QFI Quantitative Finance Exam

Exam QFIQF

MORNING SESSION

Date: Wednesday, May 1, 2019
Time: 8:30 a.m. – 11:45 a.m.

INSTRUCTIONS TO CANDIDATES

General Instructions

1. This examination has a total of 100 points. It consists of a morning session (worth 60 points) and an afternoon session (worth 40 points).

   a) The morning session consists of 10 questions numbered 1 through 10.

   b) The afternoon session consists of 6 questions numbered 11 through 16.

   The points for each question are indicated at the beginning of the question.

2. Failure to stop writing after time is called will result in the disqualification of your answers or further disciplinary action.

3. While every attempt is made to avoid defective questions, sometimes they do occur. If you believe a question is defective, the supervisor or proctor cannot give you any guidance beyond the instructions on the exam booklet.

Written-Answer Instructions

1. Write your candidate number at the top of each sheet. Your name must not appear.

2. Write on only one side of a sheet. Start each question on a fresh sheet. On each sheet, write the number of the question that you are answering. Do not answer more than one question on a single sheet.

3. The answer should be confined to the question as set.

4. When you are asked to calculate, show all your work including any applicable formulas. When you are asked to recommend, provide proper justification supporting your recommendation.

5. When you finish, insert all your written-answer sheets into the Essay Answer Envelope. Be sure to hand in all your answer sheets because they cannot be accepted later. Seal the envelope and write your candidate number in the space provided on the outside of the envelope. Check the appropriate box to indicate morning or afternoon session for Exam QFIQF.

6. Be sure your written-answer envelope is signed because if it is not, your examination will not be graded.

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BEGINNING OF EXAMINATION
Morning Session

1. (*5 points*) You are given the following stochastic differential equations:

$$dX_t = 0.06X_t + 0.25X_t dW_t \text{ with } X_0 = 1$$
$$dY_t = 0.1dt + 0.15dW_t \text{ with } Y_0 = 1$$
$$Z_t = W_t^3 - 3tW_t$$

where W_t is a standard Wiener process, in particular, $E\left[W_t^2\right] = t$ and $E\left[W_t^4\right] = 3t^2$

(a) (*0.5 points*) State the three conditions for a stochastic process to be a martingale.

(b) (*2 points*) Show that Z_t is a martingale by verifying the three conditions in part (a).

(c) (*1 point*) Calculate $E\left[\ln(X_t)\right]$ using Ito's lemma.

(d) (*0.5 points*) Explain why $\ln\left(E[X_t]\right)$ is greater than $E\left[\ln(X_t)\right]$.

(e) (*1 point*) Calculate the probability of Y_3 being greater than $\ln(X_3)$.

2. (*4 points*) Let r be the continuously compounded risk-free interest rate and $\mathbb{Q}$ the risk-neutral measure.

Assume that r is a positive constant, and for $t \geq 0$ the price S_t of a non-dividend paying stock follows Geometric Brownian Motion:

$$dS_t = rS_t dt + \sigma S_t dW_t$$

where W_t is a standard Brownian motion under $\mathbb{Q}$, and σ is a positive constant.

(a) (*1 point*) Show that the stochastic process $X_t = \ln(S_t)$ follows Arithmetic Brownian Motion.

Now consider a contract on the stock that expires at time $t = 1$ with payoff $= \frac{1}{S_1}$. Denote by $V = e^{-r} E^{\mathbb{Q}}\left[\frac{1}{S_1}\right]$ the fair value of this contract at time $t = 0$.

(b) (*1 point*)

(i) State Jensen's inequality in the context of random variables.

(ii) Explain why $E^{\mathbb{Q}}\left[\frac{1}{S_1}\right] \geq \frac{e^{-r}}{S_0}$.

(c) (*1 point*)

(i) Prove that $V = \frac{e^{\sigma^2 - 2r}}{S_0}$.

(ii) Verify the inequality in part (b) (ii).

Denote by P the price at time $t = 0$ of a European put option on the stock which expires at time $t = 1$ with strike price $K = S_0 e^r$.

(d) (*1 point*) Determine the condition on S_0 in terms of r and σ such that, as functions of S_0, P exhibits higher convexity than V.

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3. (*7 points*) Let $\{W_t : t \geq 0\}$ be a standard Wiener process under the probability space $(\Omega, F, \mathbb{P})$ where $\mathbb{P}$ is the real-world measure.

Consider a non-dividend paying stock with price S_t at time $t \geq 0$ whose dynamics is given by the SDE

$$dS_t = \mu S_t \, dt + \sigma S_t \, dW_t$$

where μ and $\sigma > 0$ are constants.

Assume that the risk-free rate is a constant r. Using the Girsanov theorem we can find the risk-neutral measure $\mathbb{Q}$ that is equivalent to $\mathbb{P}$ such that the discounted price $e^{-rt}S_t$ is a martingale under $\mathbb{Q}$, that is, for some diffusion parameter f

$$d\left(e^{-rt}S_t\right) = f d\widetilde{W}_t$$

where $\widetilde{W}_t$ is a standard Wiener process under $\mathbb{Q}$.

(a) (*2.5 points*) Derive the dynamics of S_t under $\mathbb{Q}$ (i.e., in terms of dt and $d\widetilde{W}_t$) and the Radon-Nikodym derivative $\xi(W_t) = \dfrac{d\mathbb{Q}(W_t)}{d\mathbb{P}(W_t)}$.

Let $P = P(t, S_t)$ be the price at time t of a European put option that expires at time T with $T \geq t$.

(b) (*1 point*) Derive the option price at time t as a conditional expectation under $\mathbb{Q}$ of an appropriate terminal value $P(T, S_T)$.

Now consider a self-financing portfolio Σ at any time $t \leq T$ consisting of a long position in $\theta_{1,t}$ shares of the stock and a short position in $\theta_{2,t}$ options, that is,

$$\Sigma = \theta_{1,t} S - \theta_{2,t} P$$
$$d\Sigma = \theta_{1,t} dS - \theta_{2,t} dP$$

(c) (*1 point*) Show that if $\theta_{1,t} = \dfrac{\partial P}{\partial S}\theta_{2,t}$, then the portfolio Σ is risk-free.

3. Continued

(d) (*1 point*) Derive the Black-Scholes PDE for $P(t, S_t)$ with $t < T$ based on an arbitrage argument.

(e) (*0.5 points*) State the Feynman-Kac Theorem for a one-dimensional diffusion process.

(f) (*1 point*) Verify your result in part (b) by applying the Feynman-Kac Theorem.

4. (*4 points*) Let $\{X_t : t \geq 0\}$ be the process that follows the stochastic differential equation (SDE):

$$dX_t = aX_t dt + b\, dW_t \quad \text{with } X_0 = c$$

where a, b, and c are positive constants, and W_t is a standard Wiener process.

(a) (*1.5 points*) Solve the SDE for X_t.

(b) (*0.5 points*) Determine the distribution of X_t including its expected value and variance.

(c) (*2 points*) Show that the random variable $\int_0^T X_t dt$ follows a normal distribution with expectation $\frac{c}{a}\left(e^{aT}-1\right)$ and variance $\frac{b^2}{2a^3}\left(e^{2aT}-4e^{aT}+2aT+3\right)$.

(Hint: You may assume that $\int_0^T \int_0^t f(s,t)\, dW_s\, dt = \int_0^T \int_s^T f(s,t)\, dt dW_s$.)

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5. (*8 points*) You have been asked to consider the following two interest rate models for valuing a Eurodollar future based on 3-month LIBOR:

Model 1: Black-Karasinski model for the short-term interest rate r_t:

$$dr_t = r_t\left(\theta_t + \frac{\sigma_t^2}{2} - \gamma_t \ln(r_t)\right)dt + \sigma_t\, r_t\, dX_t$$

where $\theta_t, \sigma_t,$and γ_t are functions of $t, \theta_t > 0, \sigma_t > 0$, and X_t is a standard Brownian motion.

Model 2: Hull-White model for the short-term interest rate r_t:

$$dr_t = (\theta_t - \gamma r_t)\,dt + \sigma dX_t$$

where γ and σ are positive constants.

Under Model 2, the price of a zero-coupon bond with \$1 principal at time t with maturity date T is given by

$$Z(t,T) = e^{A(t,T) - B(t,T) r_t}$$

where $A(t,T)$ is a function of t and T and

$$B(t,T) = \frac{1 - e^{-\gamma(T-t)}}{\gamma}.$$

(a) (*1.5 points*) Show that $d\ln(r_t) = (\theta_t - \gamma_t \ln(r_t))\,dt + \sigma_t dX_t$ under Model 1.

(b) (*1 point*) Compare and contrast the following properties under Model 1 and Model 2.

(i) Arbitrage vs. non-arbitrage model

(ii) Distribution of r_t

(iii) Possible values of r_t

(iv) Analytical Bond Price

5. **Continued**

Let $\sigma_Z(t,T)$ be the diffusion of $\frac{dZ(t,T)}{Z(t,T)}$ and $f(t,T)$ be the continuously compounded instantaneous forward rate under the Heath-Jarrow-Morton framework.

(c) (*0.5 points*) Describe the restriction imposed on $\sigma_Z(t,T)$.

(d) (*0.5 points*) Express the drift and the diffusion of $df(t,T)$ in terms of $\sigma_Z(t,T)$.

(e) (*1 point*) Show that $\sigma_Z(t,T) = -B(t,T)\sigma$ for Model 2 using Ito's lemma.

(f) (*1 point*) Derive an expression for $df(t,T)$ under Model 2 in terms of $B(t,T)$.

Let $f(t,\tau,T)$ and $f^{fut}(t,\tau,T)$ be the continuously compounded, forward rate and the futures rate respectively, at time t for an investment between τ and T.

(g) (*0.5 points*) Explain intuitively why $f(t,\tau,T)$ differs from $f^{fut}(t,\tau,T)$.

(h) (*2 points*) Compute $f^{fut}(0,\tau,T) - f(0,\tau,T)$ under Model 2.

6. (*6 points*) Assume the Black-Scholes (B-S) framework.

The current B-S price of a T-year European call option with strike price K on a non-dividend paying stock is:

$$C = SN(d_1) - Ke^{-rT}N(d_2)$$

where:

$N(\bullet)$ = Cumulative normal distribution
S = the current price of the stock
r = the continuously compounded risk-free interest rate
σ = the volatility of the stock's continuously compounded returns

$$d_1 = \frac{\ln\frac{S}{K} + \left(r + \frac{\sigma^2}{2}\right)T}{\sigma\sqrt{T}}$$

$$d_2 = \frac{\ln\frac{S}{K} + \left(r - \frac{\sigma^2}{2}\right)T}{\sigma\sqrt{T}}$$

(a) (*2 points*) Show that:

$$\frac{\partial C}{\partial S} = N(d_1)$$

(b) (*1 point*) Identify whether each of the following statements is true or false. Briefly justify your answer.

(i) The delta of a European call option that is out-of-the-money will converge to 0 as the expiration date approaches.

(ii) For an out-of-the-money option with the underlying having low volatility, if the volatility increases, the delta of the option increases.

6. Continued

Your company owns 100 European call options with a strike of $100 on a stock. You are given:

- The stock pays no dividends.
- The continuously compounded risk-free interest rate is 0%.
- Initially, the options are at-the-money with one year to expiration with the price of $7.97 per option.
- One week later, the price of the stock is $104, the delta of the call option is 0.62, and the price of the call option is $10.20.
- Two weeks later, the price of the stock is $100, and the price of the call option is $7.81.
- The implied volatility is 20% for both valuation and hedging.

You are asked to hedge your option portfolio using the underlying shares. Assume that you can only buy or sell whole shares and all options or shares are liquidated at the end of two weeks.

(c) (*2.5 points*) Calculate the profit or loss at the end of two weeks from delta hedging (ignore any cash withdrawal or infusion that may be needed for rebalancing) if:

(i) You rebalance your hedge only at the end of each week.

(ii) You never rebalanced your hedge.

(d) (*0.5 points*) Explain why the hedge profits or losses are different between the two hedging strategies in part (c).

7. (*9 points*) You are a portfolio manager at a small financial institution. You have a long position in a vanilla call option which is worth \$10 today when the underlying stock price is $S = \$100$.

The change in the call option price $C(S,t)$ when the underlying stock price changes by a small amount dS from time t to time $t + dt$ is:

$$C(S + dS, t + dt) - C(S, t) = \Theta dt + \Delta ds + \frac{1}{2}\Gamma dS^2$$

where

$$\Theta = \frac{\partial C}{\partial t}, \Delta = \frac{\partial C}{\partial S}, \text{and } \Gamma = \frac{\partial^2 C}{\partial S^2}$$

And the change in the value of the call over a calendar day (assuming 365 days in a year) is:

- \$5.99, when the stock price moves up by 10%, and
- -\$4.01, when the stock price moves down by 10%

Assume the option's theta Θ is -3.65 and the risk-free rate of return is 0%.

(a) (*2 points*) Determine the option's delta and gamma using the Taylor series expansion.

Assume S follows a binomial model with $\mu = 0$ and the change in the value of the stock, dS, over dt is:

- $+\sigma S\sqrt{dt}$, when the stock price moves up, and
- $-\sigma S\sqrt{dt}$, when the stock price moves down.

where σ is the realized volatility.

You are delta hedging using the implied volatility Σ.

(b) (*1.5 points*) Show that the payoff of a delta-hedged portfolio is $\Theta dt + \frac{1}{2}\Gamma dS^2$ when the stock price moves in either direction.

7. **Continued**

(c) (*1 point*) Calculate the profit or loss after time dt and explain when you would have a profit.

(d) (*1 point*) Describe one advantage and one disadvantage of delta hedging using the actual volatility.

Your colleague made the following comments on your firm's delta hedging strategy:

- Hedging discretely rather than continuously will introduce uncertainty in the hedging outcome and bias the final gain/loss.
- The firm can reduce its hedging error significantly by increasing the frequency of the hedge rebalancing if it uses the implied volatility to rebalance the position.
- The firm can cut its hedging error by half by doubling its rebalancing frequency.
- The more often the firm hedges, the smaller the hedging error and the greater the expected hedging profit, even if the firm has to incur transaction cost.

(e) (*1.5 points*) Critique each of your colleague's comments above.

Later, you decided to sell the call with the following terms.

Time to Maturity (in year)	1
Strike	110
Spot	100
Implied Volatility	20%
Dividend Rate	0%
Risk-free rate (continuously compounded)	2.5%
Bid-off spread (% of stock price)	0.2%

Assuming continuous hedging, you expect the realized volatility for this stock to be 23% over the next year.

(f) (*2 points*) Estimate the hedging error if the option is delta-hedged with weekly frequency using 20% volatility.

8. (*6 points*) You are pricing an exotic option on the S&P 500, which expires in 6 months and has a payoff depending on the value of S&P 500 at expiration.

Assume that:

- The continuously compounded risk-free rate is 2.60%
- The S&P 500 dividend rate is 2.29%

The following table summarizes current Black-Scholes prices (as % of spot) and sensitivities of various financial derivatives with the same maturity but different volatilities:

Derivative	Strike	Volatility	Price	Delta	Vega	Vanna	Volga
Put	93.57%	15.78%	1.77%	-25.00%	0.22%	-0.0111%	0.0052%
Straddle	100.39%	11.92%	6.65%	0.0000%	0.56%	0.0046%	0.0000%
Call	105.08%	9.70%	0.99%	25.00%	0.22%	0.0239%	0.0113%

(a) (*1 point*) Construct a 25-delta risk reversal strategy and a butterfly strategy with the instruments above.

(b) (*1 point*) List main features that make risk reversal and butterfly able to hedge volatility skew and convexity.

(c) (*0.5 points*) Describe the Vanna-Volga approach.

Assuming a flat volatility surface at 11.92%, you reprice the above options as well as an exotic option as follows:

Derivative	Strike	Flat Vol (σ_{50})	Price	Vanna	Volga
Put	93.57%	11.92%	0.96%	-0.0176%	1.0594%
Straddle	100.39%	11.92%	6.65%	0.0046%	0.0000%
Call	105.08%	11.92%	1.51%	0.0176%	0.6567%
Exotic		11.92%	1.50%	-0.0148%	-0.0048%

You are using the Vanna-Volga approach to make an adjustment to the exotic option price of 1.50% above.

(d) (*0.5 points*) Explain your reasons for making the adjustment.

(e) (*3 points*) Calculate the adjustment using the volatility surface given in the first table.

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9. (*6 points*) Let X be a $N \times 4$ matrix representing the normalized observations of changes of forward rates for various maturities, where N is the number of observations.

Let $V = X'X / N$ be the correlation matrix of the changes of forward rates, where X' is the transpose of X, and W be the matrix of normalized eigenvectors of V. You are given:

$$V = \begin{bmatrix} 1 & 0.5 & 0.2 & 0.3 \\ 0.5 & 1 & 0.3 & 0.2 \\ 0.2 & 0.3 & 1 & 0.5 \\ 0.3 & 0.2 & 0.5 & 1 \end{bmatrix} \quad \text{and} \quad W = \begin{bmatrix} a & 0.5 & 0.5 & -0.5 \\ b & 0.5 & c & 0.5 \\ 0.5 & -0.5 & d & -0.5 \\ 0.5 & -0.5 & 0.5 & 0.5 \end{bmatrix}$$

$P = XW$ is the matrix of principal components with the following information.

- $\lambda_1, \lambda_2, \lambda_3, \lambda_4$ are the eigenvalues corresponding to the principal components PC1, PC2, PC3 and PC4 respectively using correlation matrix.
- $\lambda_1 > \lambda_2 > \lambda_3 > \lambda_4 > 0$
- $\lambda_3 = \frac{3}{5}, \lambda_4 = \frac{2}{5}$

We define:

- An eigenvector is **pure shift** if all elements are the same and positive; **shift** if all its components are positive, strictly increasing and then strictly decreasing
- An eigenvector is **slope** if its elements are strictly increasing (strictly decreasing) presenting a unique change of sign; **weak slope** if its elements present a unique change of sign;
- An eigenvector is **curvature** if its elements are strictly decreasing (increasing) and then strictly increasing (decreasing) presenting only two changes of sign; **weak curvature** if its elements present only two changes of sign

(a) (*0.5 points*) Explain what normalization is and why forward rates need to be normalized before the analysis.

9. **Continued**

(b) (*1 point*) Show that

(i) All columns of P are uncorrelated.

(ii) The variance of the *m*-th principal component is λ_m.

(Hint: Consider $P'P$ where P' is the transpose of P.)

(c) (*1 point*) Formulate the maximization problem that is solved by the first eigenvector w_1.

(d) (*1.5 points*) Calculate the remaining two eigenvalues λ_1 and λ_2 of V, and the entries a, b, c, d of W.

(e) (*1 point*) Calculate the proportion of the total variation in the forward rates explained by each of the first four principal components.

(f) (*1 point*) Interpret the meaning for each of the first three eigenvectors.

10. (*5 points*) A Fund Manager holds a 10-year bond with $10 million face value paying a 6% semi-annual coupon. The duration of the bond is 7.8. The bond has no default risk.

You are given the following data.

Bond Data per $100 of Face Value							
Period i	**Time** T_i	**Cash Flow** *CF*	**Discount Factor** $Z(0, T_i)$	**Discounted Cash Flow** $CF * Z(0, T_i)$	**Weight** w_i	**Level of Yield Curve:** $\beta_{i,1}$	**Slope of Yield Curve:** $\beta_{i,2}$
1	0.5	3	0.9615	2.88	0.0263	1.1511	-0.1582
2	1.0	3	0.9401	2.82	0.0257	1.0987	-0.2480
3	1.5	3	0.9191	2.76	0.0251	1.0181	-0.2450
4	2.0	3	0.8937	2.70	0.0246	0.9791	-0.2965
…	…	…	…	…	…	…	…
20	10.0	103	0.6270	64.58	0.5888	0.8517	0.6635
	Total			109.68	1.0000		

(a) (*0.5 points*) Calculate the price of a 2-year floating rate bond with coupons paid semi-annually and with a 35 basis points spread, assuming that its par value is $100.

Assume that the monthly changes in the interest rates dr have a normal distribution with a mean $\mu = 6.5 * 10^{-6}$ and standard deviation $\sigma = 0.50\%$.

(b) (*0.5 points*) Calculate the 95%, 1-month VaR of the 10-year bond using the duration approximation.

(c) (*1 point*) Explain a pitfall in calculating VaR using the above approach.

(d) (*1 point*) Explain pitfalls in using the VaR as a risk measure and how to address them.

The term structure of interest rates is usually changing over time. The Factor Duration methodology can be used to compute the sensitivity of a bond portfolio to changes in the level and slope of the yield curve.

10. Continued

You are given the following factor durations for the 10-year coupon bond:

$$D_1 = \sum \left(w_i * T_i * \beta_{i,1} \right) = 6.70, \text{ and } D_2 = \sum \left(w_i * T_i * \beta_{i,2} \right) = 4.42.$$

(e) (*2 points*) Calculate the face values of a 2-year zero-coupon bond and a 10-year zero-coupon bond you should buy or short-sell to achieve factor neutrality and immunize the 10-year coupon bond from interest rate changes in the level and slope of the yield curve.

****END OF EXAMINATION****
Morning Session

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