November 2002
Course 4 solutions

Question # 1
Answer: B

$$\rho_1 = \frac{\phi_1}{1-\phi_2} = 0.5$$

$$\rho_2 = \phi_2 + \frac{\phi_1^2}{1-\phi_2} = -0.2$$

Solving simultaneously gives:

$\phi_1 = 0.8$
$\phi_2 = -0.6$

Question # 2
Answer: C

$g = [12(.45)] = [5.4] = 5; \quad h = 5.4 - 5 = 0.4.$

$\hat{\pi}_{.45} = .6x_{(5)} + .4x_{(6)} = .6(360) + .4(420) = 384.$

Question # 3
Answer: D

N is distributed $Poisson(\lambda)$
$\mu = E(\lambda) = \alpha\theta = 1(1.2) = 1.2.$
$v = E(\lambda) = 1.2; \quad a = Var(\lambda) = \alpha\theta^2 = 1(1.2)^2 = 1.44.$
$k = \frac{1.2}{1.44} = \frac{5}{6}; \quad Z = \frac{2}{2+5/6} = \frac{12}{17}.$
Thus, the estimate for Year 3 is
$\frac{12}{17}(1.5) + \frac{5}{17}(1.2) = 1.41.$

Note that a Bayesian approach produces the same answer.

Question # 4
Answer: C

At the time of the second failure,

$$\hat{H}(t) = \frac{1}{n} + \frac{1}{n-1} = \frac{23}{132} \Rightarrow n = 12.$$

At the time of the fourth failure,

$$\hat{H}(t) = \frac{1}{12} + \frac{1}{11} + \frac{1}{10} + \frac{1}{9} = .3854.$$

Question # 5
Answer: E

$$R^2 = \hat{b}^2 \frac{\sum x_i^2}{\sum y_i^2} = 2.065^2 \frac{42}{182} = .9841.$$

Question # 6
Answer: B

The likelihood is:

$$L = \prod_{j=1}^{n} \frac{r(r+1)\cdots(r+x_j-1)b^{x_j}}{x_j!(1+b)^{r+x_j}} \propto \prod_{j=1}^{n} b^{x_j}(1+b)^{-r-x_j}.$$

The loglikelihood is:

$$l = \sum_{j=1}^{n} \left[x_j \ln b - (r+x_j)\ln(1+b) \right]$$

$$l' = \sum_{j=1}^{n} \left[\frac{x_j}{b} - \frac{r+x_j}{1+b} \right] = 0$$

$$0 = \sum_{j=1}^{n} \left[x_j(1+b) - (r+x_j)b \right] = \sum_{j=1}^{n} x_j - rnb$$

$$0 = n\bar{x} - rnb; \quad \hat{b} = \bar{x}/r.$$

Question # 7
Answer: C

The Bühlmann credibility estimate is $Zx+(1-Z)\mu$ where x is the first observation. The Bühlmann estimate is the least squares approximation to the Bayesian estimate. Therefore, Z and μ must be selected to minimize

$$\frac{1}{3}[Z+(1-Z)\mu-1.5]^2+\frac{1}{3}[2Z+(1-Z)\mu-1.5]^2+\frac{1}{3}[3Z+(1-Z)\mu-3]^2.$$

Setting partial derivatives equal to zero will give the values. However, it should be clear that μ is the average of the Bayesian estimates, that is,

$$\mu=\frac{1}{3}(1.5+1.5+3)=2.$$

The derivative with respect to Z is (deleting the coefficients of 1/3):

$$2(-Z+.5)(-1)+2(.5)(0)+2(Z-1)(1)=0$$
$$Z=.75.$$

The answer is

$$.75(1)+.25(2)=1.25.$$

Question # 8
Answer: E

The confidence interval is $(\hat{S}(t_0)^{1/\theta},\hat{S}(t_0)^{\theta})$.
Taking logarithms of both endpoints gives the two equations

$$\ln .695=-.36384=\frac{1}{\theta}\ln\hat{S}(t_0)$$
$$\ln .843=-.17079=\theta\ln\hat{S}(t_0).$$

Multiplying the two equations gives

$$.06214=[\ln\hat{S}(t_0)]^2$$
$$\ln\hat{S}(t_0)=-.24928$$
$$\hat{S}(t_0)=.77936.$$

The negative square root is required in order to make the answer fall in the interval (0,1).

Question # 9
Answer: E

Because $\rho_k = \phi^k$ there are a number of ways to get the value of ϕ.

$\phi = .215^{1/2} = -.46368; \quad \phi = (-.1)^{1/3} = -.46416; \quad \phi = \dfrac{-.1}{.215} = -.46512.$

Also, because the mean is zero, δ must be zero. Then (using the first choice for ϕ),

$\hat{y}_{T+1} = -.46368(-.431) + 0 = .1998.$

Question # 10
Answer: B

The likelihood is:

$$L = \frac{\alpha 150^{\alpha}}{(150+225)^{\alpha+1}} \frac{\alpha 150^{\alpha}}{(150+525)^{\alpha+1}} \frac{\alpha 150^{\alpha}}{(150+950)^{\alpha+1}}$$
$$= \frac{\alpha^3 150^{3\alpha}}{(375 \bullet 675 \bullet 1100)^{\alpha+1}}.$$

The loglikelihood is:

$$l = 3\ln\alpha + 3\alpha\ln 150 - (\alpha+1)\ln(375\bullet 675\bullet 1100)$$
$$l' = \frac{3}{\alpha} + 3\ln 150 - \ln(375\bullet 675\bullet 1100) = \frac{3}{\alpha} - 4.4128$$
$$\hat{\alpha} = 3/4.4128 = .6798.$$

Question # 11
Answer: D

For this problem, $r = 4$ and $n = 7$. Then,

$$\hat{v} = \frac{33.60}{4(7-1)} = 1.4 \text{ and } \hat{a} = \frac{3.3}{4-1} - \frac{1.4}{7} = .9.$$

Then,

$$k = \frac{1.4}{.9} = \frac{14}{9}; \quad Z = \frac{7}{7+(14/9)} = \frac{63}{77} = .82.$$

Question # 12
Answer: A

$$r_{YX_2 \cdot X_3} = \frac{r_{YX_2} - r_{YX_3} r_{X_2X_3}}{\sqrt{1-r_{YX_3}^2}\sqrt{1-r_{X_2X_3}^2}} = \frac{.6-.5\cdot .4}{\sqrt{1-.5^2}\sqrt{1-.4^2}} = .504$$

Question # 13
Answer: C

For a mixture variable, raw moments are weighted averages of the individual moments. Thus,

$E(X) = .5m_1 + .5m_2$ and $E(X^2) = .5(2m_1^2) + .5(2m_2^2)$.

The square of the coefficient of variation is

$$\frac{Var(X)}{E(X)^2} = \frac{E(X^2) - E(X)^2}{E(X)^2} = \frac{m_1^2 + m_2^2}{.25(m_1 + m_2)^2} - 1.$$

Divide numerator and denominator by m_2^2 and let $r = m_1 / m_2$. The square of the coefficient of variation becomes

$\dfrac{r^2+1}{.25(r+1)^2} - 1$. Setting the derivative equal to zero yields $r = 1$, however, this value minimizes the function (at a value of 1). There are no other critical points. Looking at the endpoints ($r = 0$ and $r =$ infinity) the limiting value is 3, which is the maximum. Therefore, the least upper bound for the coefficient of variation is the square root of 3.

Question # 14
Answer: B

X is the random sum $Y_1 + Y_2 + ... + Y_N$.
N has a negative binomial distribution with $r = \alpha = 1.5$ and $\beta = \theta = 0.2$.

$$E(N) = r\beta = 0.3$$
$$Var(N) = r\beta(1+\beta) = 0.36$$

$$E(Y) = 5000$$
$$Var(Y) = 25,000,000$$

$E(X) = 0.3 \times 5000 = 1500$
$Var(X) = 0.3 \times 25,000,000 + 0.36 \times 25,000,000 = 16,500,000$

Number of exposures (insureds) required for full credibility
$n_{FULL} = (1.645/0.05)^2 \times 16,500,000/(1500)^2 = 7937.67$.

Number of expected claims required for full credibility
$E(N) \times n_{FULL} = 0.3 \times 7937.67 = 2381$.

Question # 15
Answer: C

The estimated relative risk is
$\dfrac{h(t \mid Z=1)}{h(t \mid Z=0)} = \dfrac{h_0(t)e^b}{h_0(t)} = e^b = 1.822 \Rightarrow b = .6.$

For the single covariate case, the Wald test for testing H_0: $\beta = 0$ reduces to:

$(b-0)^2 I(b) = (.6)^2(3.968) = 1.43.$

Question # 16
Answer: D

See pages 535-7, the bottom of page 567 and the top of page 568. The only correct statement – and the correct answer – is (D).

Question # 17
Answer: E

X	$F_n(x)$	$F_n(x^-)$	$F_0(x)$	$\lvert F_n(x) - F_0(x)\rvert$	$\lvert F_n(x^-) - F_0(x)\rvert$
29	0.2	0	0.252	0.052	0.252
64	0.4	0.2	0.473	0.073	0.273
90	0.6	0.4	0.593	0.007	0.193
135	0.8	0.6	0.741	0.059	0.141
182	1.00	0.8	0.838	0.162	0.038

where:
$\hat{\theta} = \bar{x} = 100$ and $F_0(x) = 1 - e^{-x/100}$.
The maximum value from the last two columns is 0.273.

Question # 18
Answer: E

$\mu = E(\lambda) = 1; \quad v = E(\sigma^2) = 1.25; \quad a = Var(\lambda) = 1/12.$

$k = v/a = 15; \quad Z = \dfrac{1}{1+15} = \dfrac{1}{16}.$

Thus, the estimate for Year 2 is

$\dfrac{1}{16}(0) + \dfrac{15}{16}(1) = .9375.$

Question # 19
Answer: B

Time must be reversed, so let T be the time between accident and claim report and let $R = 3 - T$. The desired probability is

$\Pr(T < 2 \mid T \le 3) = \Pr(3 - R < 2 \mid 3 - R \le 3) = \Pr(R > 1 \mid R \ge 0) = \Pr(R > 1).$

The product-limit calculation is:

R	Y	d
0	40	9
1	55	23
2	43	43

The estimate of surviving past (reversed) time 1 is (31/40)(32/55) = .4509.

Question # 20
Answer: E

The solution depends on identifying the parameter a with the overall average, which requires the convention stated in Pindyck and Rubinfeld under the table on page 140. The convention also applies to subgroups. With this convention, the solution is the following conditional expectation:

$E\left(Y \mid E = -1, F = -1, G = -1, H = -1\right) - a = -b_1 - b_2 - c_1 - c_2.$

Question # 21
Answer: E

The posterior density, given an observation of 3 is:

$$\pi(\theta\,|\,3)=\frac{f(3\,|\,\theta)\pi(\theta)}{\int_1^\infty f(3\,|\,\theta)\pi(\theta)d\theta}=\frac{\dfrac{2\theta^2}{(3+\theta)^3}\dfrac{1}{\theta^2}}{\int_1^\infty 2(3+\theta)^{-3}d\theta}$$

$$=\frac{2(3+\theta)^{-3}}{-(3+\theta)^{-2}\Big|_1^\infty}=32(3+\theta)^{-3},\quad \theta>1.$$

Then,

$$\Pr(\Theta>2)=\int_2^\infty 32(3+\theta)^{-3}d\theta=-16(3+\theta)^{-2}\Big|_2^\infty=\frac{16}{25}=.64.$$

Question # 22
Answer: B

Because all previous values are 0, previous single and double smoothed values are also zero.
Then,

$$\tilde{y}_0=.6(1)+.4(0)=.6$$
$$\tilde{y}_1=.6(1.2)+.4(.6)=.96$$
$$\tilde{y}_2=.6(1.3)+.4(.96)=1.164$$
$$\tilde{\tilde{y}}_0=.6(.6)+.4(0)=.36$$
$$\tilde{\tilde{y}}_1=.6(.96)+.4(.36)=.72$$
$$\tilde{\tilde{y}}_2=.6(1.164)+.4(.72)=.9864.$$

Question # 23
Answer: B

$$\begin{aligned}L&=F(1000)^7[F(2000)-F(1000)]^6[1-F(2000)]^7\\&=(1-e^{-1000/\theta})^7(e^{-1000/\theta}-e^{-2000/\theta})^6(e^{-2000/\theta})^7\\&=(1-p)^7(p-p^2)^6(p^2)^7\\&=p^{20}(1-p)^{13}\end{aligned}$$

where $p=e^{-1000/\theta}$. The maximum occurs at p = 20/33 and so $\hat{\theta}=-1000/\ln(20/33)=1996.90.$

Question # 24
Answer: A

$E(X \mid \theta) = \theta / 2.$

$$E(X_3 \mid 400, 600) = \int_{600}^{\infty} E(X \mid \theta) f(\theta \mid 400, 600) d\theta = \int_{600}^{\infty} \frac{\theta}{2} 3 \frac{600^3}{\theta^4} d\theta = \frac{3(600^3)}{2} \frac{\theta^{-2}}{-2} \Bigg|_{600}^{\infty}$$

$$= \frac{3(600^3)(600^{-2})}{4} = 450.$$

Question # 25
Answer: D

The data may be organized as follows:

t	Y	d	$\hat{S}(t)$
2	10	1	(9/10) = .9
3	9	2	.9(7/9) = .7
5	7	1	.7(6/7) = .6
6	5	1	.6(4/5) = .48
7	4	1	.48(3/4) = .36
9	2	1	.36(1/2) = .18

Because the product-limit estimate is constant between observations, the value of $\hat{S}(8)$ is found from $\hat{S}(7) = .36$.

Question # 26
Answer: D

As of time 10 there were 7 observed payments, so $O = 7$. For the Weibull distribution, the cumulative hazard function is $H(x) = -\ln S(x) = x^2 / 25$. Then

$$E(Z) = Var(Z) = \sum_{i=1}^{10} \frac{x_i^2}{25} = \frac{1}{25}(4+9+9+25+25+36+49+49+81+100) = 15.48.$$

The chi-squared test statistic (with one degree of freedom) is $(7-15.48)^2 / 15.48 = 4.645$.
From the tables, this leads to rejection at the 5% level, but not at the 2.5% level.

Question # 27
Answer: D

$$\text{ESS}_R = 15{,}000 - 5{,}565 = 9{,}435$$

$$R_{UR}^2 = .38 = 1 - \frac{ESS_{UR}}{TSS} \Rightarrow ESS_{UR} = .62(15{,}000) = 9{,}300$$

$$F = \frac{(9{,}435 - 9{,}300)/3}{9{,}300/3{,}114} = 15.07.$$

Question # 28
Answer: C

The maximum likelihood estimate for the Poisson distribution is the sample mean:

$$\hat{\lambda} = \overline{x} = \frac{50(0) + 122(1) + 101(2) + 92(3)}{365} = 1.6438.$$

The table for the chi-square test is:

Number of days	Probability	Expected*	Chi-square
0	$e^{-1.6438} = .19324$	70.53	5.98
1	$1.6438e^{-1.6438} = .31765$	115.94	0.32
2	$\frac{1.6438^2 e^{-1.6438}}{2} = .26108$	95.30	0.34
3+	.22803**	83.23	0.92

*365x(Probability) **obtained by subtracting the other probabilities from 1

The sum of the last column is the test statistic of 7.56. Using 2 degrees of freedom (4 rows less 1 estimated parameter less 1) the model is rejected at the 2.5% significance level but not at the 1% significance level.

Question # 29
Answer: D

$$\mu(0)=\frac{.4(0)+.1(1)+.1(2)}{.6}=.5;\quad \mu(1)=\frac{.1(0)+.2(1)+.1(2)}{.4}=1$$

$$\mu=.6(.5)+.4(1)=.7$$

$$a=.6(.5^2)+.4(1^2)-.7^2=.06$$

$$v(0)=\frac{.4(0)+.1(1)+.1(4)}{.6}-.5^2=\frac{7}{12};\quad v(1)=\frac{.1(0)+.2(1)+.1(4)}{.4}-1^2=.5$$

$$v=.6(7/12)+.4(.5)=11/20$$

$$k=v/a=55/6;\quad Z=\frac{10}{10+55/6}=\frac{60}{115}$$

Bühlmann credibility premium = $\frac{60}{115}\frac{10}{10}+\frac{55}{115}(.7)=.8565$.

Question # 30
Answer: A

All the statements about R^2 are true, but only (A) is not raised as an objection to R^2.

Question # 31
Answer: C

$$\mu=.5(0)+.3(1)+.1(2)+.1(3)=.8$$

$$\sigma^2=.5(0)+.3(1)+.1(4)+.1(9)-.64=.96$$

$$E(S_n^2)=\frac{n-1}{n}\sigma^2=\frac{3}{4}(.96)=.72$$

$$bias=.72-.96=-.24.$$

Question # 32
Answer: C

The four classes have means .1, .2, .5, and .9 respectively and variances .09, .16, .25, and .09 respectively.

Then,

$\mu = .25(.1 + .2 + .5 + .9) = .425$

$v = .25(.09 + .16 + .25 + .09) = .1475$

$a = .25(.01 + .04 + .25 + .81) - .425^2 = .096875$

$k = .1475 / .096875 = 1.52258$

$$Z = \frac{4}{4 + 1.52258} = .7243$$

The estimate is $\left[.7243(2/4) + .2757(.425)\right] \cdot 5 = 2.40$.

Question # 33
Answer: D

The lower limit is determined as the smallest value such that

$\hat{S}(t) \le .25 + 1.96\sqrt{\hat{V}[\hat{S}(t)]}$.

At t = 50 the two sides are .360 and $.25 + 1.96(.0470) = .342$ and the inequality does not hold.

At t = 54 the two sides are .293 and $.25 + 1.96(.0456) = .339$ and the inequality does hold.

The lower limit is 54.

Question # 34
Answer: A

$$Q = T\sum_{k=1}^{K} \hat{r}_k^2 = 100\left((-.01)^2 + .01^2 + \cdots + .10^2\right) = 8.96$$

Question # 35
Answer: A

The distribution used for simulation is given by the observed values.

Question # 36
Answer: B

First obtain the distribution of aggregate losses:

Value	Probability
0	1/5
25	(3/5)(1/3) = 1/5
100	(1/5)(2/3)(2/3) = 4/45
150	(3/5)(2/3) = 2/5
250	(1/5)(2)(2/3)(1/3) = 4/45
400	(1/5)(1/3)(1/3) = 1/45

$$\mu = (1/5)(0) + (1/5)(25) + (4/45)(100) + (2/5)(150) + (4/45)(250) + (1/45)(400) = 105$$

$$\begin{aligned}\sigma^2 &= (1/5)(0^2) + (1/5)(25^2) + (4/45)(100^2) + (2/5)(150^2)\\ &\quad + (4/45)(250^2) + (1/45)(400^2) - 105^2 = 8{,}100.\end{aligned}$$

Question # 37
Answer: A

Loss Range	Cum. Prob.
0 – 100	0.320
100 – 200	0.530
200 – 400	0.800
400 – 750	0.960
750 – 1000	0.980
1000 – 1500	1.000

At 400, $F(x) = 0.8 = 1 - e^{-\frac{400}{\theta}}$; solving gives $\theta = 248.53$.

Question # 38
Answer: D

The sum of the squared values of the independent variable is 2266 + $20(100)^2$ = 202,266. The value of s^2 is 5348/18 = 297.111. Then,

$$\hat{s}_{\hat{\alpha}} = s^2 \frac{\Sigma X_i^2}{N \Sigma x_i^2} = 297.111 \frac{202{,}266}{20(2266)} = 1326.$$

The 97.5th percentile of a t-distribution with 18 degrees of freedom is 2.101, so the lower limit of the symmetric 95% confidence interval for α is

$$68.73 - 2.101\sqrt{1326} = -7.78.$$

Question # 39
Answer: B

$$\Pr(class1\,|\,1) = \frac{(1/2)(1/3)}{(1/2)(1/3)+(1/3)(1/6)+(1/6)(0)} = \frac{3}{4}$$
$$\Pr(class2\,|\,1) = \frac{(1/3)(1/6)}{(1/2)(1/3)+(1/3)(1/6)+(1/6)(0)} = \frac{1}{4}$$
$$\Pr(class3\,|\,1) = \frac{(1/6)(0)}{(1/2)(1/3)+(1/3)(1/6)+(1/6)(0)} = 0$$

because the prior probabilities for the three classes are 1/2, 1/3, and 1/6 respectively.

The class means are

$$\mu(1) = (1/3)(0)+(1/3)(1)+(1/3)(2) = 1$$
$$\mu(2) = (1/6)(1)+(2/3)(2)+(1/6)(3) = 2.$$

The expectation is

$$E(X_2\,|\,1) = (3/4)(1)+(1/4)(2) = 1.25.$$

Question # 40
Answer: **E**

The first, second, third, and sixth payments were observed at their actual value and each contributes $f(x)$ to the likelihood function. The fourth and fifth payments were paid at the policy limit and each contributes $1 - F(x)$ to the likelihood function. This is answer (E).