QFI Quantitative Finance Exam

Exam QFIQF

AFTERNOON SESSION

Date: Wednesday, May 1, 2019
Time: 1:30 p.m. – 3:45 p.m.

INSTRUCTIONS TO CANDIDATES

General Instructions

1. This afternoon session consists of 6 questions numbered 11 through 16 for a total of 40 points. The points for each question are indicated at the beginning of the question.

2. Failure to stop writing after time is called will result in the disqualification of your answers or further disciplinary action.

3. While every attempt is made to avoid defective questions, sometimes they do occur. If you believe a question is defective, the supervisor or proctor cannot give you any guidance beyond the instructions on the exam booklet.

Written-Answer Instructions

1. Write your candidate number at the top of each sheet. Your name must not appear.

2. Write on only one side of a sheet. Start each question on a fresh sheet. On each sheet, write the number of the question that you are answering. Do not answer more than one question on a single sheet.

3. The answer should be confined to the question as set.

4. When you are asked to calculate, show all your work including any applicable formulas. When you are asked to recommend, provide proper justification supporting your recommendation.

5. When you finish, insert all your written-answer sheets into the Essay Answer Envelope. Be sure to hand in all your answer sheets because they cannot be accepted later. Seal the envelope and write your candidate number in the space provided on the outside of the envelope. Check the appropriate box to indicate morning or afternoon session for Exam QFIQF.

6. Be sure your written-answer envelope is signed because if it is not, your examination will not be graded.

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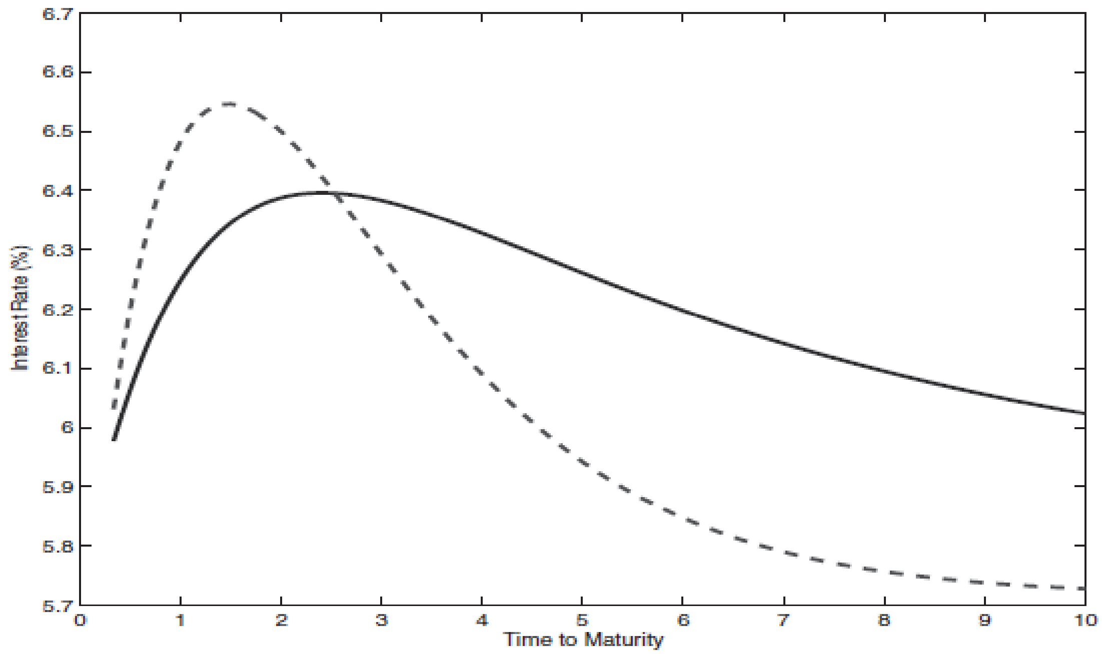
****BEGINNING OF EXAMINATION****
Afternoon Session
Beginning with Question 11

11. (*4 points*) You are provided with a continuously compounded spot curve and a one-year continuously compounded forward curve directly derived from the spot curve in the following chart.



(a) (*2 points*) Determine which curve is the spot curve and which curve is the forward curve. Justify your answer by mathematically demonstrating the key relationship between the two curves.

To project future liability cash flows on a variable annuity contract, you need to extend the interest rate curve beyond the time given above. Your actuarial colleague recommends using the extended Nelson-Siegel model to extend the curve up to 30 years.

(b) (*2 points*) Assess whether the recommendation is appropriate:

(i) If yes, describe how to use the model to extend the curve.

(ii) If no, provide your explanation.

12. (*4 points*) A leveraged inverse floater with \$100 notional amount and a 3-year maturity pays coupons on an annual basis, according to the following formula:

$$c(t) = 15\% - 3 * r(t-1)$$

where $r(t-1)$ denotes the annually compounded one-year rate at time $t-1$. Given the following data as of 12/31/2018:

Date	T	Discount factors
12/31/2019	1	0.980
12/31/2020	2	0.950
12/31/2021	3	0.915

(a) (*1.5 points*) Calculate the price of the leveraged inverse floater on 12/31/2018.

(b) (*2.5 points*) Calculate the duration of the leveraged inverse floater on 12/31/2018.

13. (*8 points*) You have been asked to use Monte Carlo simulations to estimate the value of an inverse floater under the single-factor Vasicek model:

$$dr_t = \gamma(\bar{r} - r_t)dr + \sigma dX_t$$

where γ, $\bar{r}$ and σ are positive constants, r_t is the short rate and X_t is a standard Brownian motion under the risk-neutral measure.

(a) (*2 points*) Identify the steps for estimating the fair value of an inverse floater using Monte Carlo simulation.

(b) (*1 point*) Show that the long-term rate $r_t(\tau)$ under the single factor Vasicek model follows the SDE

$$dr_t(\tau) = \frac{B(\tau)}{\tau}\gamma(\bar{r} - r_t)dt + \frac{B(\tau)}{\tau}\sigma dX_t$$

where t is the current time, T is the maturity date, $\tau = T - t$ is the time to maturity, and $B(\tau) = \dfrac{1-\exp(-\gamma\tau)}{\gamma}$.

(c) (*0.5 points*) Show that the long-term rate is less volatile than short-term rate under the single factor Vasicek model.

You have been asked to price a convertible bond.

Let $V(t, r_t, S_t)$ be the convertible bond price at time *t*, where

$$dS_t = r_t S_t dt + \sigma_E S_t dW_t$$

$$E\left[(dW_t dX_t)\right] = \rho dt$$

- W_t is a standard Brownian motion under the risk-neutral measure
- S_t is the price of the stock at time t and σ_E is the volatility of the stock
- ρ is the correlation coefficient constant

You are given that $\left\{e^{-\int_0^t r_s ds} V_t, t \geq 0\right\}$ is a martingale under the risk-neutral measure.

13. **Continued**

(d) (*2.5 points*) Derive the PDE for $V(t, r_t, S_t)$ using Ito's lemma without explicitly stating the terminal and boundary conditions.

As single factor Vasicek model cannot explain the variation of slope and curvature of the yield curve, you have been given the following two-factor Vasicek model with independent factors for studying the short-term rates.

$$d\Phi_{1,t} = \gamma_1^*\left(\overline{\Phi}_1^* - \Phi_{1,t}\right)dt + \sigma_1 dX_{1,t}$$
$$d\Phi_{2,t} = \gamma_2^*\left(\overline{\Phi}_2^* - \Phi_{2,t}\right)dt + \sigma_2 dX_{2,t}$$
$$r_t = \Phi_{1,t} + \Phi_{2,t},$$

where $\gamma_1^*, \gamma_2^*, \overline{\Phi}_1^*, \overline{\Phi}_2^*, \sigma_1$ and σ_2 are constants with $\gamma_1^* - \gamma_2^* > 0$ and $X_{1,t}$ and $X_{2,t}$ are independent standard Brownian motions.

(e) (*1 point*) Show, using Ito's lemma, that the term structure steepens as $\Phi_{2,t}$ increases, in addition to its movement implied by r_t.

(f) (*1 point*) Explain why the two-factor Vasicek model with independent factors fails to simultaneously explain the market prices of all caps and all swaptions.

14. (*9 points*) You are given the following $B(\tau)$ and $A(\tau)$ values associated with the Vasicek model.

Risk Neutral

τ in years	$B(\tau)$	$A(\tau)$
0.5	0.4477	-0.00166
1	0.8053	-0.00615

Real World

τ in years	$B(\tau)$	$A(\tau)$
0.5	0.4392	-0.00185
1	0.7762	-0.00785

From the market you observed the following prices for two short-term coupon bonds. The current short-term interest rate is r_0 = 2.10%.

Bond Reference	Time to maturity	Coupon rate (BEY)	Bid Price	Ask Price
1	0.5 years	5%	101.32	101.38
2	1 year	6%	103.73	103.87

The haircut for Treasury securities is 2.5% of the value of the treasury note.

(a) (*1 point*) Calculate the fitted bond prices for the two bonds using the Vasicek model.

(b) (*3 points*)

(i) Identify any mispricing,

(ii) Propose a relative value trade,

(iii) Calculate the replicating portfolio required in the trade.

(c) (*2 points*) Describe how the relative value trade can be carried out through the repo market and calculate the return on capital for this trade.

14. Continued

At time $t = 0.5$, a bond with 0.5 years remaining maturity with 5% coupon rate is trading at $101.50.

(d) (*2 points*) Describe the dynamic rebalancing activities and calculate the cash position of the replicating portfolio at $t = 0.5$.

(e) (*1 point*) Calculate the overall profit from time $t = 0$ to time $t = 1$.

15. (*8 points*) You would like to price interest rate derivatives, assuming that the interest rate follows the Vasicek model:

$$dr_t = \gamma(\theta - r_t)dt + \sigma dX_t$$

where γ, θ and σ are constants, r_t is the short rate and X_t is a standard Brownian motion under the risk-neutral measure.

Let $Z(r_t, t, T)$ be the price at time t of a zero-coupon bond maturing at T under the Vasicek model.

(a) (*1.5 points*) Derive formulas for the spot-rate duration and convexity of $Z(r_t, t; T)$ when

(i) $\gamma \neq 0$

(ii) $\gamma = 0$

(b) (*1 point*) Explain the impact of increasing γ on the duration and convexity.

Let $r(T, T+1)$ be the one-year effective rate observed at time T. Consider a contract that caps $r(T, T+1)$ at r_K, with the payment at time $T+1$ on a notional amount N.

(c) (*1 point*) Show that this contract can be characterized as a put option on a zero-coupon bond with appropriate notional amount and maturity.

Now you would like to price a put option with maturity T and strike price K on a zero-coupon bond maturing at time T_B $(T_B > T)$ when the risk-neutral interest rate process is as before but with $\gamma = 0$ (i.e. driftless Ho-Lee model).

Use the T-forward risk-neutral dynamics for the following.

(d) (*2 points*) Determine the distribution of r_t.

(e) (*1 point*) Calculate the mean of $Z(r_T, T; T_B)$.

(f) (*1.5 points*) Calculate the variance of $\ln[Z(r_T, T; T_B)]$.

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16. (*7 points*) QFI Capital is launching a new hedge fund focusing on shorting volatility of the US stock market.

(a) (*1 point*) Describe briefly four commonly observed characteristics of the volatility of asset returns.

To analyze the volatility, your assistant performed Auto Correlation Function (ACF) and Partial Auto Correlation Function (PACF) tests on the squared return residuals on ABC, a major US stock index. He suggested modeling the return volatility using an ARCH(m) model:

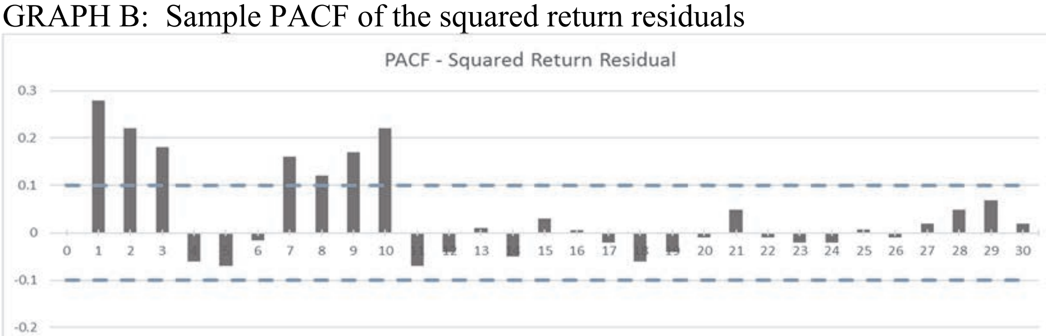
$$\sigma_t^2 = \alpha_0 + \sum_{i=1}^{m} \alpha_i a_{t-i}^2$$
$$a_t = \sigma_t \epsilon_t,$$

where

- α_0 and $\alpha_i, i = 1,2,\ldots,m,$ are constant coefficients
- $\{\epsilon_t\}$ is a sequence of i.i.d. standard normal random variables
- a_t is the residual of the return on ABC index at time t, that is, $a_t = r_t - \mu_t$; r_t is the index return and μ_t is the corresponding mean

GRAPH A: Sample ACF of the squared return residuals

GRAPH B: Sample PACF of the squared return residuals



16. **Continued**

Assume that a_t^2 is an unbiased estimate of σ_t^2 for the given sample.

(b) (*1 point*) Critique your assistant's suggestion by analyzing the pattern of graphs A and B

(c) (*0.5 points*) Recommend the order *m* of the ARCH model assuming an ARCH model is used. Justify your recommendation.

You learned that GARCH model is an alternative option in modeling the volatility.

(d) (*1.5 points*) List three strengths and three weaknesses of ARCH and GARCH models in relation to modeling the volatility.

You decide to model the return volatility on ABC with a GARCH (1,1) model:

$$\sigma_t^2 = \omega + \alpha a_{t-1}^2 + \beta \sigma_{t-1}^2$$

where ω, α, and β are constant parameters.

(e) (*1 point*) Show that $\sigma_t^2 = \sigma_1^2 \beta^{t-1} + \sum_{j=1}^{t-1} \left(\omega + \alpha a_j^2\right) \beta^{t-j-1}$ for $t = 2,3,\ldots,m$

You are to estimate the parameters with m observations using Maximum Likelihood Estimation (MLE).

(f) (*1 point*) Prove that the condition to estimate the parameters for the GARCH model is to maximize the following function:

$$-\ln \sigma_1^2 - \frac{a_1^2}{\sigma_1^2} + \sum_{t=2}^{m} \left[-\ln\left(\sigma_1^2 \beta^{t-1} + \sum_{j=1}^{t-1} \left(\omega + \alpha a_j^2\right) \beta^{t-j-1} \right) - \frac{a_t^2}{\sigma_1^2 \beta^{t-1} + \sum_{j=1}^{t-1} \left(\omega + \alpha a_j^2\right) \beta^{t-j-1}} \right]$$

Question 16 is continued on the next page.

16. Continued

To check the adequacy of the fitted GARCH (1,1) model, your assistant performed the following tests:

T Test for Estimated Coefficients		
Parameters	**t value**	**Pr(>|t|)**
ω	2.6092	0.0092
α	3.0838	0.0021
β	21.62	$< 2e^{-16}$

Ljung-Box Test for Squared Standardized Residual			
		Statistic	**p-Value**
$R\wedge 2$	Q(10)	0.98918	0.9998
$R\wedge 2$	Q(15)	11.1334	0.7431
$R\wedge 2$	Q(20)	13.4687	0.8564

Your assistant claims that the goodness of fit of GARCH(1,1) model is adequate.

(g) (*1 point*) Critique your assistant's claim.

****END OF EXAMINATION****
Afternoon Session

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