

QFI QF Model Solutions

Spring 2019

1. Learning Objectives:

1. The candidate will understand the foundations of quantitative finance

Learning Outcomes:

- (1a) Understand and apply concepts of probability and statistics important in mathematical finance.
- (1c) Understand Ito integral and stochastic differential equations.
- (1d) Understand and apply Ito's Lemma.
- (1f) Understand and apply Jensen's Inequality.
- (1h) Define and apply the concepts of martingale, market price of risk and measures in single and multiple state variable contexts.

Sources:

An Introduction to the Mathematics of Financial Derivatives, Hirta, Ali and Neftci, Salih N., 3rd Edition 2nd Printing, 2014

QFIC-113-17 Frequently Asked Questions in Quantitative Finance, Wilmott, Paul, 2nd Edition, 2009

Problems and Solutions in Mathematical Finance: Stochastic Calculus, Chin, Eric, Nel, Dian and Olafsson, Sverrir, 2014

Commentary on Question:

This question tests candidates' understanding of stochastic calculus.

Solution:

- (a) State the three conditions for a stochastic process to be a martingale.

Commentary on Question:

The candidates performed brilliantly on this section.

1. Continued

A stochastic process X_t is said to be a continuous-time martingale with respect to the family of filtration F_s if it satisfies the following conditions:

- $E(X_t|F_s) = X_s$, for all $0 \leq s \leq t$
- $E(|X_t|) < \infty$
- X_t is F_t -adapted

- (b) Show that Z_t is a martingale by verifying the three conditions in part (a).

Commentary on Question:

The candidates performed as expected on this section. To receive full credit, candidates must show verifications for each of the three conditions.

For $t \geq s$, we have:

$$\begin{aligned}
 E[Z_t|F_s] &= E[W_t^3 - 3tW_t|F_s] \\
 &= E[W_t((W_t - W_s + W_s)^2 - 3t)|F_s] \\
 &= E[W_t((W_t - W_s)^2)|F_s] + 2W_sE[W_t(W_t - W_s)|F_s] \\
 &\quad + W_s^2E[W_t|F_s] - 3tE[W_t|F_s] \\
 &= E[(W_t - W_s + W_s)(W_t - W_s)^2|F_s] \\
 &\quad + 2W_sE[(W_t - W_s + W_s)(W_t - W_s)|F_s] + W_s^3 - 3tW_s \\
 &= E[(W_t - W_s)^3|F_s] + E[W_s(W_t - W_s)^2|F_s] \\
 &\quad + 2W_sE[(W_t - W_s)^2|F_s] + 2W_sE[W_s(W_t - W_s)|F_s] + W_s^3 - 3tW_s \\
 &= W_s(t - s) + 2W_s(t - s) + W_s^3 - 3tW_s \\
 &= W_s^3 - 3sW_s
 \end{aligned}$$

Thus, the process satisfies the first condition of a martingale.

Since $|Z_t| = |W_t^3 - 3tW_t| \leq |W_t^3| + 3t|W_t|$, we have:

$$\begin{aligned}
 E(|Z_t|) &\leq E(|W_t^3|) + 3tE(|W_t|) \\
 E(|Z_t|) &\leq \sqrt{E(|W_t^2|^2)E(|W_t|^2)} + 3tE(|W_t|) \\
 E(|Z_t|) &\leq \sqrt{E(W_t^4)E(W_t^2)} + 3tE(|W_t|) \\
 E(|Z_t|) &\leq \sqrt{E(W_t^4)E(W_t^2)} + 3t\sqrt{E(W_t^2)} \\
 E(|Z_t|) &\leq 6t\sqrt{t} < \infty
 \end{aligned}$$

Thus, the process satisfies the second condition of a martingale.

Finally, the 3rd condition follows immediately because Z_t is a function of W_t .

Thus Z_t is a martingale.

1. Continued

- (c) Calculate $E[\ln(X_t)]$ using Ito's lemma.

Commentary on Question:

The candidates performed as expected on this section.

Using Ito's lemma with $V_t = \ln(X_t)$, we get:

$$dV_t = \frac{1}{X_t} (0.06X_t dt + 0.25X_t dW_t) - \frac{1}{2X_t^2} (0.0625X_t^2 dt)$$

$$dV_t = (0.06 - \frac{0.0625}{2})dt + 0.25dW_t$$

$$dV_t = 0.02875dt + 0.25dW_t$$

Integrate both sides, we have:

$$\ln(X_t) = \ln(X_0) + 0.02875t + 0.25 \int_0^t dW_t$$

$$\text{Since } E\left(\int_0^t dW_t\right) = 0 \text{ and } X_0 = 1, E(\ln(X_t)) = 0.02875t.$$

- (d) Explain why $\ln(E[X_t])$ is greater than $E[\ln(X_t)]$.

Commentary on Question:

The candidates performed well on this section.

Using Jensen's inequality and the fact that the logarithm function is concave on its domain, $\ln(E(X)) > E(\ln(X))$.

Or by continuing the development from (c):

$$X_t = \exp(0.02875t + 0.25 \int_0^t dW_t)$$

$$E(X_t) = E(\exp(0.02875t + 0.25 \int_0^t dW_t))$$

$$E(X_t) = \exp(0.06t)$$

$$\ln(E(X_t)) = 0.06t$$

Per this result and the result from (c), $\ln(E(X)) > E(\ln(X))$.

- (e) Calculate the probability of Y_3 being greater than $\ln(X_3)$.

Commentary on Question:

The candidates performed as expected on this section.

$$Y_3 = 1 + 0.1 * 3 + 0.15W_3$$

Per (c), we know that:

$$\ln X_3 = 0.02875 * 3 + 0.25W_3$$

1. Continued

We set a variable Z_3 :

$$Z_3 = Y_3 - \ln X_3$$

$$Z_3 = 1 + (0.1 - 0.02875) * 3 - 0.1W_3 \sim N(1.21375, 0.03)$$

Thus:

$$P(Y_3 > \ln X_3) = P(Y_3 - \ln X_3 > 0) = P(Z_3 > 0)$$

$$P(Y_3 > \ln X_3) = 1 - N\left(\frac{0-1.21375}{\sqrt{0.03}}\right) = N\left(\frac{0.21375}{\sqrt{0.03}}\right)$$

$$P(Y_3 > \ln X_3) \approx 0.8914$$

2. Learning Objectives:

1. The candidate will understand the foundations of quantitative finance

Learning Outcomes:

- (1d) Understand and apply Ito's Lemma.
- (1f) Understand and apply Jensen's Inequality.

Sources:

QFIQ-113-17 Frequently Asked Questions in Quantitative Finance, Wilmott, Paul, 2nd Edition, 2009, Ch. 2

An Introduction to the Mathematics of Financial Derivatives, Hirs, Ali and Neftci, Salih N., 3rd Edition 2nd Printing, 2014

Problems and Solutions in Mathematical Finance: Stochastic Calculus, Chin, Eric, Nel, Dian and Olafsson, Sverrir, 2014

QFIQ-120-19: Chapters 6 and 7 of Pricing and Hedging Financial Derivatives, Marroni, Leonardo and Perdomo, Irene, 2014

Commentary on Question:

The question tested the candidate's understanding of Ito's Lemma and Jensen's inequality/convexity, in the context of pricing of derivative contracts.

Solution:

- (a) Show that the stochastic process $X_t = \ln(S_t)$ follows Arithmetic Brownian Motion.

Commentary on Question:

This is a standard application of Ito's Lemma. Most candidates were able to provide a correct answer. Partial credit was given if the candidate didn't provide detail or had an incorrect answer.

Use Ito's Lemma to obtain the SDE of $X_t = \ln(S_t)$:

$$dX_t = (\partial X / \partial t) dt + (\partial X / \partial S) dS_t + 0.5 (\partial^2 X / \partial S^2) (dS_t)^2 = (1/S_t) dS_t - 0.5 (1/S_t^2) (dS_t)^2.$$

We are given that $dS_t = rS_t dt + \sigma S_t dW_t$ with W_t a standard Wiener process. Compute $(dS_t)^2 = \sigma^2 S_t^2 dt$. Substituting into the last equation yields:

$$dX_t = (r - 0.5\sigma^2) dt + \sigma dW_t \text{ which is ABM.}$$

2. Continued

- (b)
- (i) State Jensen's inequality in the context of random variables.
 - (ii) Explain why $E^{\mathbb{Q}} \left[\frac{1}{S_1} \right] \geq \frac{e^{-r}}{S_0}$.

Commentary on Question:

Most candidates were able to recall correctly the definition in part (i) and apply it to part (ii). Points were deducted for wrong statements, such as $E^{\mathbb{Q}}[S_1] = S_0$, or for not providing sufficient detail or a reference to justify the computation.

- (i) This is a recall question. The candidate should state in terms of expected values: Jensen's inequality states that for a convex function f and a random variable X : $E(f(X)) \geq f(E(X))$. The mathematical definition, which is that for a convex function f , points x, y in its domain and any $0 \leq \lambda \leq 1$:
 $\lambda f(x) + (1 - \lambda) f(y) \geq f(\lambda x + (1 - \lambda)y)$, doesn't earn credit since the problem is asking for the definition of Jensen inequality in the context of random variables.
- (ii) Jensen's inequality implies that $E^{\mathbb{Q}} \left[\frac{1}{S_1} \right] \geq \frac{1}{E^{\mathbb{Q}}[S_1]}$ since the function $f(x) = \frac{1}{x}$ is convex for $x > 0$. From part (a)
$$S_t = S_0 \exp((r - 0.5\sigma^2)t + \sigma W_t)$$
and for $Y \sim N(\mu, \sigma^2)$, we know that: $E(\exp(Y)) = \exp(\mu + 0.5\sigma^2)$.
Therefore $E^{\mathbb{Q}}[S_1] = S_0 \exp((r - 0.5\sigma^2) + 0.5\sigma^2) = S_0 \exp(r)$
which explains the statement. Alternatively, $E^{\mathbb{Q}}[e^{-r}S_1] = S_0$
due to the fact that the discounted stock price is a martingale with respect to the risk-neutral probability \mathbb{Q} .

- (c)
- (i) Prove that $V = \frac{e^{\sigma^2 - 2r}}{S_0}$.
 - (ii) Verify the inequality in part (b) (ii).

Commentary on Question:

Candidates did relatively well on this part. Common mistakes had to do with the application of Ito's Lemma to $C_t = 1/S_t$.

2. Continued

(i) Let $C_t = 1/S_t$ for $0 \leq t \leq 1$. Similarly to (b) (ii):

$S_t = S_0 \exp((r - 0.5\sigma^2)t + \sigma W_t)$ thus

$$C_t = \frac{\exp((0.5\sigma^2 - r)t - \sigma W_t)}{S_0}$$

Using once again the mgf of a normally distributed random variable, we obtain:

$$V = \exp(-r) E^{\mathbb{Q}}[C_1] = \exp(-r) \frac{\exp(\sigma^2 - r)}{S_0} = \frac{\exp(\sigma^2 - 2r)}{S_0}$$

Alternatively, Ito's Lemma can be used to derive the above formula for the contract price C_t .

(ii) From part (c) (i) $E^{\mathbb{Q}}[C_1] = \frac{\exp(\sigma^2 - r)}{S_0} > \frac{\exp(-r)}{S_0} = \frac{1}{E^{\mathbb{Q}}[S_1]}$

(d) Determine the condition on S_0 in terms of r and σ such that, as functions of S_0 , P exhibits higher convexity than V .

Commentary on Question:

Candidates did poorly in this part. Most were not able to identify the second derivative with respect to S_t as the convexity of the contract, and compare it against the Gamma of the put option.

Compare the Γ of the contract with the Γ of the put option, which is given as:
 $N'(d_1)/(\sigma S_0)$

The put option has higher convexity when:

$$N'(d_1)/(\sigma S_0) > (2/S_0^3) \exp(\sigma^2 - 2r)$$

Note that $d_1 = \frac{\ln(\frac{S}{K}) + (r + \frac{1}{2}\sigma^2)}{\sigma} = \frac{\sigma}{2}$ therefore: $S_0^2 > \sqrt{8\pi} \sigma \exp(\frac{9\sigma^2}{8} - 2r)$.

3. Learning Objectives:

1. The candidate will understand the foundations of quantitative finance.

Learning Outcomes:

- (1d) Understand and apply Ito's Lemma.
- (1e) Understand the Black Scholes Merton PDE (partial differential equation).
- (1i) Demonstrate understanding of the differences and implications of real-world versus risk-neutral probability measures, and when the use of each is appropriate.
- (1k) Understand the importance of the Feynman-Kac Theorem.

Sources:

An Introduction to the Mathematics of Financial Derivatives, Hirta, Ali and Neftci, Salih N., 3rd Edition 2nd Printing, 2014, Chapters 14-15.

Problems and Solutions in Mathematical Finance: Stochastic Calculus, Chin, Eric, Nel, Dian and Olafsson, Sverrir, 2014.

Commentary on Question:

This question tested the candidate's understanding of Girsanov Theorem and how martingales are used in the pricing of European options. It also tested knowledge of the derivation of the Black-Scholes PDE and understanding of Feynman-Kac Theorem for a one-dimensional diffusion as applied to risk-neutral valuation.

Solution:

- (a) Derive the dynamics of S_t under \mathbb{Q} (i.e., in terms of dt and $d\widetilde{W}_t$) and the Radon-Nikodym derivative $\xi(W_t) = \frac{d\mathbb{Q}(W_t)}{d\mathbb{P}(W_t)}$.

Commentary on Question:

Candidates were for the most part successful at deriving the dynamics of the stock price under the risk-neutral probability, but performance was mixed on the part asking them to compute the Radon-Nikodym derivative. A common mistake was getting the sign of the random component wrong.

From the product rule we have

$$\begin{aligned} d(e^{-rt}S_t) &= e^{-rt}(\mu S_t dt + \sigma S_t dW_t) - re^{-rt}S_t dt \\ &= \sigma e^{-rt}S_t \left(\frac{\mu - r}{\sigma} dt + dW_t \right) \end{aligned}$$

Given that

$$d(e^{-rt}S_t) = fd\widetilde{W}_t$$

we must have

3. Continued

$$d\widetilde{W}_t = \frac{\mu-r}{\sigma} dt + dW_t, \text{ therefore } \widetilde{W}_t = \frac{\mu-r}{\sigma} t + W_t.$$

In turn

$$\begin{aligned} dS_t &= \mu S_t dt + \sigma S_t dW_t \\ &= r S_t dt + \sigma S_t \left(\frac{\mu-r}{\sigma} dt + dW_t \right) \\ &= r S_t dt + \sigma S_t d\widetilde{W}_t \end{aligned}$$

Applying $X_u = -\frac{\mu-r}{\sigma}$ to Equation (14.84) in Neftci and calculating $\xi(W_t)$ using Equation (14.76) we have

$$\xi(W_t) = e^{\left(\int_0^t X_u dW_u - \frac{1}{2} \int_0^t X_u^2 du\right)} = e^{\left(-\int_0^t \frac{\mu-r}{\sigma} dW_u - \frac{1}{2} \int_0^t \left(\frac{\mu-r}{\sigma}\right)^2 du\right)} = e^{-\frac{\mu-r}{\sigma} W_t - \frac{(\mu-r)^2}{2\sigma^2} t}$$

Alternative solution for $\xi(W_t)$: From the formula sheet, we have

$$d\mathbb{P}(W_t) = \frac{1}{\sqrt{2\pi t}} e^{-\frac{(W_t)^2}{2t}} dW_t.$$

By definition, \widetilde{W}_t is a standard Wiener process under \mathbb{Q} , hence

$$d\mathbb{Q}(\widetilde{W}_t) = \frac{1}{\sqrt{2\pi t}} e^{-\frac{(\widetilde{W}_t)^2}{2t}} d\widetilde{W}_t.$$

We now compute

$$\xi(W_t) = \frac{d\mathbb{Q}(W_t)}{d\mathbb{P}(W_t)} = \frac{\frac{1}{\sqrt{2\pi t}} e^{-\frac{(\widetilde{W}_t)^2}{2t}} d\widetilde{W}_t}{\frac{1}{\sqrt{2\pi t}} e^{-\frac{(W_t)^2}{2t}} dW_t} = e^{-\frac{(\mu-r)W_t}{\sigma} - \frac{(\mu-r)^2 t}{2\sigma^2}}.$$

- (b) Derive the option price at time t as a conditional expectation under \mathbb{Q} of an appropriate terminal value $P(T, S_T)$.

Commentary on Question:

Candidates did worse than expected on this part. Many omitted specifying a suitable information set or the terminal condition.

Since this is a European put option expiring at time T , the terminal condition is $P(T, S_T) = \max\{0, K - S_T\}$, where K is the option strike.

Since $e^{-rt} S_t$ is a martingale under \mathbb{Q} we have

$$e^{-rt} P(t, S_t) = E_{\mathbb{Q}}(e^{-rT} \max\{0, K - S_T\} | F_t)$$

3. Continued

Therefore, the option price at time t is:

$$P(t, S_t) = E_{\mathbb{Q}}(e^{-r(T-t)} \max\{0, K - S_T\} | F_t).$$

- (c) Show that if $\theta_{1,t} = \frac{\partial P}{\partial S} \theta_{2,t}$, then the portfolio Σ is risk-free.

Commentary on Question:

Candidates did relatively well on this part. Partial credit was given for answers that assumed the portfolio is risk-free and concluded that $\theta_1 = \frac{\partial P}{\partial S} \theta_2$.

By Ito's Lemma:

$$dP = \frac{\partial P}{\partial S} dS + \frac{\partial P}{\partial t} dt + \frac{(\sigma S)^2}{2} \frac{\partial^2 P}{\partial S^2} dt$$

$$d\Sigma = \theta_1 dS - \theta_2 dP = (\theta_1 - \theta_2 \frac{\partial P}{\partial S}) dS - \theta_2 \left(\frac{\partial P}{\partial t} + \frac{(\sigma S)^2}{2} \frac{\partial^2 P}{\partial S^2} \right) dt$$

If $\theta_1 = \theta_2 \frac{\partial P}{\partial S}$, then $d\Sigma = -\theta_2 \left(\frac{\partial P}{\partial t} + \frac{(\sigma S)^2}{2} \frac{\partial^2 P}{\partial S^2} \right) dt$ thus the portfolio Σ is risk-free because the diffusion term vanishes

- (d) Derive the Black-Scholes PDE for $P(t, S_t)$ with $t < T$ based on an arbitrage argument.

Commentary on Question:

Candidates did well on this part.

Since Σ is risk-free, it will grow at rate r , i.e. $d\Sigma = r\Sigma dt$, which implies:

$$r \left(\theta_2 \frac{\partial P}{\partial S} S_t - \theta_2 P \right) dt + \theta_2 \left(\frac{\partial P}{\partial t} + 0.5 \left(\frac{\partial^2 P}{\partial S^2} \right) \sigma^2 S_t^2 \right) dt = 0$$

or

$$\frac{\partial P}{\partial t} + 0.5 \frac{\partial^2 P}{\partial S^2} \sigma^2 S_t^2 + \frac{\partial P}{\partial S} r S_t = rP(t, S_t).$$

3. Continued

- (e) State the Feynman-Kac Theorem for a one-dimensional diffusion process.

Commentary on Question:

Candidates did poorly on this part. Most could not recall the Feynman-Kac Theorem. Some candidates listed one or more formulas from the formula sheet (21.90-21.92) without explanation of what these formulas mean. It should be noted that Feynman-Kac is presented in distinct ways in the two source texts listed above. Although the statement of the question was a clear reference to the Feynman-Kac Theorem as it appears in Chin et al., candidates who were able to provide a full and correct statement from Hirsu-Neftci were given full credit.

Feynman-Kac states that given SDE:

$$dS_t = \mu(S_t, t) dt + \sigma(S_t, t) dW_t$$

with W_t a standard Wiener process under probability space (Ω, F, \mathbb{P}) , if $V(t, S_t)$ satisfies the PDE:

$$\frac{\partial V}{\partial t} + 0.5 \left(\frac{\partial^2 V}{\partial S^2} \right) \sigma(S_t, t)^2 + \left(\frac{\partial V}{\partial S} \right) \mu(S_t, t) = r(t) V(t, S_t)$$

with boundary condition $V(T, S_T) = \Psi(S_T)$, then it is given by:

$$V(t, S_t) = E_{\mathbb{P}}(e^{-\int_t^T r(u) du} \Psi(S_T) | F_t).$$

- (f) Verify your result in part (b) by applying the Feynman-Kac Theorem.

Commentary on Question:

Candidates who did not correctly identify the statement of Feynman-Kac Theorem were not able to successfully complete this part. Partial credit was given for incomplete answers with correct statements in them.

From (a), we can write the dynamics of dS_t in the form:

$$dS_t = rS_t dt + \sigma S_t d\widetilde{W}_t$$

where \widetilde{W}_t is a standard Wiener process under the probability space (Ω, F, \mathbb{Q}) . Then, by applying Feynman-Kac to $r(u) = r$ and $V(t, S_t) = P(t, S_t)$ that satisfies the PDE in part (d) with the boundary condition $\Psi(S_T) = \max\{0, K - S_T\}$, we obtain:

3. Continued

$$P(t, S_t) = E_{\mathbb{Q}}(e^{-r(T-t)} \max\{0, K - S_T\} | F_t)$$

which is identical to the answer in part (b).

4. Learning Objectives:

1. The candidate will understand the foundations of quantitative finance

Learning Outcomes:

- (1c) Understand Ito integral and stochastic differential equations.
- (1d) Understand and apply Ito's Lemma.

Sources:

An Introduction to the Mathematics of Financial Derivatives, Neftci, Hirsra, 3rd Edition, Ch. 7-10

Problems and Solutions in Mathematical Finance, Chin et al pp126-127

Commentary on Question:

This question required only basic knowledge of stochastic calculus and the Ito's lemma. As a result, most candidates did well on this question.

Solution:

- (a) Solve the SDE for X_t .

Commentary on Question:

This question tested the candidates basic understanding of Ito's lemma and the ability to solve stochastic differential equations. Most candidates did well on this part. The candidates that did not do well generally did not choose Y_t appropriately or did not understand that the SDE for X_t could not be integrated directly using standard calculus techniques.

By choosing $Y_t = X_t e^{-at}$ and applying product rule, we get:

$$\begin{aligned} dY_t &= -aX_t e^{-at} dt + e^{-at} (aX_t dt + b dW_t) \\ dY_t &= b e^{-at} dW_t \end{aligned}$$

By integrating on both sides we get:

$$\begin{aligned} \int_0^t dY_s &= \int_0^t b e^{-as} dW_s \\ Y_t &= Y_0 + b \int_0^t e^{-as} dW_s \end{aligned}$$

Finally, by substituing Y_t by $X_t e^{-at}$ and noting that $Y_0 = X_0 = c$, we get:

$$X_t = c e^{at} + b \int_0^t e^{a(t-s)} dW_s$$

4. Continued

- (b) Determine the distribution of X_t including its expected value and variance.

Commentary on Question:

This question tested understanding of Ito integrals. Most candidates were able to calculate the expected value and the variance based on the result from part (a). However, many candidates failed to mention the distribution of X_t .

Using results from (a) and the fact that $\int_0^t (e^{a(t-s)})^2 ds = \frac{e^{2at}-1}{2a}$, we get that X_t follows a normal distribution with an expected value of ce^{at} and a variance of $\frac{b^2(e^{2at}-1)}{2a}$.

- (c) Show that the random variable $\int_0^T X_t dt$ follows a normal distribution with expectation $\frac{c}{a}(e^{aT} - 1)$ and variance $\frac{b^2}{2a^3}(e^{2aT} - 4e^{aT} + 2aT + 3)$.

(Hint: You may assume that $\int_0^T \int_0^t f(s, t) dW_s dt = \int_0^T \int_s^T f(s, t) dt dW_s$.)

Commentary on Question:

This question further tested understanding of Ito calculus. Most candidates were able to figure out the expectation and the variance based on the hint. However, many candidates did not mention the distribution of the random variable.

We integrate the solution from (a) between 0 and T :

$$\int_0^T X_t dt = \int_0^T ce^{at} dt + \int_0^T b \int_0^t e^{a(t-s)} dW_s dt$$

By switching the order of integration, using the hint we get:

$$\begin{aligned} \int_0^T X_t dt &= \frac{e^{aT} - 1}{a} c + \int_0^T b \int_s^T e^{a(t-s)} dt dW_s \\ \int_0^T X_t dt &= \frac{e^{aT} - 1}{a} c + \int_0^T b \frac{e^{a(T-s)} - 1}{a} dW_s \end{aligned}$$

We finally get that $\int_0^T X_t dt$ follows a normal distribution with an expected value of $\frac{e^{aT}-1}{a} c$ and a variance of

$$\frac{b^2}{a^2} \int_0^T (e^{a(T-s)} - 1)^2 ds = \frac{b^2}{a^2} \int_0^T (e^{2a(T-s)} - 2e^{a(T-s)} + 1) ds$$

which simplifies to $\frac{b^2}{2a^3}(e^{2aT} - 4e^{aT} + 3 + 2aT)$.

5. Learning Objectives:

1. The candidate will understand the foundations of quantitative finance.
2. The candidate will understand the fundamentals of fixed income markets and traded securities.
3. The candidate will understand:
 - The Quantitative tools and techniques for modeling the term structure of interest rates.
 - The standard yield curve models.
 - The tools and techniques for managing interest rate risk.

Learning Outcomes:

- (1d) Understand and apply Ito's Lemma.
- (2d) Understand the characteristics and uses of interest rate forwards, swaps, futures, and options.
- (3b) Understand and apply various one-factor interest rate models.
- (3i) Understand and apply the Heath-Jarrow-Morton approach including the Libor Market Model.

Sources:

An Introduction to the Mathematics of Financial Derivatives, Hirta, Ali and Neftci, Salih N., 3rd Edition 2nd Printing, 2014 (Ch. 10)

Fixed Income Securities: Valuation, Risk, and Risk Management, Veronesi, Pietro, 2010 (Ch. 19)

An Introduction to the Mathematics of Financial Derivatives, Hirta, Ali and Neftci, Salih N., 3rd Edition 2nd Printing, 2014 (Ch. 19)

Fixed Income Securities: Valuation, Risk, and Risk Management, Veronesi, Pietro, 2010 (Ch. 21)

Commentary on Question:

This question tested understanding of both one-factor short rate models and the Heath-Jarrow-Morton framework. The question also tested the difference between forward and future rates within the context of interest rate models. Most candidates were not able to earn full credit for this question. However, many candidates earned partial credit. Candidates generally did well on parts (a) and (b) and struggled with parts (c), (g), and (h).

5. Continued

Solution:

- (a) Show that $d\ln(r_t) = (\theta_t - \gamma_t \ln(r_t))dt + \sigma_t dX_t$ under Model 1.

Commentary on Question:

This question was a relatively straight-forward application of Ito's lemma within the context of the Black-Karasinski model. Most candidates were able to derive the formula by applying Ito's lemma.

Note that

$$\frac{dr_t}{r_t} = \left(\theta_t + \frac{\sigma_t^2}{2} - \gamma_t \ln(r_t) \right) dt + \sigma_t dX_t$$

By Ito's lemma

$$\begin{aligned} d\ln(r_t) &= \frac{1}{r_t} dr_t - \frac{1}{2r_t^2} (dr_t)^2 \\ &= \left(\theta_t + \frac{\sigma_t^2}{2} - \gamma_t \ln(r_t) \right) dt + \sigma_t dX_t - \frac{1}{2r_t^2} \sigma_t^2 r_t^2 dt \\ &= (\theta_t - \gamma_t \ln r_t) dt + \sigma_t dX_t \end{aligned}$$

- (b) Compare and contrast the following properties under Model 1 and Model 2.
- (i) Arbitrage vs. non-arbitrage model
 - (ii) Distribution of r_t
 - (iii) Possible values of r_t
 - (iv) Analytical Bond Price

Commentary on Question:

This question tested understanding of two basic short rate models: the Hull-White model and the Black-Karasinski model. Most candidates compared the two models correctly for at least three of the four properties. Some candidates indicated that the Black-Karasinski model permitted arbitrage, which is incorrect. The time varying parameters (θ_t and γ_t) can be calibrated to reproduce the current yield curve.

Model 1: Black and Karasinski is a "no arbitrage" model.

Model 2: Hull and White model is a "no arbitrage" model.

Model 1: Black and Karasinski model: r_t is log-normally distributed.

Model 2: Hull and White model: r_t is normally distributed.

5. Continued

Model 1: Black and Karasinski model:

Unlike Hull-White model, it assumes that the logarithm of interest rates is normally distributed, implying that interest rates are always positive.

Model 2: Hull and White model: r_t can be negative.

Model 1: Black and Karasinski model: This model does not allow for a closed form representation of bond prices. The standard methodology to price securities is through trees or Monte Carlo simulations

Model 2: Hull and White model: Has analytical formulas for bond prices, call and put options on bonds.

- (c) Describe the restriction imposed on $\sigma_Z(t, T)$.

Commentary on Question:

This question tested understanding of the assumptions underlying the Heath-Jarrow-Morton framework. Most candidates were not able to give the correct condition. Answers such as the volatility function should be differentiable earned only partial credit.

The HJM framework starts with the following SDE for a zero coupon bond with maturity T

$$\frac{dZ(t, T)}{Z(t, T)} = r_t dt + \sigma_Z(t, T) dX_t$$

where r_t is the current instantaneous interest rate, and $\sigma_Z(t, T)$ is some volatility function with the restriction that $\sigma_Z(T, T) = 0$.

- (d) Express the drift and the diffusion of $df(t, T)$ in terms of $\sigma_Z(t, T)$.

Commentary on Question:

This question tested understanding of the relationship between the forward rate and zero-coupon bond under the Heath-Jarrow-Morton framework. Many candidates were able to figure out the stochastic differential equation of $f(t, T)$. Some candidates provided the correct diffusion term, but with an incorrect sign. Other candidates provided the drift and the diffusion in terms of $\sigma_f(t, T)$, instead of $\sigma_Z(t, T)$.

It can be shown that instantaneous forward rate $f(t, T) = f(t, T, T)$ follows the SDE given by

$$df(t, T) = \left(\sigma_Z(t, T) \frac{\partial \sigma_Z(t, T)}{\partial T} \right) dt - \frac{\partial \sigma_Z(t, T)}{\partial T} dX_t$$

5. Continued

- (e) Show that $\sigma_Z(t, T) = -B(t, T)\sigma$ for Model 2 using Ito's lemma.

Commentary on Question:

This question was a straight-forward application of Ito's lemma to the analytical price of a zero-coupon bond under the Hull-White model. Many candidates were able to prove this equation by applying Ito's lemma. Candidates did not need to evaluate the drift term to receive full credit. Some candidates incorrectly indicated that $\frac{\partial Z(t, T)}{\partial t} = 0$, which is incorrect, since $A(t, T)$ and $B(t, T)$ are functions of t .

Applying Ito's lemma we have

$$dZ(t, T) = \left(\frac{\partial Z(t, T)}{\partial t} + \frac{\partial Z(t, T)}{\partial r_t} (\theta_t - \gamma r_t) + \frac{\sigma^2}{2} \frac{\partial^2 Z(t, T)}{\partial r_t^2} \right) dt + \frac{\partial Z(t, T)}{\partial r_t} \sigma dX_t$$

From $Z(t, T) = e^{A(t, T) - B(t, T)r_t}$ we calculate

$$\begin{aligned} \frac{\partial Z(t, T)}{\partial r_t} &= -B(t, T)Z(t, T) \\ \frac{\partial^2 Z(t, T)}{\partial r_t^2} &= B(t, T)^2 Z(t, T) \\ \frac{\partial Z(t, T)}{\partial t} &= \left(\frac{\partial A(t, T)}{\partial t} - \frac{\partial B(t, T)}{\partial t} r_t \right) Z(t, T) \end{aligned}$$

It follows that

$$dZ = \left(\frac{\partial A}{\partial t} - \frac{\partial B}{\partial t} r_t - B(\theta_t - \gamma r_t) + \frac{\sigma^2}{2} B^2 \right) Z dt - BZ\sigma dX_t$$

Thus

$$\begin{aligned} \frac{dZ(t, T)}{Z(t, T)} &= \left(\frac{\partial A(t, T)}{\partial t} - \frac{\partial B(t, T)}{\partial t} r_t - B(t, T)(\theta_t - \gamma r_t) + \frac{\sigma^2 B(t, T)^2}{2} \right) dt \\ &\quad - B(t, T)\sigma dX_t \end{aligned}$$

This implies $\sigma_Z(t, T) = -B(t, T)\sigma$

5. Continued

- (f) Derive an expression for $df(t, T)$ under Model 2 in terms of $B(t, T)$.

Commentary on Question:

This question asked candidates to connect the SDE of the forward rate under the HJM framework with the analytical bond price formula under the Hull-White model. Only a few candidates were able to derive the expression correctly. Many candidates substituted for $B(t, T)$, which was unnecessary. Full credit was awarded only to correct answers given in terms of $B(t, T)$, as specified in the question.

$$df(t, T) = m(t, T)dt + \sigma_f(t, T)dX_t$$

$$df(t, T) = \left(\sigma_f(t, T) \int_t^T \sigma_f(t, \tau) d\tau \right) dt + \frac{\partial B(t, T)}{\partial T} \sigma dX_t$$

$$df(t, T) = \left(\frac{\partial B(t, T)}{\partial T} \sigma \int_t^T \frac{\partial B(t, \tau)}{\partial \tau} \sigma d\tau \right) dt + \frac{\partial B(t, T)}{\partial T} \sigma dX_t$$

By Fundamental theorem of Calculus and noting $B(t, t) = 0$ we find

$$df(t, T) = \frac{\partial B(t, T)}{\partial T} \sigma^2 B(t, T) dt + \frac{\partial B(t, T)}{\partial T} \sigma dX_t$$

Alternatively,

Apply $\sigma_z(t, T) = -\sigma B(t, T)$ in the text (21.53) or part (e) to obtain the above equation directly.

$$df(t, T) = \left(\sigma_z(t, T) \frac{\partial \sigma_z(t, T)}{\partial T} \right) dt - \frac{\partial \sigma_z(t, T)}{\partial T} dX_t$$

$$df(t, T) = \left((-B(t, T)\sigma) \frac{\partial B(t, T)}{\partial T} \sigma \right) dt - \left(-\frac{\partial B(t, T)}{\partial T} \right) \sigma dX_t$$

$$df(t, T) = \frac{\partial B(t, T)}{\partial T} \sigma^2 B(t, T) dt + \frac{\partial B(t, T)}{\partial T} \sigma dX_t$$

- (g) Explain intuitively why $f(t, \tau, T)$ differs from $f^{fut}(t, \tau, T)$.

Commentary on Question:

This question tested theoretical understanding of the difference between forward rates and future rates. Only a few candidates were able to explain the difference by mentioning the convexity that exists between the value of a bond and its yield. Many candidates listed practical differences between futures and forward rate contracts, which do not intuitively explain the difference in rates.

5. Continued

The forward risk-neutral methodology implies that the expected future price of the bond used as numeraire equals its current forward price.

If a derivative security's payoff depends on the yield of the bond instead of its price, then the forward risk-neutral expected future yield is approximated well by the forward rate plus a convexity adjustment.

The latter takes into account the natural convexity that exists between the value of a bond and its yield.

The convexity adjustment is zero when there is a "natural" lag between cash flow formation and payment date.

- (h) Compute $f^{fut}(0, \tau, T) - f(0, \tau, T)$ under Model 2.

Commentary on Question:

This question tested the model-dependent convexity adjustment needed for interest rate futures. Only few candidates were able to compute the difference correctly. A few candidates knew the correct formula but did not evaluate the formula correctly. Many candidates did not recall the formula or did not attempt the question.

$$f^{fut}(0, \tau, T) - f(0, \tau, T) = \int_0^\tau \frac{\sigma_Z(t, T)^2 - \sigma_Z(t, \tau)^2}{2(T - \tau)} dt$$

Since $\sigma_Z(t, T)^2 = (B(t, T)\sigma)^2 = \frac{(1 - 2e^{-\gamma(T-t)} + e^{-2\gamma(T-t)})\sigma^2}{\gamma^2}$ we have

$$\begin{aligned} \sigma_Z(t, T)^2 - \sigma_Z(t, \tau)^2 &= \frac{(2e^{-\gamma(\tau-t)} - 2e^{-\gamma(T-t)} + e^{-2\gamma(T-t)} - e^{-2\gamma(\tau-t)})\sigma^2}{\gamma^2} \\ &= \frac{2e^{\gamma t}(e^{-\gamma\tau} - e^{-\gamma T})\sigma^2}{\gamma^2} - \frac{e^{2\gamma t}(e^{-2\gamma\tau} - e^{-2\gamma T})\sigma^2}{\gamma^2} \end{aligned}$$

Therefore the integral is

$$\begin{aligned} &\frac{1}{2(T - \tau)} \int_0^\tau \frac{2e^{\gamma t}(e^{-\gamma\tau} - e^{-\gamma T})\sigma^2}{\gamma^2} - \frac{e^{2\gamma t}(e^{-2\gamma\tau} - e^{-2\gamma T})\sigma^2}{\gamma^2} dt \\ &= \frac{(e^{\gamma\tau})(e^{-\gamma\tau} - e^{-\gamma T})\sigma^2}{\gamma^3(T - \tau)} \Big|_0^\tau - \frac{(e^{2\gamma t})(e^{-2\gamma\tau} - e^{-2\gamma T})\sigma^2}{4\gamma^3(T - \tau)} \Big|_0^\tau \\ &= \frac{(e^{\gamma\tau} - 1)(e^{-\gamma\tau} - e^{-\gamma T})\sigma^2}{\gamma^3(T - \tau)} - \frac{(e^{2\gamma\tau} - 1)(e^{-2\gamma\tau} - e^{-2\gamma T})\sigma^2}{4\gamma^3(T - \tau)} \end{aligned}$$

6. Learning Objectives:

4. The candidate will understand:
- How to apply the standard models for pricing financial derivatives.
 - The implications for option pricing when markets do not satisfy the common assumptions used in option pricing theory.
 - How to evaluate risk exposures and the issues in hedging them.

Learning Outcomes:

- (4a) Demonstrate an understanding of option pricing techniques and theory for equity derivatives.
- (4b) Identify limitations of the Black-Scholes-Merton pricing formula
- (4d) Demonstrate an understanding of how to delta hedge, and the interplay between hedging assumptions and hedging outcomes.
- (4e) Analyze the Greeks of common option strategies.
- (4f) Appreciate how hedge strategies may go awry.

Sources:

Pricing and Hedging Financial Derivatives, Marroni and Perdomo, 2014, Ch. 6

The Volatility Smile, Derman, Miller, and Park, 2016, Ch. 7

Commentary on Question:

This question tested understanding of delta-hedging and the Greeks.

Solution:

- (a) Show that:

$$\frac{\partial C}{\partial S} = N(d_1)$$

Commentary on Question:

The candidates performed below expectation on this section. To receive full credit, candidates needed to show (not just state) $SN'(d_1) - Ke^{-rT}N'(d_2) = 0$.

6. Continued

$$\begin{aligned}
 \frac{\partial C}{\partial S} &= \frac{\partial [SN(d_1) - Ke^{-rT}N(d_2)]}{\partial S} \\
 &= N(d_1) + S \frac{\partial N(d_1)}{\partial S} - Ke^{-rT} \frac{\partial N(d_2)}{\partial S} \\
 &= N(d_1) + S \frac{\partial N(d_1)}{\partial d_1} \frac{\partial d_1}{\partial S} - Ke^{-rT} \frac{\partial N(d_2)}{\partial d_2} \frac{\partial d_2}{\partial S} \\
 &= N(d_1) + SN'(d_1) \frac{1}{S\sigma\sqrt{T}} - Ke^{-rT} N'(d_2) \frac{1}{S\sigma\sqrt{T}} \\
 &= N(d_1) + \frac{1}{S\sigma\sqrt{T}} [SN'(d_1) - Ke^{-rT}N'(d_2)] \\
 Ke^{-rT}N'(d_2) &= Ke^{-rT} \frac{1}{\sqrt{2\pi}} e^{-0.5d_2^2} \\
 &= Ke^{-rT} \frac{1}{\sqrt{2\pi}} e^{-0.5(d_1 - \sigma\sqrt{T})^2} \\
 &= Ke^{-rT} \frac{1}{\sqrt{2\pi}} e^{-0.5(d_1^2 - 2d_1\sigma\sqrt{T} + \sigma^2T)} \\
 &= Ke^{-rT} \frac{1}{\sqrt{2\pi}} e^{-0.5d_1^2 + d_1\sigma\sqrt{T} - 0.5\sigma^2T} \\
 &= Ke^{-rT} N'(d_1) e^{d_1\sigma\sqrt{T}} e^{-0.5\sigma^2T} \\
 &= Ke^{-rT} N'(d_1) e^{\ln\left(\frac{S}{K}\right) + (r+0.5\sigma^2)T} e^{-0.5\sigma^2T} \\
 &= Ke^{-rT} N'(d_1) \frac{S}{K} e^{(r+0.5\sigma^2)T} e^{-0.5\sigma^2T} \\
 &= SN'(d_1)
 \end{aligned}$$

Therefore, $\frac{\partial C}{\partial S} = N(d_1)$

- (b) Identify whether each of the following statements is true or false. Briefly justify your answer.
- (i) The delta of a European call option that is out-of-the-money will converge to 0 as the expiration date approaches.
 - (ii) For an out-of-the-money option with the underlying having low volatility, if the volatility increases, the delta of the option increases.

Commentary on Question:

The candidates performed above expectation on this section. To receive full credit, candidates must identify whether each statement is true or false and also provide an appropriate brief justification.

6. Continued

- (i) True. If the call option is out-of-the-money, the delta will approach to 0. In this case, the option will not be exercised, and consequently, there will be no dependency of the option price on the price of the underlying security.
- (ii) True. Due to low volatility, the probability that the underlying price will cross the strike before the expiry is relatively low. If the volatility starts to increase, we would expect a higher probability of the option expiring in-the-money. This increases the dependency of the option price on the price of the underlying, thus increasing the delta of the option.
- (c) Calculate the profit or loss at the end of two weeks from delta hedging (ignore any cash withdrawal or infusion that may be needed for rebalancing) if:
- (i) You rebalance your hedge only at the end of each week.
- (ii) You never rebalanced your hedge.

Commentary on Question:

The candidates performed as expected on this section. To receive full credit, candidates must calculate the initial delta and the profits correctly.

$$\Delta_0 = N(d_1) = N \left[\frac{\ln \frac{100}{100} + \left(0 + \frac{0.2^2}{2}\right) (1)}{0.2\sqrt{1}} \right] = N(0.10) = 0.5398 \approx 0.54$$

Initially, you are long 100 options and short 54 shares.

After one week, the delta of the call option is 0.62.

- (i) $\text{Profit}_{\text{rebal}} = 100[7.81 - 7.97] - 54[104 - 100] - 62[100 - 104] = \boxed{16}$
- (ii) $\text{Profit}_{\text{no rebal}} = 100[7.81 - 7.97] - 54[100 - 100] = \boxed{-16}$
- (d) Explain why the hedge profits or losses are different between the two hedging strategies in part (c).

Commentary on Question:

The candidates performed poorly on this section. To receive full credit, candidates needed to mention the gamma impact.

6. Continued

For strategy (i), Gamma impact is positive and it will generate a profit for this strategy. For strategy (ii), loss from Theta impact with no offset from Gamma impact which is 0 since the underlying stock did not move from $T = 0$ to the end of week 2.

7. Learning Objectives:

4. The candidate will understand:
 - How to apply the standard models for pricing financial derivatives.
 - The implications for option pricing when markets do not satisfy the common assumptions used in option pricing theory.
 - How to evaluate risk exposures and the issues in hedging them.

Learning Outcomes:

- (4c) Demonstrate an understating of the different approaches to hedging – static and dynamic.
- (4d) Demonstrate an understanding of how to delta hedge, and the interplay between hedging assumptions and hedging outcomes.
- (4e) Analyze the Greeks of common option strategies.
- (4f) Appreciate how hedge strategies may go awry.
- (4g) Describe and explain some approaches for relaxing the assumptions used in the Black-Scholes-Merton formula.
- (4h) Compare and contrast the various kinds of volatility, e.g, actual, realized, implied and forward, etc.

Sources:

The Volatility Smile, Derman, Miller, and Park, 2016 Ch 3, 5, 6, 7

Pricing and Hedging Financial Derivatives, Marroni and Perdomo, 2014, Ch 6

QFIC-114-17: Frequently Asked Questions in Quantitative Finance, Wilmott, Paul, 2nd Edition

Commentary on Question:

This question challenged candidates understanding of the material and computational capacity. Most candidates attempted most parts of the question and got at least partial marks by demonstrating their understanding of delta-neutral positions, comparison between implied and actual volatilities and their impact on the hedge effectiveness. For some parts of the question, candidates failed to clearly show the derivation but memorized formulas, in which case they earned only partial marks.

Solution:

- (a) Determine the option's delta and gamma using the Taylor series expansion.

7. Continued

Commentary on Question:

Most candidates got marks from this part. One common mistake was to calculate the change of stock price as 0.1 instead of 10, resulting in $\Delta = 50$ and $\Gamma = 200$, in which case candidates got partial marks

Using Taylor series expansion:

$$C(S + dS, t + dt) = C(S, t) + \Theta dt + \Delta dS + \frac{1}{2} \Gamma dS^2$$

For stock prices going up by 10% (i.e. \$10):

$$5.99 = \frac{-3.65}{365} + 10\Delta + \frac{1}{2} \Gamma (100)$$

For stock prices going down by 10% (i.e. -\$10):

$$-4.01 = \frac{-3.65}{365} + (-10)\Delta + \frac{1}{2} \Gamma (100)$$

Thus, solving the system of equations we find:

$$\Delta = 0.5 \text{ and } \Gamma = 0.02$$

- (b) Show that the payoff of a delta-hedged portfolio is $\Theta dt + \frac{1}{2} \Gamma dS^2$ when the stock price moves in either direction.

Commentary on Question:

Majority of candidates were able to recognize the construction of a delta-neutral portfolio and eliminate the delta term.

$$\text{Given } dS = \pm \sigma S \sqrt{dt} \Rightarrow dS^2 = \sigma^2 S^2 dt$$

As stock prices change:

$$C(S \pm \sigma S \sqrt{dt}, t + dt) - C(S, t) = \Theta dt \pm \Delta \sigma S \sqrt{dt} + \frac{1}{2} \Gamma \sigma^2 S^2 dt$$

Except for the term $\pm \Delta \sigma S \sqrt{dt}$, the payoffs on the call are the same. If we can eliminate the term $\pm \Delta \sigma S \sqrt{dt}$, the payoffs will be the same regardless of the stock movement.

To do so, we need to short Δ shares of the underlying stock S .

Thus, with $V = C(S, t) - \Delta S$, it is a delta-hedge portfolio.

7. Continued

When the underlying stock price changes by a small amount dS during time dt :

The change in call option value is (1):

$$C(S \pm dS, t + dt) - C(S, t) = \Theta dt \pm \Delta \sigma S \sqrt{dt} + \frac{1}{2} \Gamma \sigma^2 S^2 dt$$

and the value change in the short position of Δ shares is (2):

$$\Delta * S(t + dt) - \Delta * S(t) = \pm \Delta * \sigma S \sqrt{dt}$$

Then, the change in value of a delta-hedge position is (1) – (2):

$$dV(S, t) = \Theta dt + \frac{1}{2} \Gamma \sigma^2 S^2 dt$$

- (c) Calculate the profit or loss after time dt and explain when you would have a profit.

Commentary on Question:

Most candidates were able to identify the profit or loss (P&L) and drew the conclusion on the condition to be profitable. However, with a few exceptions they didn't clearly show the derivation of P&L and thus got only partial marks.

In a long option position, with:

$$dV(S, t) = \Theta dt + \frac{1}{2} \Gamma \sigma^2 S^2 dt = 0$$

When hedging with implied volatility Σ , you replace σ with Σ , and the amount we expect to lose due to time decay during time dt is:

$$\Theta dt = -\frac{1}{2} \Gamma \Sigma^2 S^2 dt$$

The gain from convexity, if stock moves by an amount $\pm \Delta \sigma S \sqrt{dt}$ with realized volatility σ , is $\frac{1}{2} \Gamma \sigma^2 S^2 dt$

So, the profit and loss (P&L) after time dt is the difference between $-\frac{1}{2} \Gamma \Sigma^2 S^2 dt$ and $\frac{1}{2} \Gamma \sigma^2 S^2 dt$.

$$\text{Profit} = \frac{1}{2} \Gamma S^2 (\sigma^2 - \Sigma^2) dt$$

Therefore the daily profit is deterministic, and to make a profit, all required is the actual volatility σ is greater than the implied volatility Σ , which is a one direction battle.

7. Continued

- (d) Describe one advantage and one disadvantage of delta hedging using the actual volatility.

Commentary on Question:

Candidates generally did well in this part. Among the three disadvantages listed below candidates only needed to provide one to get full marks.

When hedge with actual volatility, the daily change of portfolio value is:

$$dV = 0.5 (\sigma^2 - \sigma^{i2}) S^2 \hat{\Gamma}^2 dt + S (\Delta^i - \Delta) ((\mu - r) dt + \sigma dX)$$

Pros:

- Profit is guaranteed at expiry

Cons:

- How to achieve the guaranteed profit is unknown due to the dX term
- Local fluctuation in profit/loss
- Actual volatility is not easy to observe from market

- (e) Critique each of your colleague's comments above.

Commentary on Question:

Most candidates did well on the comments 3 and 4, where they were able to clearly point out the inappropriateness of the statements. Candidates did relatively less consistently well on the first two statements.

(i) False. Hedging discretely rather than continuously will introduce uncertainty in the hedging outcome but does not bias the final P&L. The expected value of final P&L is zero.

(ii) False. The firm can reduce its hedging error significantly by increasing the frequency of the hedge rebalancing if the firm uses realized volatility to rebalance the position.

(iii) False. The firm can cut its hedging error by half by quadrupling its rebalancing frequency.

(iv) False. In the presence of transaction cost, the more often you hedge, the smaller the hedging error and the smaller the expected hedging profit.

7. Continued

- (f) Estimate the hedging error if the option is delta-hedged with weekly frequency using 20% volatility.

Commentary on Question:

This was a computational intense part and candidates did poorly in this part. Very few candidates were able to utilize the Leland's adjustment for the transaction cost. Most candidates did not attempt this part at all.

Leland's adjustment can be used to compute adjusted vol using the transaction costs of 0.10% (half the bid-offer spread)

For a short position, the cost can be estimated by using:

$$\hat{\sigma}^2 = \sigma^2 + 2\sigma k \sqrt{\frac{2}{\pi dt}}$$

$$\hat{\sigma}^2 = .23^2 + 2 * .23 * .0010 \sqrt{\frac{2}{\pi \left(\frac{1}{52}\right)}}$$

$$\hat{\sigma} = 0.2357$$

Option price with this vol = 6.51
(d1 = -0.1805, d2 = -0.4162, sigma = 23.57%)

V(I, 0) = 6.28
(d1 = -0.1907, d2 = -0.4207, sigma = 23.00%)

Hence, the amount lost in transaction costs = 6.51 – 6.28 = 0.23

8. Learning Objectives:

4. The candidate will understand:
 - How to apply the standard models for pricing financial derivatives.
 - The implications for option pricing when markets do not satisfy the common assumptions used in option pricing theory.
 - How to evaluate risk exposures and the issues in hedging them.

Learning Outcomes:

- (4a) Demonstrate an understanding of option pricing techniques and theory for equity derivatives.
- (4b) Identify limitations of the Black-Scholes-Merton pricing formula
- (4c) Demonstrate an understanding of the different approaches to hedging – static and dynamic.
- (4f) Appreciate how hedge strategies may go awry.

Sources:

Pricing and Hedging Financial Derivatives Ch. 7

The Volatility Smile (Derman) Ch. 3

Commentary on Question:

This question tested understanding of volatility hedging use with Black-Scholes pricing. And one-third of candidates didn't attempt to answer the question.

Solution:

- (a) Construct a 25-delta risk reversal strategy and a butterfly strategy with the instruments above.

Commentary on Question:

Most candidates who wrote an answer were able to determine which components to use and received at least partial credit, but few candidates calculated the market price.

Risk reversal:

Buy 25-delta call and short 25-delta put
Market price of RR = 0.99% - 1.77% = -0.78%

Butterfly:

Buy 25-delta call and put, short 50-delta neutral straddle
Market price of BFLY = (1.77% + 0.99%)/2 - 6.65% = -5.27%

8. Continued

- (b) List main features that make risk reversal and butterfly able to hedge volatility skew and convexity.

Commentary on Question:

About half the candidates who attempted the question answered this correctly.

Risk reversal has high level of $dVega/dSpot$ (Vanna) so is very sensitive to Vega with respect to changes of underlying. Therefore it can take positions of steepness/flatness of volatility skew.

Butterfly has high level of $dVega/dVol$ (Volga) so is very sensitive to slope of volatility skew. Therefore it is usually used to take positions on volatility convexity.

- (c) Describe the Vanna-Volga approach.

Commentary on Question:

Most candidates received at least partial credit for this part.

The Vanna-Volga approach quantifies $dVega/dSpot$, $dVega/dVol$ exposures of exotic options, so that traders can adjust their prices accordingly.

- (d) Explain your reasons for making the adjustment.

Commentary on Question:

Few candidates received points for this part. Many left it blank.

Prices of a risk reversal and a butterfly are wrong if the actual volatility smile is ignored.

Only the 50-delta volatility is used to price all options

- (e) Calculate the adjustment using the volatility surface given in the first table.

Commentary on Question:

Many left this part blank and very few actually computed for an answer.

Using the vanna-volga approach we use the 25-delta risk-reversal and the butterfly with 25-50-25 delta to estimate slope & convexity effects.

Market price of RR at $\sigma_{50} = 1.51\% - 0.96\% = 0.55\%$

Market price of BFLY at $\sigma_{50} = (0.96\% + 1.51\%)/2 - 6.65\% = -5.42\%$

Vanna of RR = $0.0239\% - (-0.0111\%) = 0.0350\%$

Volga of BFLY = $(0.0052\% + 0.0113\%)/2 - 0\% = 0.0083\%$

8. Continued

$$\text{Vanna Ratio} = -0.0148\% / 0.0350\% = -0.4229$$

$$\text{Volga Ratio} = -0.0048\% / 0.0083\% = -0.5818$$

The ratios can quantify how much Vanna/Volga of the RR and BFLY can hedge for the specific exotic option.

Use the market values from part (a)

$$\text{Vanna contribution} = -0.78\% - 0.55\% = -1.33\%$$

$$\text{Volga contribution} = -5.27\% + 5.42\% = 0.15\%$$

The two contributions can measure how “wrong” those prices of RR and BFLY are if actual volatility smile is ignored and only σ_{50} is used to price all options.

$$\text{Vanna Cost} = -0.4229 * -1.33\% = 0.56\%$$

$$\text{Volga Cost} = -0.5818 * 0.15\% = -0.087\%$$

The two costs measure how “wrong” the price of exotic is within Black-Scholes framework ignoring volatility skew and convexity. Therefore we can use the two amounts to adjust the price of exotic option.

$$\text{Vanna Volga Price} = 1.5\% + 0.56\% - 0.087\% = 1.97\%$$

9. Learning Objectives:

2. The candidate will understand the fundamentals of fixed income markets and traded securities.
5. The candidate will understand important quantitative techniques relating to financial time series, volatility modeling, and stochastic modeling.

Learning Outcomes:

- (2c) Understand measures of interest rate risk including duration, convexity, slope, and curvature.
- (5c) Apply various techniques for analyzing factor models including principal component analysis (PCA) and statistical factor analysis.

Sources:

QFIA-125-16: Market Models: A Guide for Financial Data Analysis, Ch. 6, Principal Component Analysis.

Fixed Income Securities: Valuation, Risk, and Risk Management, Veronesi, Pietro, 2010 (Ch. 4).

Commentary on Question:

The overall scores were low. The candidates did better in calculating the λ factors and the proportion of the total variation in the forward rates explained by the principal components.

However, parts (b), (c), and (f) had very low scores. The candidates were not successful in using the correct theoretical formulae and using the matrix calculations to arrive at the required demonstration.

In part (f), it was not sufficient to identify the factors of the shift, slope and curvature, but they had to be qualified as a pure shift, weak slope, and weak curvature given the definition of those in the question itself.

Solution:

- (a) Explain what normalization is and why forward rates need to be normalized before the analysis.

The data should be normalized or standardized by subtracting the variable mean and dividing by the variable standard deviation.

The forward rates will need to be normalized before the analysis, otherwise the first principal component will be dominated by the input variable with greatest volatility.

9. Continued

(b) Show that

(i) All columns of P are uncorrelated.

(ii) The variance of the m -th principal component is λ_m .

(Hint: Consider $P'P$ where P' is the transpose of P .)

$$P'P = W'X'XW = nW'W\Lambda$$

where the last equality holds since $V = X'X/n$ and $VW = W\Lambda$ with Λ being the 4 x 4 diagonal matrix of eigenvalues of V

$$P'P = nW^{-1}W\Lambda \text{ as } W \text{ is an orthogonal matrix } W' = W^{-1} \\ = n\Lambda \text{ as } W^{-1}W = I \text{ where } I \text{ is the Identity matrix}$$

Since this is a diagonal matrix, the columns of P are uncorrelated (i.e. all the non-diagonal entries of $P'P$ are 0), and the variance of the m -th principal component is λ_m

(c) Formulate the maximization problem that is solved by the first eigenvector w_1 .

The objective of PCA is to find coefficients $w_{11}, w_{12}, \dots, w_{1n}$ to maximize the variance of $\Delta\phi_1^{PCA}(t)$:

$$\max_{\alpha_{11}, \alpha_{12}, \dots, \alpha_{1n}} \text{Var}(\Delta\phi_1^{PCA}) = \sum_{k=1}^n \sum_{l=1}^n w_{1k} w_{1l} \sigma_{kl}$$

subject to the optimization constraint of $\sum_{j=1}^n w_{1j}^2 = 1$

In matrix form, $\sum_{k=1}^n \sum_{l=1}^n w_{1k} w_{1l} \sigma_{kl} = w_1' V w_1$ and $\sum_{j=1}^n w_{1j}^2 = w_1' w_1$

Thus, the Lagrangian or Lagrange function for this optimization/maximization problem is

$$L(w_1) = w_1' V w_1 - \lambda(w_1' w_1 - 1)$$

Taking the derivative, the first order condition for solving this Lagrangian is

$$\frac{\partial L(w_1)}{\partial w_1} = 2(V w_1 - \lambda w_1) = 0$$

Therefore, the above condition leads to the solution $V w_1 = \lambda w_1$, as required.

9. Continued

- (d) Calculate the remaining two eigenvalues λ_1 and λ_2 of V , and the entries a, b, c, d of W .

From $VW = W\Lambda$, we have

$$\begin{bmatrix} 1 & .5 & .2 & .3 \\ .5 & 1 & .3 & .2 \\ .2 & .3 & 1 & .5 \\ .3 & .2 & .5 & 1 \end{bmatrix} \begin{bmatrix} a & .5 & .5 & -.5 \\ b & .5 & c & .5 \\ .5 & -.5 & d & -.5 \\ .5 & -.5 & .5 & .5 \end{bmatrix} = \begin{bmatrix} a & .5 & .5 & -.5 \\ b & .5 & c & .5 \\ .5 & -.5 & d & -.5 \\ .5 & -.5 & .5 & .5 \end{bmatrix} \begin{bmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & .6 & \\ & & & .4 \end{bmatrix}$$

Using the entry (1,2) on the first row and the second column we have

$$1 * 0.5 + 0.5 * 0.5 + 0.2 * (-0.5) + 0.3 * (-0.5) = 0.5 * \lambda_2$$

Thus $\lambda_2 = 1$

Since $\lambda_1 + \lambda_2 + \lambda_3 + \lambda_4 = 4$,

we find $\lambda_1 = 4 - (1 + .6 + .4) = 2$

In turn using the entries (1,1) and (2,1) we have

$$1 * a + 0.5 * b + 0.2 * 0.5 + 0.3 * 0.5 = 2a$$

and

$$0.5 * a + 1 * b + 0.3 * 0.5 + 0.2 * 0.5 = 2b$$

That is, $a - .5b = .25$ and $-.5a + b = .25$.

It follows that $a = b = .5$

Similarly, using the entries (1,3) and (2,3) we have

$$1 * 0.5 + 0.5 * c + 0.2 * d + 0.3 * 0.5 = 0.5 * 0.6$$

and

$$0.5 * 0.5 + 1 * c + 0.3 * d + 0.2 * 0.5 = c * 0.6$$

That is, $0.5c + 0.2d = -0.35$ and $0.4c + 0.3d = -0.35$.

It follows that $c = d = -.5$

- (e) Calculate the proportion of the total variation in the forward rates explained by each of the first four principal components.

The proportion of the total variation in the forward rates explained by the first principal component:

$$\frac{2}{2 + 1 + \frac{3}{5} + \frac{2}{5}} = 50\%$$

9. Continued

The proportion of the total variation in the forward rates explained by the second principal component:

$$\frac{1}{2 + 1 + \frac{3}{5} + \frac{2}{5}} = 25\%$$

The proportion of the total variation in the forward rates explained by the third principal component:

$$\frac{\frac{3}{5}}{2 + 1 + \frac{3}{5} + \frac{2}{5}} = 15\%$$

The proportion of the total variation in the forward rates explained by the fourth principal component:

$$\frac{\frac{2}{5}}{2 + 1 + \frac{3}{5} + \frac{2}{5}} = 10\%$$

(f) Interpret the meaning for each of the first three eigenvectors.

$$\begin{aligned}W_1' &= (0.5, 0.5, 0.5, 0.5), \\W_2' &= (0.5, 0.5, -0.5, -0.5), \\W_3' &= (0.5, -0.5, -0.5, 0.5), \\W_4' &= (-0.5, 0.5, -0.5, 0.5)\end{aligned}$$

Based on the information provided and the values for the eigenvectors, the first three eigenvectors that are respectively pure shift, weak slope and weak curvature.

50% of the total variation in the forward rates explained by pure shift

25% of the total variation in the forward rates explained by weak slope

15% of the total variation in the forward rates explained by weak curvature

10. Learning Objectives:

2. The candidate will understand the fundamentals of fixed income markets and traded securities.

Learning Outcomes:

- (2a) Understand the characteristics of fixed rate, floating rate, and zero-coupon bonds.
- (2b) Bootstrap a yield curve.
- (2c) Understand measures of interest rate risk including duration, convexity, slope, and curvature.

Sources:

Fixed Income Securities Valuation, Risk, and Risk Management, Veresoni, Pietro 2010, CH. 2-3-4.

Commentary on Question:

The results were not as good as expected, particularly for part (e), but even the simple calculation of the 1-month VaR was not very well done. Perhaps candidates may have not practiced applying the formulae to calculate VaR.

For part (e), it was an application of a Factor neutrality concept against changes in level and slope of the yield curve over time. Although some candidates calculated correctly the factor durations with respect to the level and the slope, they were not able to apply the calculation formula for k_s and k_l .

Solution:

- (a) Calculate the price of a 2-year floating rate bond with coupons paid semi-annually and with a 35 basis points spread, assuming that its par value is \$100.

$$\begin{aligned}\text{Price} &= 100 + .35\% * (Z(0, 0.5) + Z(0, 1) + Z(0, 1.5) + Z(0, 2)) \\ &= 100 + .35\% * (0.9615 + 0.9401 + 0.9191 + 0.8937) = 100.013\end{aligned}$$

- (b) Calculate the 95%, 1-month VaR of the 10-year bond using the duration approximation.

$$\begin{aligned}\mu_P &= -7.8 * 100 * 6.5 * 10^{-6} = 0.00507 & \sigma_P &= 7.8 * 100 * 0.50\% = 3.90 \\ \text{VaR} &= -(\mu_P - 1.645 * \sigma_P) = -(-0.00507 - 1.645 * 3.90) = 6.42 \\ &10 \text{ M} * 6.42 / 100 = 642,000\end{aligned}$$

- (c) Explain a pitfall in calculating VaR using the above approach.

The duration is appropriate for small parallel changes in the level of interest rates. However, VaR is concerned with large changes. Therefore, the duration approximation method is internally inconsistent.

10. Continued

Result of the VaR is only an approximation. The VaR does not say anything about how large the losses are expected to be if they occur.

It depends on the assumption that the underlying losses are normally distributed. If the underlying distribution is not normal, the equation does not hold. It also depends on the sample used for calculation.

The lack of subadditivity of VaR makes VaR not a consistent risk measure.

The VaR measure could be especially problematic for portfolios that include derivative securities.

- (d) Explain pitfalls in using the VaR as a risk measure and how to address them.

VaR measure does not capture well the risk embedded in the tails of the distribution.

Recommend using expected shortfall to address the pitfalls with VaR:

Expected shortfall measures how large can we expect the loss of a portfolio to be when the loss is higher than VaR. [Or a CTE at a percentage of probabilities].

The expected shortfall is very useful for addressing situations in which the portfolio losses are not expected to be normally distributed.

It is particularly important when the underlying factor has a fat-tail distribution or when the portfolio contains highly nonlinear derivative securities.

- (e) Calculate the face values of a 2-year zero-coupon bond and a 10-year zero-coupon bond you should buy or short-sell to achieve factor neutrality and immunize the 10-year coupon bond from interest rate changes in the level and slope of the yield curve.

Given data: D_1 , D_2 , and from the table, P_p , P_z^2 , P_z^{10} , $\beta_{2,1}$, $\beta_{2,2}$, $\beta_{10,1}$, $\beta_{10,2}$

Calculate:

$$D_{z1}^2 = 2 * \beta_{2,1} = 2 * 0.9791 = 1.9582$$

$$D_{z2}^2 = 2 * \beta_{2,2} = 2 * -0.2965 = -0.5930$$

$$D_{z1}^{10} = 10 * \beta_{10,1} = 10 * .8517 = 8.5170$$

$$D_{z2}^{10} = 10 * \beta_{10,2} = 10 * .6635 = 6.6350$$

10. Continued

$$K_2 = - (P_p/P_z^2) * (D_1 \times D_{z2}^{10} - D_2 \times D_{z1}^{10}) / (D_{z1}^2 \times D_{z2}^{10} - D_{z2}^2 \times D_{z1}^{10})$$

$$K_{10} = - (P_p/P_z^{10}) * (D_1 \times D_{z2}^2 - D_2 \times D_{z1}^2) / (D_{z1}^{10} \times D_{z2}^2 - D_{z2}^{10} \times D_{z1}^2)$$

$$K_2 = - (109.68/89.87) * (6.70 \times 6.635 - 4.42 \times 8.517) / (1.9582 \times 6.635 - (-0.593) \times 8.517)$$

$$K_{10} = - (109.68/62.70) * (6.70 \times (-0.5930) - 4.42 \times 1.9582) / (8.517 \times (-0.5930) - 6.6350 \times 1.9582)$$

$$K_2 = -0.4606$$

$$K_{10} = -1.22431121$$

11. Learning Objectives:

2. The candidate will understand the fundamentals of fixed income markets and traded securities.

Learning Outcomes:

- (2a) Understand the characteristics of fixed rate, floating rate, and zero-coupon bonds.
- (2c) Understand measures of interest rate risk including duration, convexity, slope, and curvature.

Sources:

Fixed Income Securities: Valuation, Risk, and Risk Management, Veronesi, Pietro, 2010, Ch. 2 & 5.

Commentary on Question:

Commentary listed underneath question component.

Solution:

- (a) Determine which curve is the spot curve and which curve is the forward curve. Justify your answer by mathematically demonstrating the key relationship between the two curves.

Commentary on Question:

Most candidates did well on this part. Other than what was provided in the solution below, candidates who provided reasonable mathematical justification for the key relationship between the two curves also received at least partial credits. Candidate who identified the correct curves without providing justification received a smaller partial credit.

The solid curve is the spot curve, and the dotted curve is the one-year forward curve derived from the spot curve.

When the spot curve is increasing, the forward curve is above the spot curve.
When the spot curve is decreasing, the forward curve is below the spot curve.
When the spot curve is flat, the forward curve equals the spot curve.

The forward rate can be denoted by:

$$f(0, T, T + \Delta) = -\frac{\ln(F(0, T, T + \Delta))}{\Delta}$$

11. Continued

The definition of the forward discount factor is given as:

$$\begin{aligned}F(0, T, T + \Delta) &= \frac{Z(0, T + \Delta)}{Z(0, T)} \\&= \frac{e^{-r(0, T + \Delta)(T + \Delta)}}{e^{-r(0, T)T}} \\&= e^{-r(0, T + \Delta)(T + \Delta) + r(0, T)T} \\&= e^{-r(0, T)\Delta - (r(0, T + \Delta) - r(0, T))(T + \Delta)}\end{aligned}$$

By substituting the forward discount factor into the forward rate formula, we have:

$$f(0, T, T + \Delta) = r(0, T) + (T + \Delta) \times \frac{r(0, T + \Delta) - r(0, T)}{\Delta}$$

This equation shows the key relation between forward rates and spot rates. For any maturity T , the forward rate between T and $T + \Delta$ equals the spot rate with maturity T plus a term that is positive if the spot curve is rising, and it is negative if the spot curve is declining at T . To see this last point, note that the numerator of the equation is the difference in the spot curve between the two maturities T and $T + \Delta$. If this difference is positive and Δ is sufficiently small, then the term structure is increasing at T . If that difference is instead negative, the spot curve is decreasing at T .

- (b) Assess whether the recommendation is appropriate:
- (i) If yes, describe how to use the model to extend the curve.
 - (ii) If no, provide your explanation.

Commentary on Question:

Overall candidates did poorly on this part. Most failed to describe how to use the model to extend the curve. Candidates who only answered “yes” or “no” without providing explanation did not receive any credit.

Yes, using the extended Nelson-Siegel model to extend the curve is appropriate. It has the flexibility to capture severe non-linearities in the shape of the term structure of interest rates.

11. Continued

To use the model to extend the curve:

(1) First, the model parameters used in the model have to be estimated from the current bond data.

(2) The continuously compounded yield with maturity T is given by

$$r(0, T) = \theta_0 + (\theta_1 + \theta_2) \frac{1 - e^{-\frac{T}{\lambda_1}}}{\frac{T}{\lambda_1}} - \theta_2 e^{-\frac{T}{\lambda_1}} + \theta_3 \left(\frac{1 - e^{-\frac{T}{\lambda_2}}}{\frac{T}{\lambda_2}} - e^{-\frac{T}{\lambda_2}} \right)$$

(3) For given parameter values $(\theta_0, \theta_1, \theta_2, \theta_3, \lambda_1, \lambda_2)$, it is possible to compute the value of bond prices implied by the Nelson Siegel model.

(4) An optimization process can be implemented to minimize the fitting errors of these bond prices.

(5) After all the model parameters are determined, the curve can then be constructed and extended by plugging in time of maturities up to 30 years.

12. Learning Objectives:

2. The candidate will understand the fundamentals of fixed income markets and traded securities.

Learning Outcomes:

- (2a) Understand the characteristics of fixed rate, floating rate, and zero-coupon bonds.
- (2c) Understand measures of interest rate risk including duration, convexity, slope, and curvature.

Sources:

Fixed Income Securities: Valuation, Risk, and Risk Management, Veronesi, Pietro, 2010
Ch. 2, Ch.3, and Ch.4

Commentary on Question:

This question tested understanding on the inverse floater. The key insight for candidates to grasp was that the inverse floater can be replicated by a portfolio of zero-coupon bonds, coupon bond, and floating rate bonds. To get maximum points, a candidate needed to correctly state the replicating portfolio (long three 3-year zero-coupon bonds and one 3-year coupon bond, and short three 3-year floating rate bonds) and correctly calculate the price of each bond. Alternatively, candidates could get maximum points by correctly calculating $r(1)$, $r(2)$, $r(3)$ then $c(1)$, $c(2)$, $c(3)$ to get the desired answer.

For part (b), candidates could get the portfolio duration by calculating the duration of each bond, and needed to demonstrate that the inverse floater can be replicated by a portfolio of zero-coupon bonds, coupon bond, and floating rate bonds as well. If candidate calculated part (a) using the alternative method and applied the same technique to calculate duration, then duration in (b) wouldn't be correct.

Solution:

- (a) Calculate the price of the leveraged inverse floater on 12/31/2018.

$$\text{Price leveraged inverse floater} = 3 * P_z(0,3) + P_c(0,3) - 3 * P_{FR}(0,3)$$

$P_z(0,3)$, $P_c(0,3)$, and $P_{FR}(0,3)$ denote the prices of a zero coupon bond, a coupon bond, and a floating rate bond with three years to maturity.

Given the discount factor $Z(0, 3) = 0.915$

$$P_z(0,3) = 100 * 0.915 = \$91.50$$

Given the discount factors $Z(0, 1)$, $Z(0, 2)$, $Z(0, 3)$ and $c = 15\%$, we can calculate the price of the fixed coupon bond $P_c(0,3)$.

12. Continued

Date	T	Discount Factors	Cash Flow	Disc. CF
12/31/2019	1	0.98	\$15.00	\$14.70
12/31/2020	2	0.95	\$15.00	\$14.25
12/31/2021	3	0.915	\$115.00	\$105.23
				\$134.18

Thus, $P_c(0,3) = \$134.18$.

Therefore

$$\begin{aligned} \text{Price leveraged inverse floater} &= 3 * P_z(0,3) + P_c(0,3) - 3 * P_{FR}(0,3) \\ &= \$274.50 + 134.18 - \$300 = \$108.675 \end{aligned}$$

Alternatively:

Date	T	Discount Factors	r(t)	c(t)	Discount CF
12/31/2019	1	0.98	$\frac{1}{0.98} - 1 = 0.02041$	8.888	8.700
12/31/2020	2	0.95	$\frac{0.98}{0.95} - 1 = 0.03158$	5.526	5.250
12/31/2021	3	0.915	$\frac{0.95}{0.915} - 1 = 0.03825$	103.525	94.725
					108.675

- (b) Calculate the duration of the leveraged inverse floater on 12/31/2018.

Commentary on Question:

The duration of the inverse floater is calculated from durations of the replicating portfolio.

To compute the duration of the leveraged inverse floater, we need to compute the duration of the coupon bond. The numbers in the last two columns below need to be computed.

Date	T	Discount Factors	Cash Flow	Disc. CF	Weight w	W * T
12/31/2019	1	0.98	\$15.00	\$14.70	0.1096	0.1096
12/31/2020	2	0.95	\$15.00	\$14.25	0.1062	0.2124
12/31/2021	3	0.915	\$115.00	\$105.23	0.7842	2.3527
				\$134.18		2.6747

12. Continued

Consider simplifying the calculations:

Security	Value	Weight w	Duration D	D * w
3 x $P_z(3)$	\$274.50	253%	3.00	7.58
$P_c(3)$	\$134.18	123%	2.6747	3.30
-3 x $P_{FR}(3)$	-\$300.00	-276%	1.00	-2.76
	\$108.68			8.12

So the leveraged inverse floater has a duration of 8.12, almost three times its maturity.

13. Learning Objectives:

1. The candidate will understand the foundations of quantitative finance.
2. The candidate will understand the fundamentals of fixed income markets and traded securities.
3. The candidate will understand:
 - The Quantitative tools and techniques for modeling the term structure of interest rates.
 - The standard yield curve models.
 - The tools and techniques for managing interest rate risk.

Learning Outcomes:

- (1d) Understand and apply Ito's Lemma.
- (1h) Define and apply the concepts of martingale, market price of risk and measures in single and multiple state variable contexts.
- (2a) Understand the characteristics of fixed rate, floating rate, and zero-coupon bonds.
- (3b) Understand and apply various one-factor interest rate models.
- (3h) Understand the application of Monte Carlo simulation to risk neutral pricing of interest rate securities.
- (3j) Understand and apply multifactor interest rate models including the two-factor Hull-White model.

Sources:

Fixed Income Securities: Valuation, Risk, and Risk Management, Veronesi, Pietro, 2010 (Ch. 2, 15, 17, 22)

An Introduction to the Mathematics of Financial Derivatives, Hirsa, Ali and Neftci, Salih N., 3rd Edition 2nd Printing, 2014 (Ch. 10)

Commentary on Question:

This question tested candidates' knowledge of Ito's Lemma, martingale, interest rate modelling, and one-factor and two-factor Vasicek models. Most candidates performed below expectation on this question.

Solution:

- (a) Identify the steps for estimating the fair value of an Inverse Floater using Monte Carlo simulation.

13. Continued

Commentary on Question:

In general, candidates performed below expectation on this part. Most candidates only included “Simulating path of interest rates...” and/or “Taking average of present values...” in the answers. They did not provide details with correct formulas of simulations of path of interest rates and how the present values of cashflows were calculated. Only a few candidates included the formulas for computing the cashflow under inverse floater.

1) Simulating one path of interest rates r_t^j from time 0 to maturity. Repeating the simulation of interest rates J times, thus obtaining J paths of interest rates where

$$r_{i+1}^j = r_i^j + \gamma^*(\bar{r} - r_i^j)\delta + \sigma\sqrt{\delta} \epsilon_{i+1}^j \text{ where } i \text{ denotes the step}$$
$$\delta = \frac{1}{360}$$

Although, these are daily rates, we treat them as continuously compounded quarterly rates as approximation error is small.

2) For each interest rate path j , $j = 1, \dots, J$, we compute the sequence of quarterly cash flows by applying formula in

$$CF(t) = \text{Notional} \times \text{Max}(0, (\text{Base Interest Rate} - LIBOR_{t-1})) \times \text{Accrual Factor}$$

(between $t - 1$ and t) / Number of days

Because the cash flow depends on the LIBOR, while we are simulating a continuously compounded (short term) interest rate, we need to transform it to LIBOR, according to the formula

$$LIBOR_i^j = \frac{360}{90} \left(e^{r_i^j \times 0.25} - 1 \right)$$

3) Given the daily sequence of $LIBOR_i^j$, we can compute the number of days the simulated $LIBOR_i^j$.

Let i^* be a coupon date in the simulation, and assume 360 days in one year, so that a coupon date is every 90 days. Denote the coupon date sequentially, for simplicity i.e. $i^*=1,2,3,\dots,12$ although each occurs every 90 days.

$$CF_{i^*}^j = \text{Notional} \times \text{Max}((\text{Base Interest Rate} - LIBOR_{i^*-1}^j), 0) \times \text{Accrual Factor}$$

(between i^*-1 and i^*) / 360

13. Continued

4) For each simulation j complete the present value of the sequence of cash flows for each simulation

$$V_0^j = \sum_{i^*=1}^{12} CF_{i^*} / \left[\left(1 + \frac{LIBOR_0^j}{4}\right) x \dots x \left(1 + \frac{LIBOR_2^j}{4}\right) x \dots x \left(1 + \frac{LIBOR_{i^*-1}^j}{4}\right) \right]$$

where summation is carried out over all coupon cashflows.

5) Taking the average of the present values

$$\hat{V} = \frac{1}{J} \sum_{j=1}^J V_0^j$$

where the hat “^” indicates that this value is an approximation of the true price of the asset obtained by Monte Carlo simulations.

(b) Show that the long-term rate $r_t(\tau)$ under the single factor Vasicek model follows the SDE

$$dr_t(\tau) = \frac{B(\tau)}{\tau} \gamma (\bar{r} - r_t) dt + \frac{B(\tau)}{\tau} \sigma dX_t$$

where t is the current time, T is the maturity date, $\tau = T - t$ is the time to maturity, and $B(\tau) = \frac{1 - \exp(-\gamma\tau)}{\gamma}$.

Commentary on Question:

In general candidates performed as expected on this part. Full credits were given for the alternative solution.

$$dr_t = \gamma(\bar{r} - r_t)dt + \sigma dX_t$$

$$Z(r_t; t, T) = \exp(A(t; T) - r_t B(t; T))$$

Write $\tau = T - t$

$$Z(r_t, \tau) = Z(r_t; t, T) = \exp(A(0; \tau) - r_t B(0; \tau))$$

Write $A(\tau) = A(0; \tau), B(\tau) = B(0; \tau)$

$$Z(r_t, \tau) = \exp(A(\tau) - r_t B(\tau))$$

$$r_t(\tau) = -\frac{\ln(Z(r_t, \tau))}{B(\tau)} = -\frac{A(\tau)}{B(\tau)} + \frac{B(\tau)}{B(\tau)} r_t$$

13. Continued

Applying Ito's lemma to $r_t(\tau)$ and keep τ as fixed in this exercise

$$\frac{\partial r_t(\tau)}{\partial r_t} = \frac{B(\tau)}{\tau}, \frac{\partial^2 r_t(\tau)}{\partial r_t^2} = 0, \frac{\partial r_t(\tau)}{\partial t} = 0$$

$$dr_t(\tau) = [0 + \frac{\partial r_t(\tau)}{\partial r_t} \gamma(\bar{r} - r_t) + \frac{\partial^2 r_t(\tau)}{\partial r_t^2} \sigma^2 / 2] dt + \frac{\partial r_t(\tau)}{\partial r_t} \sigma dX_t$$

$$dr_t(\tau) = \frac{B(\tau)}{\tau} \gamma(\bar{r} - r_t) dt + \frac{B(\tau)}{\tau} \sigma dX_t$$

$$\text{Alternatively, } r_t(\tau) = -\frac{\ln(Z(r_t, \tau))}{\tau} = -\frac{A(\tau)}{\tau} + \frac{B(\tau)}{\tau} r_t$$

$$dr_t(\tau) = \frac{B(\tau)}{\tau} dr_t = \frac{B(\tau)}{\tau} \gamma(\bar{r} - r_t) dt + \frac{B(\tau)}{\tau} \sigma dX_t$$

- (c) Show that the long-term rate is less volatile than short-term rate under the single factor Vasicek model.

Commentary on Question:

Most candidates performed as expected on this part. Credits were given if they mentioned $\frac{B(\tau)}{\tau} < 1$ or $\frac{B(\tau)}{\tau}$ decreases with maturity or provided correct proof of

$$\frac{d\frac{B(\tau)}{\tau}}{d\tau} < 0 \text{ or } \lim_{\tau \rightarrow \infty} \frac{B(\tau)}{\tau} = 0$$

Based on part (b), the short term volatility = σ ,

the long term volatility = $\frac{B(\tau)}{\tau} \sigma$

As $\frac{B(\tau)}{\tau} < 1$ and $\frac{B(\tau)}{\tau}$ decreases with maturity, implies that longer-term rates move more slowly than the short-term rate.

- (d) Derive the PDE for $V(t, r_t, S_t)$ using Ito's lemma without explicitly stating the terminal and boundary conditions.

Commentary on Question:

In general, candidates performed below expectation on this part. Most candidates could provide the correct formula under multivariate Ito's lemma. However, only some candidates could carry out the derivation.

$$dr_t^2 = \sigma^2 dt$$

$$dS_t^2 = \sigma_E^2 S_t^2$$

$$dS_t dr_t = \rho \sigma \sigma_E S_t dt$$

13. Continued

In addition, some candidates did not evaluate properly for $d\left(e^{-\int_0^t r_s ds} V_t\right)$.

By multivariate Ito's lemma,

$$\begin{aligned} & dV(t, r_t, S_t) \\ &= \frac{\partial V}{\partial t} dt + \frac{\partial V}{\partial r_t} dr_t + \frac{\partial V}{\partial S_t} dS_t + \frac{1}{2} \frac{\partial^2 V}{\partial r_t^2} dr_t^2 + \frac{1}{2} \frac{\partial^2 V}{\partial S_t^2} dS_t^2 + \frac{\partial^2 V}{\partial S_t \partial r_t} dS_t dr_t \\ &= \left[\frac{\partial V}{\partial t} + \frac{\partial V}{\partial r_t} \gamma(\bar{r} - r_t) + \frac{\partial V}{\partial S_t} r_t S_t + \frac{1}{2} \frac{\partial^2 V}{\partial r_t^2} \sigma^2 + \frac{1}{2} \frac{\partial^2 V}{\partial S_t^2} \sigma_E^2 S_t^2 + \frac{\partial^2 V}{\partial S_t \partial r_t} \rho \sigma \sigma_E S_t \right] dt + \\ & \quad \frac{\partial V}{\partial r_t} \sigma dX_t + \frac{\partial V}{\partial S_t} \sigma_E S_t dW_t \end{aligned}$$

where $dW_t^2 = dX_t^2 = dt$ and $dt^2 = dt dW_t = dt dX_t = 0$ in the mean square limits

$$\begin{aligned} dr_t^2 &= \gamma^2(\bar{r} - r_t)^2 dt^2 + \sigma^2 dX_t^2 + 2\gamma(\bar{r} - r_t) dt \sigma dX_t = 0 + \sigma^2 dt + 0 \\ dS_t^2 &= r_t^2 S_t^2 dt^2 + \sigma_E^2 S_t^2 dW_t^2 + 2r_t S_t dt \sigma_E S_t dW_t = 0 + \sigma_E^2 S_t^2 dt + 0 \\ dS_t dr_t &= \rho \sigma \sigma_E S_t dt \end{aligned}$$

as $E[(dW_t dX_t)] = \rho dt$ and

$$dW_t^2 = dX_t^2 = dt \text{ and } dt^2 = dt dW_t = dt dX_t = 0$$

We note that under risk-neutral measure, $\{e^{-\int_0^t r_s ds} V_t, t \geq 0\}$ is a martingale.

$$\begin{aligned} & d\left(e^{-\int_0^t r_s ds} V_t\right) = -r_t \left(e^{-\int_0^t r_s ds} V_t\right) dt + \left(e^{-\int_0^t r_s ds} dV_t\right) \\ &= e^{-\int_0^t r_s ds} \left\{ \left[-r_t V_t + \frac{\partial V}{\partial t} + \frac{\partial V}{\partial r_t} \gamma(\bar{r} - r_t) + \frac{\partial V}{\partial S_t} r_t S_t + \frac{1}{2} \frac{\partial^2 V}{\partial r_t^2} \sigma^2 + \frac{1}{2} \frac{\partial^2 V}{\partial S_t^2} \sigma_E^2 S_t^2 \right. \right. \\ & \quad \left. \left. + \frac{\partial^2 V}{\partial S_t \partial r_t} \rho \sigma \sigma_E S_t \right] dt + \frac{\partial V}{\partial r_t} \sigma dX_t + \frac{\partial V}{\partial S_t} \sigma_E S_t dW_t \right\} \end{aligned}$$

By martingale property, the drift term = 0

$$\begin{aligned} & -r_t V_t + \frac{\partial V}{\partial t} + \frac{\partial V}{\partial r_t} \gamma(\bar{r} - r_t) + \frac{\partial V}{\partial S_t} r_t S_t + \frac{1}{2} \frac{\partial^2 V}{\partial r_t^2} \sigma^2 + \frac{1}{2} \frac{\partial^2 V}{\partial S_t^2} \sigma_E^2 S_t^2 \\ & \quad + \frac{\partial^2 V}{\partial S_t \partial r_t} \rho \sigma \sigma_E S_t = 0 \end{aligned}$$

This is the PDE satisfied by the instrument price.

- (e) Show, using Ito's lemma, that the term structure steepens as $\Phi_{2,t}$ increases, in addition to its movement implied by r_t .

13. Continued

Commentary on Question:

In general, candidates performed below expectation on this part. Most candidates did not include the following equality. Only a few candidates had the proper explanation.

$$dr_t = [\gamma_1^*(\bar{\phi}_1^* - r_t) + \gamma_2^*\bar{\phi}_2 + (\gamma_1^* - \gamma_2^*)\phi_{2,t}]dt + \sigma_1 dX_{1,t} + \sigma_2 dX_{2,t}$$

Using Ito's Lemma on $r_t = \phi_{1,t} + \phi_{2,t}$, we obtain

$$\begin{aligned} dr_t &= d\phi_{1,t} + d\phi_{2,t} \\ dr_t &= \gamma_1^*(\bar{\phi}_1^* - \phi_{1,t})dt + \sigma_1 dX_{1,t} + \gamma_2^*(\bar{\phi}_2^* - \phi_{2,t})dt + \sigma_2 dX_{2,t} \\ dr_t &= [\gamma_1^*(\bar{\phi}_1^* - (r_t - \phi_{2,t})) + \gamma_2^*(\bar{\phi}_2^* - \phi_{2,t})]dt + \sigma_1 dX_{1,t} + \sigma_2 dX_{2,t} \\ dr_t &= [\gamma_1^*(\bar{\phi}_1^* - r_t) + \gamma_2^*\bar{\phi}_2 + (\gamma_1^* - \gamma_2^*)\phi_{2,t}]dt + \sigma_1 dX_{1,t} + \sigma_2 dX_{2,t} \end{aligned}$$

Assuming $\gamma_1^* - \gamma_2^* > 0$, for instance, then for given current state r_t , when the second factor $\phi_{2,t}$ increases, the risk neutral expectation of future short-term rate increases, which in turn implies a steepening of the term structure of interest rates (i.e. $\phi_{2,t}$ affects the slope of the term structure, in addition to its movement implied by r_t)

- (f) Explain why the two-factor Vasicek model with independent factors fails to simultaneously explain the market prices of all caps and all swaptions.

Commentary on Question:

In general, candidates performed poorly on this part. Most candidates only mentioned that two-factor Vasicek model with independent factors does not explain correlation between short and long rates. A small group of candidates mentioned that the model does not capture the joint movement of the two factors or does not fit volatility well.

This stems from the inability of two factor Vasicek model with independent factors to generate low enough correlation between changes in adjacent forward rates to match the market data.

Two-factor Vasicek model with independent factors does not capture the joint movement of the two factors.

While Vasicek model with independent factors fits the two extremes of the volatility, it does not do so well for intermediate maturities.

14. Learning Objectives:

3. The candidate will understand:
 - The Quantitative tools and techniques for modeling the term structure of interest rates.
 - The standard yield curve models.
 - The tools and techniques for managing interest rate risk.

Learning Outcomes:

- (3b) Understand and apply various one-factor interest rate models.

Sources:

Fixed Income Securities: Valuation, Risk, and Risk Management, Veronesi, Pietro, 2010

- Ch. 15.1-15.6 (exclude Appendices)
- Ch. 16.1-16.8 (exclude Appendix)

Commentary on Question:

The question tested candidates' understanding of one-factor interest rate models in a practical context. The question was meant to be straightforward, however, only a small portion of the candidates demonstrated a good understanding of relative value trade and dynamic rebalancing.

Solution:

- (a) Calculate the fitted bond prices for the two bonds using the Vasicek model.

Commentary on Question:

Candidates performed brilliantly on this part.

Use formula below to first calculate $Z(T)$:

$$Z(r_t, t; T) = e^{A(t;T) - B(t;T)r_t}$$

where risk neutral parameters are used in $A(T)$ and $B(T)$.

As given in the tables, use $B(r_0, 0; 0.5) = 0.4477$ and $A(r_0, 0; 0.5) = -0.00166$

$Z(r_0, 0; 0.5) = 98.90$ for \$100 principal

$$P(r_0, 0; 0.5) = 100 * c / 2 * Z(r_0, 0; 0.5) + 100 * Z(r_0, 0; 0.5) = (2.5 * 98.90 + 100 * 98.90) / 100 = 101.37$$

Use $B(r_0, 0; 1) = 0.8053$, $A(r_0, 0; 1) = -0.00615$

$Z(r_0, 0; 1) = 97.72$ for \$100 principal

$$P(r_0, 0; 1) = 100 * c / 2 * Z(r_0, 0; 0.5) + 100 * c / 2 * Z(r_0, 0; 1) + 100 * Z(r_0, 0; 1) = (3 * 98.90 + 103 * 97.72) / 100 = 103.62$$

14. Continued

- (b)
- (i) Identify any mispricing,
 - (ii) Propose a relative value trade,
 - (iii) Calculate the replicating portfolio required in the trade.

Commentary on Question:

Most candidates were able to identify the mispricing and proposed the correct relative value trade. Not many candidates demonstrated sufficient understanding of the replicating portfolio.

Invoice price = (bid price + ask price) / 2
(Comparing against ask/sell price is also acceptable for the mispricing analysis.)

Bond Reference	Price	Invoice Price
#1	101.37	101.35
#2	103.62	103.80

Mispricing = invoice price - fitted price = 103.80 - 103.62 = 18 bps for bond#2

Bond #2 is overpriced by the market.

Sell bond #2, buy a portfolio that replicates Vasicek bond#2 using bond #1 in the replicating portfolio.

Calculate the derivative of P:

$$\frac{\partial P(r, 0)}{\partial r} = -\frac{100c}{2} \sum_{i=1}^n B(0; T_i) Z(r, 0; T_i) - 100B(0; T_n) Z(r, 0; T_n)$$

$$\frac{\partial P_{1\text{-year}}(r, 0)}{\partial r} = -3(B(0; 0.5)Z(0; 0.5) + B(0; 1)Z(0; 1)) - 100B(0; 1)Z(0; 1)$$

$$= -82.3803$$

$$\frac{\partial P_{0.5\text{-year}}(r, 0)}{\partial r} = -2.5B(0; 0.5)Z(0; 0.5) - 100B(0; 0.5)Z(0; 0.5) = -45.3885$$

Calculate the initial position required in the 0.5 year bond:

$$\Delta = \frac{\frac{\partial P_{1\text{-year}}(r, 0)}{\partial r}}{\frac{\partial P_{0.5\text{-year}}(r, 0)}{\partial r}} = \frac{82.3803}{45.3885} = 1.82$$

14. Continued

Calculate the cash position:

For each bond #2 sold, we should purchase 1.82 unit of the 0.5 year note, therefore the borrowed cash position is:

$$C = -103.62 + 1.82 * 101.35 = 80.83$$

- (c) Describe how the relative value trade can be carried out through the repo market and calculate the return on capital for this trade.

Commentary on Question:

Only were some candidates able to correctly describe the mechanics of repo market. Very few correctly applied the concept of the return on capital introduced in the textbook.

To short the note, borrow the note from a repo dealer, sell it to the market, and post cash collateral with the repo dealer, earning the rate on deposit.

A=calculate the initial lock in profit = Sell note at invoice price and buy the replicating portfolio (the Vasicek price of the bond)

$$\text{Initial profit} = A = 103.80 - 103.62 = 0.18$$

B=calculate the effective capital in the trade using the haircut.

2.5%*unit of bond purchased*unit price of bond purchased

$$B = 2.5\% * 1.82 * 101.35 = 4.61$$

C=Return on capital = A/B

$$0.18 / 4.61 = 3.9\%$$

- (d) Describe the dynamic rebalancing activities and calculate the cash position of the replicating portfolio at $t = 0.5$.

Commentary on Question:

This part was not meant to be tricky, but many candidates did not attempt it. Very few candidates were able to describe any of the rebalancing activities.

At $t = 0.5$, we receive the principal and coupon payment of $1.82 * 102.5 = 186.55$ in cash. We have to pay for the coupon of \$3 from the bond we are shorting.

We borrowed cash of 80.83 at time 0. With interest, this cash position is $80.83 / Z(0.5) = 80.83 / 0.9890 = 81.73$ (net borrow)

We also need to buy one unit of the 0.5 year at market price of 101.20

14. Continued

New replicating portfolio is a long 0.5 year bond and a cash position.

The cash position in the replicating portfolio is $-81.73 + 183.55 - 101.50 = 0.32$

- (e) Calculate the overall profit from time $t = 0$ to time $t = 1$.

Commentary on Question:

Very few candidates attempted to calculate the overall profit.

At time 1, the principal and coupon from the long bond in the replicating portfolio will cover the short bond we shorted at time 0 exactly with no profit or loss.

We will make interest on the positive cash position between time 0.5 and time 1. The forward rate is $Z(0.5)/Z(1) = 98.90/97.72 = 1.012$. The cash will grow to $0.32 * (1.012) = 0.32$

The initial profit also earned interest over the full year.
 $0.18/Z(1) = 0.18/0.9772 = 0.184$

The overall profit is the sum of the items above = $0.32 + 0.184 = 0.50$

15. Learning Objectives:

1. The candidate will understand the foundations of quantitative finance.
2. The candidate will understand the fundamentals of fixed income markets and traded securities.
3. The candidate will understand:
 - The Quantitative tools and techniques for modeling the term structure of interest rates.
 - The standard yield curve models.
 - The tools and techniques for managing interest rate risk.

Learning Outcomes:

- (1c) Understand Ito integral and stochastic differential equations.
- (1h) Define and apply the concepts of martingale, market price of risk and measures in single and multiple state variable contexts.
- (1i) Demonstrate understanding of the differences and implications of real-world versus risk-neutral probability measures, and when the use of each is appropriate.
- (1k) Understand the importance of the Feynman-Kac Theorem.
- (2c) Understand measures of interest rate risk including duration, convexity, slope, and curvature.
- (2d) Understand the characteristics and uses of interest rate forwards, swaps, futures, and options.
- (3a) Understand and apply the concepts of risk-neutral measure, forward measure, normalization, and the market price of risk, in the pricing of interest rate derivatives.
- (3b) Understand and apply various one-factor interest rate models.
- (3e) Demonstrate understanding of option pricing theory and techniques for interest rate derivatives.
- (3f) Apply the models to price common interest sensitive instruments including: callable bonds, bond options, caps, floors, and swaptions.

Sources:

Fixed Income Securities: Valuation, Risk, and Risk Management, Veronesi, Pietro, 2010 chapters 14-21

15. Continued

Commentary on Question:

The question tested knowledge and application of interest rate model in its derivation of distribution parameters and sensitivities of interest rate on the bond. The candidates who demonstrated analytic capability and understanding of forward risk neutral measure technique could get a high credits.

Solution:

(a) Derive formulas for the spot-rate duration and convexity of $Z(r_t, t; T)$ when

(i) $\gamma \neq 0$

(ii) $\gamma = 0$

Commentary on Question:

Many candidates didn't get part (ii).

(i) $\gamma \neq 0$

From the formula, the zero coupon duration and convexity

$$D = -\frac{1}{P} \frac{\partial P}{\partial r}$$
$$\frac{\partial P}{\partial r} = -B(t; T) \exp[A(t; T) - B(t; T) \times r]$$
$$D = B(t; T) = \frac{1}{\gamma} (1 - e^{-\gamma(T-t)})$$
$$C = \frac{1}{P} \frac{\partial^2 P}{\partial r^2}$$
$$\frac{\partial^2 P}{\partial r^2} = B(t; T)^2 \exp[A(t; T) - B(t; T) \times r]$$
$$C = B(t; T)^2$$

(ii) $\gamma = 0$

The stochastic process for the interest rate is

$$dr = \sigma dX$$

This is the drift less Ho-Lee model and the bond pricing formula is given in (14.42):

$$P = \exp\left(\frac{\sigma^2}{6} (T-t)^3 - (T-t)r_t\right) \text{ and } B(t, T) = (T-t)$$

$$D = -\frac{1}{P} \frac{\partial P}{\partial r} = (T-t)$$
$$C = \frac{1}{P} \frac{\partial^2 P}{\partial r^2} = (T-t)^2$$

Another method would be to calculate limiting value of $B(t, T)$ of Vasicek at $\gamma \rightarrow 0$

15. Continued

- (b) Explain the impact of increasing γ on the duration and convexity.

Commentary on Question:

Most candidates explained when $\gamma > 0$.

Comparing durations

$$\frac{1}{\gamma}(1 - e^{-\gamma(T-t)}) < (T - t)$$

when $\gamma > 0$. Impacts are:

- Duration for the mean reverting interest rate always lower than non mean reverting model and so as convexity.
- The intensity of mean reversion reduces duration so as convexity. The more intense the mean reversion, less the price movement

When $\gamma < 0$,

$$\frac{1}{\gamma}(1 - e^{-\gamma(T-t)}) > (T - t)$$

the impacts are opposite.

In overall, the bond price movement on interest rate depends on level of mean reversion and models to choose.

- (c) Show that this contract can be characterized as a put option on a zero-coupon bond with appropriate notional amount and maturity.

Commentary on Question:

Many candidates did right.

Pay off at time $T + 1$, of the caplet = $N(r(T, T + 1) - r_K)_+$

$$r(T, T + 1) = \frac{1}{Z(r_T, T; T + 1)} - 1$$

The discounted value of the payoff at time T

$$\begin{aligned} & Z(r_T, T; T + 1)N(r(T, T + 1) - r_K)_+ \\ &= Z(r_T, T; T + 1)N\left(\frac{1}{Z(r_T, T; T + 1)} - (1 + r_K)\right)_+ \\ &= N(1 + r_K)\left(\frac{1}{1 + r_K} - Z(r_T, T; T + 1)\right)_+ \end{aligned}$$

15. Continued

- (d) Determine the distribution of r_t .

Commentary on Question:

Most candidates got right on the variance only and under prepared for the forward risk-neutral or forward measure technique.

$$dr_t = \sigma dX_t$$

$$\sigma_Z(r, t) = \frac{1}{Z} \frac{\partial Z}{\partial r} \sigma = -(T - t)\sigma$$

Using Text (21.9) under T_0 forward risk-neutral dynamics dr_t is with

$$m^*(r, t) = 0, \quad \sigma_Z(r, t) = -(T - t)\sigma, \quad s(r, t) = \sigma$$

$$dr_t = -(T - t)\sigma^2 dt + \sigma dX_t$$

$$r_t = r_0 + \frac{(T - t)^2}{2} \sigma^2 - \frac{T^2}{2} \sigma^2 + \sigma dX_t$$

Therefore the interest rate r_t has a normal distribution with mean

$$r_0 + \frac{\sigma^2 t}{2} (t - 2T)$$

and variance $\sigma^2 t$.

- (e) Calculate the mean of $Z(r_T, T; T_B)$.

Commentary on Question:

This part is a continuation of part (d). Likewise, most candidates didn't get right for the mean. Candidates could get credits by showing the final parametric form or deriving and explaining forward price.

Under T-forward risk neutral dynamics $\frac{Z(r_t, t; T_B)}{Z(r_t, t; T)}$, $t \leq T < T_B$ is a martingale.

$$\mathbb{E}_f^* \left[\frac{Z(r_T, T; T_B)}{Z(r_T, T; T)} \right] = \frac{Z(0, T_B)}{Z(0, T)} = F(0, T, T_B)$$

where $F(0, T, T_B)$ is the forward price for $Z(r_T, T; T_B)$. Or it can be shown as

$$Z(0, T_B) = e^{A(0, T_B) - T_B \times r_0}, \quad Z(0, T) = e^{A(0, T) - T \times r_0},$$

$$\frac{Z(0, T_B)}{Z(0, T)} = e^{A(0, T_B) - A(0, T) - (T_B - T) \times r_0}$$

$$A(0, T) = \frac{\sigma^2}{6} T^3,$$

$$\frac{Z(0, T_B)}{Z(0, T)} = e^{\frac{\sigma^2}{6} (T_B^3 - T^3) - (T_B - T) \times r_0}$$

15. Continued

- (f) Calculate the variance of $\ln[Z(r_T, T; T_B)]$.

Commentary on Question:

This part could be done without knowledge of forward measure. Many candidates got some credits.

$$Z(r_T, T; T_B) = \exp\left(\frac{\sigma^2}{6}(T_B - T)^3 - (T_B - T)r_T\right)$$

$$\ln(Z(r_T, T; T_B)) = \left(\frac{\sigma^2}{6}(T_B - T)^3 - (T_B - T)r_T\right)$$

Since r_T is normal with variance $\sigma^2 T$

$$S_Z(T, T_B)^2 = \text{Var}(\ln(Z(r_T, T; T_B))) = (T_B - T)^2 \text{Var}(r_T) = (T_B - T)^2 \sigma^2 T$$

16. Learning Objectives:

5. The candidate will understand important quantitative techniques relating to financial time series, volatility modeling, and stochastic modeling.

Learning Outcomes:

- (5a) Understand and apply various techniques for analyzing conditional heteroscedastic models including ARCH and GARCH.
- (5b) Demonstrate an understanding of the concept of a factor model in the context of financial time series.

Sources:

Analysis of Financial Time Series, Tsay, Ruey S., 3rd Edition, 2010

QFIC-109-15: Chapter 9 of Risk Management and Financial Institutions, Hull, 2nd Edition

Commentary on Question:

For part (a) candidates were expected to know the key characteristics for volatility.

For parts (b) & (c)

1. Candidates should be able to read ACF and PACF pattern, understand their meaning and know the difference of the two.

2. They were also expected to justify the usage of ARCH model (instead of an AR model) in the modeling of return volatility given the heteroscedastic on the squared residual of the return.

3. They were expected to know PACF is a powerful tool in selecting ARCH order.

For part (d) candidates were expected to know common characteristics for GARCH and ARCH model.

For parts (e) & (f)

1. Candidates were expected to know how to estimate the parameters using MLE.

2. Partial credits were given if candidates were unable to get to the final answer but showed their understanding of MLE.

For part (g) candidates were expected to know how to read the result of t-Test and Ljung-Box test.

Solution:

- (a) Describe briefly four commonly observed characteristics of the volatility of asset returns.

Volatility Clusters: volatility is high for certain time periods and low for certain periods.

Jump is rare: volatilities evolves over time in a constant manner, and the jump is rare.

Volatility is finite: volatility doesn't diverge to infinity, instead it varies within a fixed range.

Leverage Effect: volatility react differently to big price drop or big price increase.

16. Continued

- (b) Critique your assistant's suggestion by analyzing the pattern of graphs A and B

Graph A: The ACF pattern shows strong serial correlations at lag 1-4, 7-11 for the squared return residuals

Graph B: The PACF pattern suggests strong linear dependence at lag 1-3 and 7-10. The PACF pattern implies

$$a_t^2 = \alpha_0 + \sum_{i=1}^m \alpha_i a_{t-i}^2$$

As a_t^2 is an unbiased estimate of σ_t^2 , for a given sample, we expect σ_t^2 to be linearly related to a_t^2 , where $i=1$ to m . This leads to an ARCH(m) model

$$\sigma_t^2 = \alpha_0 + \sum_{i=1}^m \alpha_i a_{t-i}^2$$

The assistant's suggestion is appropriate.

- (c) Recommend the order m of the ARCH model assuming an ARCH model is used. Justify your recommendation.

PACF pattern suggests a ARCH (10) model as strong linear dependence is observed at lag 1-3 and 7-10.

- (d) List three strengths and three weaknesses of ARCH and GARCH models in relation to modeling the volatility.

Strengths:

1. Reflect key volatility characteristics: mean reversion (p. 189 of QFIC-109-15: Ch 9)
2. Reflect key volatility characteristics: volatility clustering (p. 116)
3. Tail distribution are heavier than normal distribution (p. 118, 133)

Weaknesses:

1. Response equally to negative and positive shocks, which is in contrast with how the price of financial assets responds in practice (p. 119)
2. Doesn't provide new insights for understanding the source of variances of the return on ABC (p. 119)
3. Some restrictions to parameters to ensure the existence of certain moments (e.g., $0 \leq \alpha_1^2 < \frac{1}{3}$ for ARCH(1)) (p. 118-119)

Note that candidates can also provide other advantages/disadvantages that are stated in the reference texts.

16. Continued

- (e) Show that $\sigma_t^2 = \sigma_1^2 \beta^{t-1} + \sum_{j=1}^{t-1} (\omega + \alpha a_j^2) \beta^{t-j-1}$ for $t = 2, 3, \dots, m$

Commentary on Question:

Full points were given to candidates showing complete derivation of the steps. Partial points were given to showing only subset of the steps. Full proof by induction was also awarded full points.

$$\begin{aligned}
 \text{Given } \sigma_t^2 &= \omega + \alpha a_{t-1}^2 + \beta \sigma_{t-1}^2 \\
 \sigma_t^2 &= \omega + \alpha a_{t-1}^2 + \beta \{ \omega + \alpha a_{t-2}^2 + \beta \sigma_{t-2}^2 \} \\
 &= \omega + \alpha a_{t-1}^2 + \beta \{ \omega + \alpha a_{t-2}^2 + \beta [\omega + \alpha a_{t-3}^2 + \beta \sigma_{t-3}^2] \} \\
 &= \omega + \alpha a_{t-1}^2 + \beta \{ \omega + \alpha a_{t-2}^2 + \beta [\omega + \alpha a_{t-3}^2 + \beta (\omega + \alpha a_{t-4}^2 + \sigma_{t-4}^2)] \} \\
 &= \beta^0 (\omega + \alpha a_{t-1}^2) + \beta^1 (\omega + \alpha a_{t-2}^2) + \beta^2 (\omega + \alpha a_{t-3}^2) + \beta^3 (\omega + \alpha a_{t-4}^2) + \beta^4 \sigma_{t-4}^2 \\
 &= \beta^{t-1-(t-1)} (\omega + \alpha a_{t-1}^2) + \beta^{t-1-(t-2)} (\omega + \alpha a_{t-2}^2) + \beta^{t-1-(t-3)} (\omega + \alpha a_{t-3}^2) \\
 &\quad + \beta^{t-1-(t-4)} (\omega + \alpha a_{t-4}^2) + \beta^{t-1-(t-5)} \sigma_{t-4}^2 \\
 &= \beta^{t-1-(t-1)} (\omega + \alpha a_{t-1}^2) + \beta^{t-1-(t-2)} (\omega + \alpha a_{t-2}^2) + \beta^{t-1-(t-3)} (\omega + \alpha a_{t-3}^2) \\
 &\quad + \beta^{t-1-(t-4)} (\omega + \alpha a_{t-4}^2) + \beta^{t-1-(t-5)} (\omega + \alpha a_{t-5}^2) \\
 &\quad + \dots + \beta^{t-1-(2-1)} \sigma_2^2 \\
 &= \beta^{t-1-(t-1)} (\omega + \alpha a_{t-1}^2) + \beta^{t-1-(t-2)} (\omega + \alpha a_{t-2}^2) + \beta^{t-1-(t-3)} (\omega + \alpha a_{t-3}^2) \\
 &\quad + \beta^{t-1-(t-4)} (\omega + \alpha a_{t-4}^2) + \beta^{t-1-(t-5)} (\omega + \alpha a_{t-5}^2) \\
 &\quad + \dots + \beta^{t-1-(2-1)} (\omega + \alpha a_1^2) + \beta^{t-1} \sigma_1^2 \\
 &= \sigma_1^2 \beta^{t-1} + \sum_{j=1}^{t-1} (\omega + \alpha a_j^2) \beta^{t-j-1}, \quad t = 2, 3, \dots, m
 \end{aligned}$$

Alternative of derivation:

Multiplying β^{-j} to $\sigma_j^2 - \beta \sigma_{j-1}^2 = \omega + \alpha a_{j-1}^2$ we have

$$\beta^{-j} \sigma_j^2 - \beta^{-(j-1)} \sigma_{j-1}^2 = (\omega + \alpha a_{j-1}^2) \beta^{-j}$$

Taking summation on j from 2 to t we have

$$\sum_{j=2}^t (\beta^{-j} \sigma_j^2 - \beta^{-(j-1)} \sigma_{j-1}^2) = \sum_{j=2}^t (\omega + \alpha a_{j-1}^2) \beta^{-j}$$

16. Continued

That is,

$$\beta^{-t}\sigma_t^2 - \beta^{-1}\sigma_1^2 = \sum_{j=1}^{t-1} (\omega + \alpha a_j^2) \beta^{-(j+1)}$$

Thus

$$\sigma_t^2 - \beta^{t-1}\sigma_1^2 = \sum_{j=1}^{t-1} (\omega + \alpha a_j^2) \beta^{t-(j+1)}$$

- (f) Prove that the condition to estimate the parameters for the GARCH model is to maximize the following function:

$$-\ln \sigma_1^2 - \frac{a_1^2}{\sigma_1^2} + \sum_{t=2}^m \left[-\ln \left(\sigma_1^2 \beta^{t-1} + \sum_{j=1}^{t-1} (\omega + \alpha a_j^2) \beta^{t-j-1} \right) - \frac{a_t^2}{\sigma_1^2 \beta^{t-1} + \sum_{j=1}^{t-1} (\omega + \alpha a_j^2) \beta^{t-j-1}} \right]$$

Commentary on Question:

It's important to show that a_t follows a normal distribution. Partial marks were rewarded for partial steps but not rewarded if candidates were just trying to make up an answer for the function that was needed to be proved.

$(a_{m+1}, a_{m+2}, \dots, a_T \mid \omega, \alpha, \beta, a_m \dots a_1) \sim \text{Normal Distribution},$

Therefore, $F[a_1, a_2, \dots, a_m] = \prod_{t=1}^m \frac{1}{\sqrt{2\pi\sigma_t^2}} \exp\left(\frac{-a_t^2}{2\sigma_t^2}\right)$

Maximizing an expression is equivalent to maximizing its logarithm.

Thus, take Logarithms $\rightarrow \sum_{t=1}^m \left[-\ln \sigma_t^2 - \frac{a_t^2}{\sigma_t^2} \right]$

From (e), $\sigma_t^2 = \sigma_1^2 \beta^{t-1} + \sum_{j=1}^{t-1} (\omega + \alpha a_j^2) \beta^{t-j-1}$, $t = 2, 3, \dots, m$

$$\sum_{t=1}^m \left[-\ln \sigma_t^2 - \frac{a_t^2}{\sigma_t^2} \right] = -\ln \sigma_1^2 - \frac{a_1^2}{\sigma_1^2} + \sum_{t=2}^m \left[-\ln \left(\sigma_1^2 \beta^{t-1} + \sum_{j=1}^{t-1} (\omega + \alpha a_j^2) \beta^{t-j-1} \right) - \frac{a_t^2}{\sigma_1^2 \beta^{t-1} + \sum_{j=1}^{t-1} (\omega + \alpha a_j^2) \beta^{t-j-1}} \right]$$

16. Continued

- (g) Your assistant claims that the goodness of fit of GARCH(1,1) model is adequate.

Critique your assistant's claim.

Commentary on Question:

To get full points, candidate needed to analyze both the t-Test and Ljung-Box Statistics, and concluded that GARCH model was an adequate fit. If reasonings provided conflicted with the conclusion, no marks were rewarded.

t-Test shows all estimated coefficient are significant at 1% level of confidence.

→ The estimates are appropriate for ω , α and β

Ljung-Box Statistics of squared standardized residual of the fitted model are non-significant at 1% confidence

→ Accept the null hypothesis that the first 10, 15 and 20 lags of ACF of the squared standardized residual of the fitted model are zero (i.e., no serial correlation on the squared residual of the fitted model). In other words, the Ljung-Box statistics of the squared standardized residual give $Q(10) = 0.98918$ with p-value of

→ This GARCH model is an adequate fit, which captures the ARCH impact with the squared residual on ABC return.

Assistant's claim is appropriate.