mean-variance optimal portfolios computed using parameters estimated from a history of observed returns.*

For investors who are taking on risk, as in the case when $\tau = .2$ shown in panel (b), the mean-variance optimal portfolios computed using estimated parameters can lie far away from the efficient frontier when only 5 years of data are used. In fact, in panel (b) we find that there are many cases where the equal-weight portfolio dominates the reportedly optimal portfolios generated using estimated parameters. This is because the equally-weighted behavioral portfolio does not require the specification of the market parameters using historical data and hence does not suffer from parameter sampling risk.

If the market parameters are stationary, we can reduce the parameter estimation risk by sampling a longer history of asset returns. In panel (c) we use 20 years (240 monthly samples) of observed data to estimate the expected returns and covariances and see that portfolios now lie much closer to the efficient frontier. It is very unlikely that the market parameters would remain stationary over a twenty year window, making this brute force approach inappropriate.

It has been noted by Best and Grauer [4] and Broadie [9] that, comparatively speaking, it is more difficult to obtain reliable estimates of the expected returns than it is to obtain reliable estimates of the variance-covariance matrix using historical data. We illustrate that it is the the mis-specification of the means that causes the actual performances of the estimated portfolios to lie off of the efficient frontier in panel (d). In this panel we show that the computed portfolios lie very close to the efficient frontier if the vector of expected returns is assumed to be known, while estimating the variance-covariance matrix using only 5 years of data. This implies that investors should take extra care to accurately specify their views of the expected returns on the assets. We will discuss a procedure for improving the estimates of expected returns in section 7.

In summary, we have observed that, at least for the true market parameters studied in this paper, using a reasonable amount of historical data (5 years), the performance of the equally-weighted portfolio is comparable with the performance of the mean-variance optimal portfolios once we account for the impact of parameter estimation risk. Simply using a longer window of historical samples is not a realistic fix to this problem because of the non-stationarity of the market parameters. We also demonstrated that the mis-specification of the expected returns has the most impact on the performance of the mean-variance optimized portfolios.

7 Using Shrinkage Estimators to Improve Estimates of Returns

We have seen that the mis-specification of the expected returns has a dramatic negative impact on the performance of the mean-variance optimized portfolios. In this section we use a Bayesian shrinkage model suggested by Jorion [14] to quantify how much useful information is contained in a sample of historical returns. This, in turn, will allow us to quantify the amount of information that is disregarded by investors who utilize the simple equally-weighted diversification heuristic.

In section 4 we defined a market:

$$\boldsymbol{\mu}_0 = \bar{\mu}\mathbf{e} \ , \tag{19}$$

$$\boldsymbol{\Sigma}_0 = \bar{\sigma}^2\mathbf{I} \ , \tag{20}$$

in which the assets are statistically indistinguishable. In this market the optimal risk-reward trade-off is obtained with the equally-weighted portfolio, a result which is independent of the specified average return $\bar{\mu}$ and average volatility $\bar{\sigma}$. At the other extreme we have an investor who uses a sample of historical returns to estimate the relevant market parameters, $\hat{\boldsymbol{\mu}}$ and $\hat{\boldsymbol{\Sigma}}$. We have

Months of Data Used	α Mean	α Std. Dev.
60	.4995	.1611
240	.7028	.1048

TABLE 4: *The mean and standard deviation of the weighting parameter α, defined as (22) using 5 years and 20 years of historical data. These statistics were estimated from a sample of 1000 scenarios.*

seen that our estimate of the expected returns, $\hat{\boldsymbol{\mu}}$, can be particularly noisy and, due to the error maximizing property of the mean-variance optimization, can result in poor portfolio selection.

For simplicity we assume that the sample variance-covariance matrix, $\hat{\boldsymbol{\Sigma}}$, provides an accurate estimate of the true covariances and we focus on quantifying the benefits of including information contained in a sample of historical data when specifying the expected return vector. Consider the Bayes-Stein weighted learning model described by Jorion [14]:

$$\hat{\boldsymbol{\mu}}_{BS} = \boldsymbol{\mu}_0 + \alpha(\hat{\boldsymbol{\mu}} - \boldsymbol{\mu}_0) \ , \tag{21}$$

where:

$$\alpha = 1 - \frac{N+2}{N+2+M(\hat{\boldsymbol{\mu}} - \boldsymbol{\mu}_0)^T \boldsymbol{\Sigma}^{-1} (\hat{\boldsymbol{\mu}} - \boldsymbol{\mu}_0)} \ , \tag{22}$$

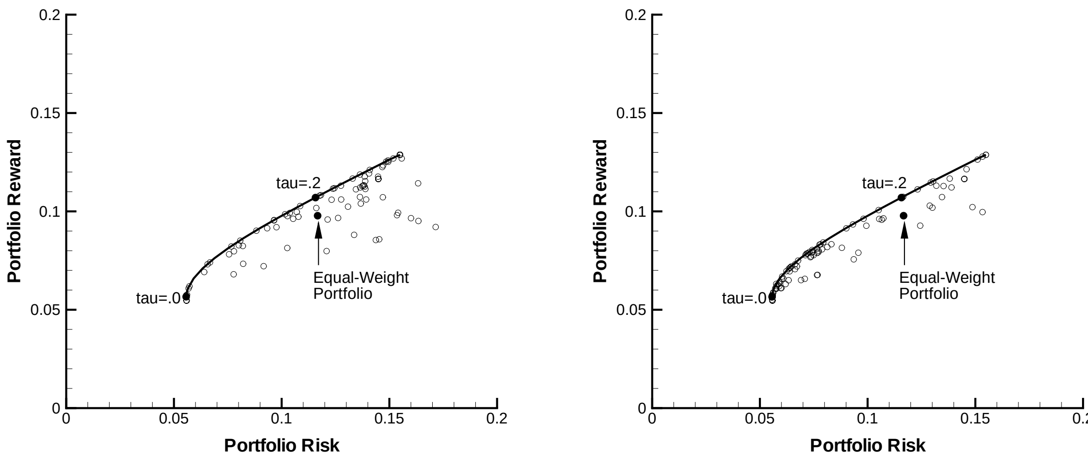
and where N is the number of assets and M is the number of historical data points in the sample. The parameter, $\alpha \in [0, 1]$, specifies the relative importance of the prior information, $\boldsymbol{\mu}_0$ and the information contained in the historical sample, $\hat{\boldsymbol{\mu}}$. This estimator is often called a shrinkage estimator because the estimated expected returns tend to $\boldsymbol{\mu}_0$ as $\alpha \to 0$. Although it is possible to consider other priors, in this paper the level of the expected returns used in the prior, $\bar{\mu}$, is chosen to be the mean of the returns on the minimum-variance portfolio:

$$\bar{\mu} = \frac{\hat{\boldsymbol{\mu}}^T \boldsymbol{\Sigma}^{-1} \mathbf{e}}{\mathbf{e}^T \boldsymbol{\Sigma}^{-1} \mathbf{e}} \ . \tag{23}$$

As an aside we note the resemblance between the Bayes-Stein estimator in equation (21) with the linear credibility approach for estimating insurance premiums.

In our case we are interested in determining the confidence level, α, in a set of historical data. If α is consistently low then there is little information contained in the sample mean and investors who establish equally-weighted portfolios are in fact doing something that is somewhat reasonable.[8] In Table 4 we show the mean and standard deviation of the weighting parameter, α, for 60 and 240 months of historical data. Using 5 years of data, the Bayes-Stein estimator places approximately equal weight on the prior and the sample means, indicating that there is surprisingly little useful information about the expected returns contained in the sample means of the historical data, even if the parameters are stationary. As more historical data is used, the quality of the sample mean improves and α increases, but of course the benefits will be spurious if the market parameters are not stationary.

[8] However, as pointed out by Brennan and Torous [7], they are still missing out on some of the diversification benefits that can be obtained by using the correlations between assets, which can be estimated with adequate precision from a reasonable number of historical samples.



(a) *Sample Estimate* (b) *Bayes-Stein Estimate*

FIGURE 4: *The actual risk-reward tradeoff under the true market parameters, measured in units of standard deviations and expected returns respectively, of a sample of 100 mean-variance optimal portfolios computed using parameters estimated from a history of observed returns. The risk tolerance level was $\tau = .2$ and 60 months of historical data were available.*

In Figure 4 we plot the risk-reward tradeoff for a sample of portfolios constructed using the sample means and those constructed using the Bayes-Stein estimators. We find that more of the portfolios lie along the efficient frontier when the shrinkage estimators are used.

8 Conclusions

In this paper we have explored some of the rationale behind the equally-weighted portfolio that is popular with some defined contribution pension plan participants. If the parameters of the return distribution are assumed to be known this simple heuristic leads to portfolios that lie below the efficient frontier and consequently are sub-optimal. We show that the equally-weighted portfolio is optimal in a market where the assets are indistinguishable and uncorrelated. This leads us to question how much investors lose by implicitly specifying the market parameters in this simple manner.

In practice we do not know the true future asset return distributions and often these are estimated using samples of historical data while assuming that the parameters are stationary. If we take into account the impact of parameter estimation risk then the computed portfolios using mean-variance optimization no longer dominate the equally-weighted portfolio. Using a Bayes-Stein learning model we show that, using a sufficiently small window of historical data over which we might hope that the parameters are stationary, little weighting is placed on the sample means. The prior for the expected returns used in the Bayes-Stein model is identical in form to the naive specification implicitly used by investors who utilize the $1/n$ heuristic. This indicates that it is surprisingly difficult to improve on this simple diversification rule.

The numerical results given in this paper, and the conclusions based on them, were derived from a set of market parameters that we feel are representative of the choices offered to many investors within their pension plan. The advantages of the equally weighted portfolio when there is estimation risk have been noted by other authors using different data sets and different time periods. For example Jobson and Korkie [12] state that *"naive formation rules such as the equal weight rule can outperform the Markowitz rule."* Michaud [16] also notes because of that estimation risk *"an equally weighted portfolio may often be substantially closer to the true MV optimality than an optimized portfolio."*

Defined contribution plans are becoming the dominant vehicle for providing retirement income. In this connection the portfolio strategies of participants are a critical factor since the asset allocation decision determines the ultimate benefits available under these plans. Here we examined the $1/n$ rule and provided a justification. Of course the performance of the equally-weighted heuristic will depend on the asset choices available to the plan members. Given that some participants use the equally-weighted heuristic to select their portfolios, perhaps pension plan sponsors should take this into account when selecting the available asset classes. For example they might consider selecting the asset choices so that this diversification rule is approximately optimal.

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