NOVEMBER 2004
COURSE 3 SOLUTIONS

Question #1
Key: D

$$E\left[Z\right] = \int_0^\infty b_t v^t \, {}_t p_x \; \mu\left(x+t\right) dt = \int_0^\infty e^{0.06\,t} e^{-0.08\,t} e^{-0.05\,t} \frac{1}{20} dt$$

$$= \frac{1}{20}\left(\frac{100}{7}\right)\left[-e^{-0.07t}\right]_0^\infty = \frac{5}{7}$$

$$E\left[Z^2\right] = \int_0^\infty \left(b_t v^t\right)^2 {}_t p_x \mu\left(x+t\right) dt = \int_0^\infty e^{0.12t} e^{-0.16t} e^{-0.05t} \frac{1}{20} dt = \frac{1}{20}\int_0^\infty e^{-0.09t} dt$$

$$= \frac{1}{20}\left(\frac{100}{9}\right)\left[e^{-0.09t}\right]_0^\infty = \frac{5}{9}$$

$$Var\left[Z\right] = \frac{5}{9} - \left(\frac{5}{7}\right)^2 = 0.04535$$

Question #2
Key: C

Let *ns* = nonsmoker and *s* = smoker

$k =$	$q_{x+k}^{(ns)}$	$p_{x+k}^{(ns)}$	$q_{x+k}^{(s)}$	$p_{x+k}^{(s)}$
0	.05	0.95	0.10	0.90
1	.10	0.90	0.20	0.80
2	.15	0.85	0.30	0.70

$$A_{x:\overline{2}|}^{1(ns)} = \quad v \quad q_x^{(ns)} + \quad v^2 \quad p_x^{(ns)} \quad q_{x+1}^{(ns)}$$

$$\frac{1}{1.02}\left(0.05\right) \quad \frac{1}{1.02^2} \quad 0.95 \times 0.10 = 0.1403$$

$$A_{x:\overline{2}|}^{1(s)} \quad v \quad q_x^{(s)} \quad + \quad v^2 \quad p_x^{(s)} \quad q_{x+1}^{(s)}$$

$$\frac{1}{1.02} \quad \left(0.10\right) + \frac{1}{\left(1.02\right)^2} \; 0.90 \times 0.20 = 0.2710$$

$A_{x:\overline{2}|}^{1}$ = weighted average = (0.75)(0.1403) + (0.25)(0.2710)
= 0.1730

Question #3
Key: A

$$\bar{P}\left(\bar{A}_x\right) = \mu = 0.03$$

$${}^2\bar{A}_x = 0.20 = \frac{\mu}{2\delta + \mu} = \frac{0.03}{2\delta + 0.03}$$

$$\Rightarrow \delta = 0.06$$

$$\text{Var}\left({}_0L\right) = \frac{{}^2\bar{A}_x - \left(\bar{A}_x\right)^2}{\left(\delta\,\bar{a}\right)^2} = \frac{0.20 - \left(\frac{1}{3}\right)^2}{\left(\frac{0.06}{0.09}\right)^2} = 0.20$$

$$\text{where} \quad A = \frac{\mu}{\mu + \delta} = \frac{0.03}{0.09} = \frac{1}{3} \qquad \bar{a} = \frac{1}{\mu + \delta} = \frac{1}{0.09}$$

Question #4
Key: B

$$s\left(60\right) = \frac{e^{-(0.1)(60)} + e^{-(0.08)(60)}}{2}$$

$$= 0.005354$$

$$s\left(61\right) = \frac{e^{-(0.1)(61)} + e^{-(0.08)(61)}}{2}$$

$$= 0.00492$$

$$q_{60} = 1 - \frac{0.00492}{0.005354} = 0.081$$

Question #5
Key: B

For Ω, $0.4 = F(\omega) = \left(\frac{\omega}{80}\right)^2$

$$0.6325 = \frac{\omega}{80}$$
$$\omega = 50.6$$

For $T(0)$ using De Moivre, $0.7 = F(t) = \frac{t}{\omega} = \frac{t}{50.6}$

$$t = (0.7)(50.6) = 35.42$$

Question #6
Key: C

$E[N] = mq = 1.8 \Rightarrow q = \frac{1.8}{3} = 0.6$

x	$f_N(x)$	$F_N(x)$
0	0.064	0.064
1	0.288	0.352
2	0.432	0.784
3	0.216	1.000

First:	$0.432 < 0.7 < 0.784$ so N = 2.	Use 0.1 and 0.3 for amounts
Second:	$0.064 < 0.1 < 0.352$ so N = 1	Use 0.9 for amount
Third:	$0.432 < 0.5 < 0.784$ so N = 2	Use 0.5 and 0.7 for amounts

Discrete uniform $\Rightarrow F_X(x) = 0.2x, \quad x = 1, 2, 3, 4, 5$

$$0.4 < 0.5 < 0.6 \Rightarrow x_1 = 3$$
$$0.6 < 0.7 < 0.8 \Rightarrow x_2 = 4$$

Aggregate claims = 3+4 = 7

Question #7
Key: C

$$E(X \wedge x) = \frac{\theta}{\alpha - 1}\left[1 - \left(\frac{\theta}{x+\theta}\right)^{\alpha-1}\right] = \frac{2000x}{x+2000}$$

x	$E(X \wedge x)$
∞	2000
250	222
2250	1059
5100	1437

$$0.75\big(E(X \wedge 2250) - E(X \wedge 250)\big) + 0.95\big(E(X) - E(X \wedge 5100)\big)$$
$$0.75(1059 - 222) + 0.95(2000 - 1437) = 1162.6$$

The 5100 breakpoint was determined by when the insured’s share reaches 3600:

3600 = 250 + 0.25 (2250 – 250) + (5100 – 2250)

Question #8
Key: D

Since each time the probability of a heavy scientist is just half the probability of a success, the distribution is binomial with $q = 0.6 \times 0.5 = 0.3$ and $m = 8$.

$$f(2) = (8 \times 7/2) \times (0.3 \wedge 2) \times (0.7 \wedge 6) = 0.30$$

Question #9
Key: A

$$\mu_{xy}(t) = \mu_x(t) + \mu_y(t) = 0.08 + 0.04 = 0.12$$
$$\bar{A}_x = \mu_x(t) / \left(\mu_x(t) + \delta\right) = 0.5714$$
$$\bar{A}_y = \mu_y(t) / \left(\mu_y(t) + \delta\right) = 0.4$$
$$\bar{A}_{xy} = \mu_{xy}(t) / \left(\mu_{xy}(t) + \delta\right) = 0.6667$$
$$\bar{a}_{xy} = 1 / \left(\mu_{xy}(t) + \delta\right) = 5.556$$

$$\bar{A}_{\overline{xy}} = \bar{A}_x + \bar{A}_y - \bar{A}_{xy} = 0.5714 + 0.4 - 0.6667 = 0.3047$$
Premium = 0.304762/5.556 = 0.0549

Question #10
Key: B

$$P_{40} = A_{40} / \ddot{a}_{40} = 0.16132 / 14.8166 = 0.0108878$$

$$P_{42} = A_{42} / \ddot{a}_{42} = 0.17636 / 14.5510 = 0.0121201$$

$$a_{45} = \ddot{a}_{45} - 1 = 13.1121$$

$$\begin{aligned} E\left[{}_3L \middle| K(42) \geq 3\right] &= 1000A_{45} - 1000P_{40} - 1000P_{42}\, a_{45} \\ &= 201.20 - 10.89 - (12.12)(13.1121) \\ &= 31.39 \end{aligned}$$

Many similar formulas would work equally well. One possibility would be $1000\,{}_3V_{42} + \left(1000P_{42} - 1000P_{40}\right)$, because prospectively after duration 3, this differs from the normal benefit reserve in that in the next year you collect $1000P_{40}$ instead of $1000P_{42}$.

Question #11
Key: E

For De Moivre's Law:

$$\mathring{e}_x = \frac{\omega - x}{2}$$

$$_{k|}q_x = \frac{1}{\omega - x}$$

$$A_x = \sum_{k=b}^{\omega-x-1} v^{k+1} \, _{k|}q_x = \frac{1}{\omega - x} \sum_{k=b}^{\omega-x-1} v^{k+1}$$

$$A_x = \frac{a_{\overline{\omega-x}|}}{\omega - x}$$

$$\ddot{a}_x = \frac{1 - A_x}{d}$$

$\mathring{e}_{50} = 25 \Rightarrow \omega = 100$ for typical annuitants

$\mathring{e}_y = 15 \Rightarrow y =$ Assumed age = 70

$$A_{70} = \frac{a_{\overline{30}|}}{30} = 0.45883$$

$$\ddot{a}_{70} = 9.5607$$

$$500000 = b \, \ddot{a}_{20} \Rightarrow b = 52,297$$

Question #12
Key: D

$$p_x^{(\tau)} = p'^{(1)}_x p'^{(2)}_x = 0.8(0.7) = 0.56$$

$$q_x^{(1)} = \left[\frac{\ln\left(p'^{(1)}_x\right)}{\ln\left(p_x^{(\tau)}\right)}\right] q_x^{(\tau)} \text{ since UDD in double decrement table}$$

$$= \left[\frac{\ln(0.8)}{\ln(0.56)}\right] 0.44$$

$$= 0.1693$$

$$_{0.3}q_{x+0.1}^{(1)} = \frac{0.3q_x^{(1)}}{1-0.1q_x^{(\tau)}} = 0.053$$

To elaborate on the last step:

$$_{0.3}q_{x+0.1}^{(1)} = \frac{\begin{pmatrix}\text{Number dying from cause} \\ \text{1 between } x+0.1 \text{ and } x+0.4\end{pmatrix}}{\text{Number alive at } x+0.1}$$

Since UDD in double decrement,

$$= \frac{l_x^{(\tau)}(0.3)q_x^{(1)}}{l_x^{(\tau)}\left(1-0.1q_x^{(\tau)}\right)}$$

Question #13
Key: B

non absorbing matrix $T = \begin{pmatrix} 0.7 & 0.1 \\ 0.3 & 0.6 \end{pmatrix}$, the submatrix excluding "Terminated", which is an absorbing state.

$$I - T = \begin{pmatrix} 0.3 & -0.1 \\ -0.3 & 0.4 \end{pmatrix}$$

$$\left(I - T\right)^{-1} = \begin{pmatrix} \frac{0.4}{0.09} & \frac{0.1}{0.09} \\ \frac{0.3}{0.09} & \frac{0.3}{0.09} \end{pmatrix} = \begin{pmatrix} 4.4\overline{4} & 1.1\overline{1} \\ 3.3\overline{3} & 3.3\overline{3} \end{pmatrix}$$

Future costs for a healthy = $4.4\overline{4} \times 500 + 1.1\overline{1} \times 3000$
= 5555

Question #14
Key: D

$$T = \begin{pmatrix} 0.7 & 0.1 & 0.2 \\ 0.3 & 0.6 & 0.1 \\ 0 & 0 & 1 \end{pmatrix} \qquad T^2 = \begin{pmatrix} 0.52 & 0.13 & 0.35 \\ 0.39 & 0.39 & 0.22 \\ 0 & 0 & 1 \end{pmatrix}$$

Actuarial present value (A.P.V.) prem = 800(1 + (0.7 + 0.1) + (0.52 + 0.13)) = 1,960
A.P.V. claim = 500(1 + 0.7 + 0.52) + 3000(0 + 0.1 + 0.13) = 1800
Difference = 160

Question #15
Key: A

Let $N_1, \quad N_2$ denote the random variable for # of claims for Type I and II in 2 years

$X_1, \quad X_2$ denote the claim amount for Type I and II

$S_1 =$ total claim amount for type I in 2 years

$S_2 =$ total claim amount for Type II at time in 2 years

$S = S_1 + S_2 =$ total claim amount in 2 years

$\{S_1\} \rightarrow$ compound poisson $\lambda_1 = 2 \times 6 = 12 \quad X_1 \sim U(0,1)$

$\{S_2\} \rightarrow$ compound poisson $\lambda_2 = 2 \times 2 = 4 \quad X_2 \sim U(0,5)$

$$E(N_1) = Var(N_1) = 2 \times 6 = 12$$

$$E(S_1) = E(N_1)E(X_1) = (12)(0.5) = 6$$

$$Var(S_1) = E(N_1)Var(X_1) + Var(N_1)\left(E(X_1)\right)^2$$

$$= (12)\frac{(1-0)}{12} + (12)(0.5)^2$$

$$= 4$$

$$E(N_2) = Var(N_2) = 2 \times 2 = 4$$

With formulas corresponding to those for S_1,

$$E(S_2) = 4 \times \frac{5}{2} = 10$$

$$Var(S_2) = 4 \times \frac{(5-0)^2}{12} + 4\left(\frac{5}{2}\right)^2 = 33.\overline{3}$$

$$E(S) = E(S_1) + E(S_2) = 6 + 10 = 16$$

Since S_1 and S_2 are independent,

$$Var(S) = Var(S_1) + Var(S_2) = 4 + 33.\overline{3} = 37.\overline{3}$$

$$\Pr(S > 18) = \Pr\left(\frac{S-16}{\sqrt{39.\overline{3}}} > \frac{2}{\sqrt{37.\overline{3}}} = 0.327\right)$$

Using normal approximation

$$\Pr(S > 18) = 1 - \Phi(0.327)$$

$$= 0.37$$

Question #16
Key: D

Since the rate of depletion is constant there are only 2 ways the reservoir can be empty sometime within the next 10 days.

Way #1:
There is no rainfall within the next 5 days

Way #2
There is one rainfall in the next 5 days
And it is a normal rainfall
And there are no further rainfalls for the next five days

Prob (Way #1) = Prob(0 in 5 days) = exp(-0.2*5) = 0.3679
Prob (Way #2) = Prob(1 in 5 days) × 0.8 × Prob(0 in 5 days)
= 5*0.2 exp(-0.2* 5)* 0.8 * exp(-0.2* 5)
= 1 exp(-1) * 0.8 * exp(-1) = 0.1083

Hence Prob empty at some time = 0.3679 + 0.1083 = 0.476

Question #17
Key: C

Let X be the loss random variable,
So $(X-5)_+$ is the claim random variable.

$$E(X)=\frac{10}{2.5-1}=6.\overline{6}$$

$$E(X\wedge 5)=\left(\frac{10}{2.5-1}\right)\left[1-\left(\frac{10}{5+10}\right)^{2.5-1}\right]$$
$$=3.038$$

$$E(X-5)_+=E(X)-E(X\wedge 5)$$
$$=6.\overline{6}-3.038$$
$$=3.629$$

Expected aggregate claims $=E(N)E(X-5)_+$
$$=(5)(3.629)$$
$$=18.15$$

Question #18
Key: B

A Pareto $(\alpha = 2,\ \theta = 5)$ distribution with 20% inflation becomes Pareto with $\alpha = 2,\ \ \theta = 5 \times 1.2 = 6$

In 2004, $E(X) = \dfrac{6}{2-1} = 6$

$$E(X \wedge 10) = \frac{6}{2-1}\left(1 - \left(\frac{6}{10+6}\right)^{2-1}\right) = 3.75$$

$$E(X-10)_+ = E(X) - E(X \wedge 10)$$
$$= 6 - 3.75 = 2.25$$

$$\text{LER} = 1 - \frac{E(X-10)_+}{E(X)} = 1 - \frac{2.25}{6} = 0.625$$

Question #19
Key: A

Let X = annual claims

$$E(X) = (0.75)(3) + (0.15)(5) + (0.1)(7)$$
$$= 3.7$$

$$\pi = \text{Premium } = (3.7)\left(\frac{4}{3}\right) = 4.93$$

Change during year $= 4.93 - 3 = +1.93$ with $p = 0.75$
$= 4.93 - 5 = -0.07$ with $p = 0.15$
$= 4.93 - 7 = -2.07$ with $p = 0.10$

Since we start year 1 with surplus of 3, at end of year 1 we have 4.93, 2.93, or 0.93 (with associated probabilities 0.75, 0.15, 0.10).

We cannot drop more then 2.07 in year 2, so ruin occurs only if we are at 0.93 after 1 and have a drop of 2.07.

$$\text{Prob} = (0.1)(0.1) = 0.01$$

Question #20
Key: E

$$0.96 = e^{-(\mu_1+\lambda)}$$
$$\mu_1 + \lambda = -\ln(0.96) = 0.04082$$
$$\mu_1 = 0.04082 - \lambda = 0.04082 - 0.01 = 0.03082$$

Similarly

$$\mu_2 = -\ln(0.97) - \lambda = 0.03046 - 0.01 = 0.02046$$
$$\mu_{xy} = \mu_1 + \mu_2 + \lambda = 0.03082 + 0.02046 + 0.01 = 0.06128$$
$${}_5 p_{xy} = e^{-(5)(0.06128)} = e^{-0.3064} = 0.736$$

Question #21
Key: C

$A_{60} = 0.36913 \qquad d = 0.05660$

${}^2A_{60} = 0.17741$

and $\sqrt{{}^2A_{60} - A_{60}^2} = 0.202862$

Expected Loss on one policy is $E\left[L(\pi)\right] = \left(100{,}000 + \frac{\pi}{d}\right)A_{60} - \frac{\pi}{d}$

Variance on one policy is $\operatorname{Var}\left[L(\pi)\right] = \left(100{,}000 + \frac{\pi}{d}\right)^2 \left({}^2A_{60} - A_{60}^2\right)$

On the 10000 lives,

$$E[S] = 10{,}000 E\left[L(\pi)\right] \text{ and } \operatorname{Var}[S] = 10{,}000 \operatorname{Var}\left[L(\pi)\right]$$

The π is such that $0 - E[S]/\sqrt{\operatorname{Var}[S]} = 2.326$ since $\Phi(2.326) = 0.99$

$$\frac{10{,}000\left(\frac{\pi}{d} - \left(100{,}000 + \frac{\pi}{d}\right)A_{60}\right)}{100\left(100{,}000 + \frac{\pi}{d}\right)\sqrt{{}^2A_{60} - A_{60}^2}} = 2.326$$

$$\frac{100\left(\frac{\pi}{d} - \left(100{,}000 + \frac{\pi}{d}\right)\right)(0.36913)}{\left(100{,}000 + \frac{\pi}{d}\right)(0.202862)} = 2.326$$

$$\frac{0.63087\frac{\pi}{d} - 36913}{100{,}000 + \frac{\pi}{d}} = 0.004719$$

$$0.63087\frac{\pi}{d} - 36913 = 471.9 = 0.004719\frac{\pi}{d}$$

$$\frac{\pi}{d} = \frac{36913 + 471.9}{0.63087 - 0.004719}$$

$$= 59706$$

$$\pi = 59706 \times d = 3379$$

Question #22
Key: C

$$ {}_1V = \left({}_0V + \pi\right)(1+i) - \left(1000 + {}_1V - {}_1V\right) \times q_{75} $$
$$ = 1.05\pi - 1000 q_{75} $$

Similarly,

$$ {}_2V = \left({}_1V + \pi\right) \times 1.05 - 1000 q_{76} $$
$$ {}_3V = \left({}_2V + \pi\right) \times 1.05 - 1000 q_{77} $$
$$ 1000 = {}_3V = \left(1.05^3 \pi + 1.05^2 \cdot \pi + 1.05\pi\right) - 1000 \times q_{75} \times 1.05^2 - 1000 \times 1.05 \times q_{76} - 1000 \times q_{77} \quad * $$
$$ \pi = \frac{1000 + 1000\left(1.05^2 q_{75} + 1.05 q_{76} + q_{77}\right)}{(1.05)^3 + (1.05)^2 + 1.05} $$
$$ = \frac{1000^x \left(1 + 1.05^2 \times 0.05169 + 1.05 \times 0.05647 + 0.06168\right)}{3.310125} $$
$$ = \frac{1000 \times 1.17796}{3.310125} = 355.87 $$

* This equation is algebraic manipulation of the three equations in three unknowns $\left({}_1V,\ {}_2V,\ \pi\right)$. One method – usually effective in problems where benefit = stated amount plus reserve, is to multiply the ${}_1V$ equation by 1.05^2, the ${}_2V$ equation by 1.05, and add those two to the ${}_3V$ equation: in the result, you can cancel out the ${}_1V$, and ${}_2V$ terms. Or you can substitute the ${}_1V$ equation into the ${}_2V$ equation, giving ${}_2V$ in terms of π, and then substitute that into the ${}_3V$ equation.

Question #23
Key: D

Actuarial present value (APV) of future benefits =

$$= (0.005\times 2000 + 0.04\times 1000)/1.06 + (1-0.005-0.04)(0.008\times 2000+0.06\times 1000)/1.06^2$$
$$= 47.17 + 64.60$$
$$= 111.77$$

APV of future premiums $= \left[1+(1-0.005-0.04)/1.06\right]50$

$$= (1.9009)(50)$$
$$= 95.05$$

$$E\left[{}_1L \,\middle|\, K(55)\geq 1\right] = 111.77 - 95.05 = 16.72$$

Question #24
Key: D

$$\mathring{e}_0 = \mathring{e}_{0:\overline{20}|} + {}_{20}p_0 \; \mathring{e}_{20}$$

$$\mathring{e}_0 = E[T] = \frac{50}{3-1} = 25$$

$$\mathring{e}_{0:\overline{20}|} = E[T \wedge 20] = \frac{50}{3-1}\left(1-\left(\frac{50}{50+20}\right)^{3-1}\right)$$

$$= 12.245$$

$${}_{20}p_0 = 1 - F_T(20) = 1 - \left(1-\left(\frac{50}{50+20}\right)^{3}\right)$$

$$= 0.3644$$

$$25 = 12.245 + 0.3644\,\mathring{e}_{20}$$

$$\mathring{e}_{20} = 35$$

Alternate approach: if losses are Pareto with $\theta = 50$ and $\alpha = 3$, then claim payments per payment with an ordinary deductible of 20 are Pareto with $\theta = 50+20$ and $\alpha = 3$.

Thus $E\left(T(20)\right) = \dfrac{50+20}{3-1} = 35$

This alternate approach was shown here for educational reasons: to reinforce the idea that many life contingent models and non-life models can have similar structure. We doubt many candidates would take that approach, especially since it involves specific properties of the Pareto distribution.

Question #25
Key: E

$Q \geq P$ since in Q you only test at intervals; surplus below 0 might recover before the next test. In P, ruin occurs if you are ever below 0.
$R \geq P$ since you are less likely to have surplus below 0 in the first N years (finite horizon) than forever.

Add the inequalities

$Q + R \geq 2P$

Also (why other choices are wrong)
$S \geq Q$ by reasoning comparable to $R \geq P$. Same testing frequency in S and Q, but Q tests forever.
$S \geq R$ by reasoning comparable to $Q \geq P$. Same horizon in S and R; R tests more frequently.
$S \geq P$ P tests more frequently, and tests forever.

Question #26
Key: A

This is a nonhomogeneous Poisson process with intensity function

λ (t) = 3+3t, $0 \leq t \leq 2$, where t is time after noon

$$\text{Average } \lambda = \frac{\int_1^2 \lambda(t)\,dt}{1} = \int_1^2 (3+3t)\,dt$$
$$= \left[3t + \frac{3t^2}{2}\right]\Bigg|_1^2$$
$$= 7.5$$

$$f(2) = \frac{e^{-7.5}\ 7.5^2}{2!} = 0.0156$$

Question #27
Key: E

$X(t)-Y(t)$ is Brownian motion with initial value –2 and $\sigma^2=0.5+1=1.5$

By formula 10.6, the probability that $X(t)-Y(t)\geq 0$ at some time between 0 and 5 is

$$
\begin{aligned}
&= 2\times \text{Prob}\left[X(5)-Y(5)\geq 0\right] \\
&= 2\times\left[1-\Phi\left(\frac{2}{\sqrt{5(1.5)}}\right)\right] = 0.4652
\end{aligned}
$$

The 2 in the numerator of $\frac{2}{\sqrt{5(1.5)}}$ comes from $X(0)-Y(0)=-2$; the process needs to move 2 to reach $X(t)-Y(t)\geq 0$.

Question #28
Key: B

$${}_{2|}q_{\overline{80:84}} = {}_{2|}q_{80} + {}_{2|}q_{84} - {}_{2|}q_{80:84}$$

$$= 0.5 \times 0.4 \times (1 - 0.6) + 0.2 \times 0.15 \times (1 - 0.1)$$
= 0.10136

Using new p_{82} value of 0.3

$$0.5 \times 0.4 \times (1 - 0.3) + 0.2 \times 0.15 \times (1 - 0.1)$$
= 0.16118

Change = 0.16118 – 0.10136 = 0.06

Alternatively,

$${}_2p_{80} = 0.5 \times 0.4 = 0.20$$
$${}_3p_{80} = {}_2p_{80} \times 0.6 = 0.12$$
$${}_2p_{84} = 0.20 \times 0.15 = 0.03$$
$${}_3p_{84} = {}_2p_{84} \times 0.10 = 0.003$$
$${}_2p_{\overline{80:84}} = {}_2p_{80} + {}_2p_{84} - {}_2p_{80}\ {}_2p_{84}$$ since independent
$$= 0.20 + 0.03 - (0.20)(0.03) = 0.224$$
$${}_3p_{\overline{80:84}} = {}_3p_{80} + {}_3p_{84} - {}_3p_{80}\ {}_3p_{84}$$
$$= 0.12 + 0.003 - (0.12)(0.003) = 0.12264$$
$${}_{2|}q_{\overline{80:84}} = {}_2p_{\overline{80:84}} - {}_3p_{\overline{80:84}}$$
$$= 0.224 - 0.12264 = 0.10136$$

Revised

$${}_3p_{80} = 0.20 \times 0.30 = 0.06$$
$${}_3p_{\overline{80:84}} = 0.06 + 0.003 - (0.06)(0.003)$$
$$= 0.06282$$
$${}_{2|}q_{\overline{80:84}} = 0.224 - 0.06282 = 0.16118$$

change $= 0.16118 - 0.10136 = 0.06$

Question #29
Key: B

$$e_x = p_x + p_x e_{x+1} \Rightarrow p_x = \frac{e_x}{1+e_{x+1}} = \frac{8.83}{9.29} = 0.95048$$

$$\ddot{a}_x = 1 + v p_x + v^2 \, {}_2p_x + ...$$

$$\ddot{a}_{x:\overline{2|}} = 1 + v + v^2 \, {}_2p_x + ...$$

$$\ddot{a}_{x:\overline{2|}} - \ddot{a}_x = v q_x = 5.6459 - 5.60 = 0.0459$$

$$v\left(1 - 0.95048\right) = 0.0459$$

$$v = 0.9269$$

$$i = \frac{1}{v} - 1 = 0.0789$$

Question #30
Key: A

Let π be the benefit premium
Let ${}_kV$ denote the benefit reserve a the end of year *k*.
For any $n, \left({}_nV+\pi\right)\left(1+i\right)=\left(q_{25+n}\times {}_{n+1}V+p_{25+n}\times {}_{n+1}V\right)$

$$= {}_{n+1}V$$

Thus ${}_1V=\left({}_0V+\pi\right)\left(1+i\right)$

$${}_2V=\left({}_1V+\pi\right)\left(1+i\right)=\left(\pi\left(1+i\right)+\pi\right)\left(1+i\right)=\pi\ddot{s}_{\overline{2}|}$$
$${}_3V=\left({}_2V+\pi\right)\left(1+i\right)=\left(\pi\ddot{s}_{\overline{2}|}+\pi\right)\left(1+i\right)=\pi\ddot{s}_{\overline{3}|}$$

By induction (proof omitted)
${}_nV=\pi\ddot{s}_{\overline{n}|}$
For $n=35$, ${}_nV=\ddot{a}_{60}$ (actuarial present value of future benefits; there are no future premiums)
$\ddot{a}_{60}=\pi\ \ddot{s}_{\overline{35}|}$
$$\pi=\frac{\ddot{a}_{60}}{\ddot{s}_{\overline{35}|}}$$
For $n=20$, ${}_{20}V=\pi\ \ddot{s}_{\overline{20}|}$
$$=\left(\frac{\ddot{a}_{60}}{\ddot{s}_{\overline{35}|}}\right)\ddot{s}_{\overline{20}|}$$

Alternatively, as above
$\left({}_nV+\pi\right)\left(1+i\right)={}_{n+1}V$
Write those equations, for $n=0$ to $n=34$
$0:\left({}_0V+\pi\right)\left(1+i\right)={}_1V$
$1:\left({}_1V+\pi\right)\left(1+i\right)={}_2V$
$2:\left({}_2V+\pi\right)\left(1+i\right)={}_3V$
$\vdots$
$34:\left({}_{34}V+\pi\right)\left(1+i\right)={}_{35}V$
Multiply equation *k* by $\left(1+i\right)^{34-k}$ and sum the results:
$$\left({}_0V+\pi\right)\left(1+i\right)^{35}+\left({}_1V+\pi\right)\left(1+i\right)^{34}+\left({}_2V+\pi\right)\left(1+i\right)^{33}+\cdots+\left({}_{34}V+\pi\right)\left(1+i\right)=$$
$${}_1V\left(1+i\right)^{34}+{}_2V\left(1+i\right)^{33}+{}_3V\left(1+i\right)^{32}+\cdots+{}_{34}V\left(1+i\right)+{}_{35}V$$

For $k = 1, 2, \cdots, 34,$ the ${}_kV\left(1+i\right)^{35-k}$ terms in both sides cancel, leaving

$${}_0V\left(1+i\right)^{35} + \pi\left[\left(1+i\right)^{35} + \left(1+i\right)^{34} + \cdots + \left(1+i\right)\right] = {}_{35}V$$

Since ${}_0V = 0$

$$\pi\ \ddot{s}_{\overline{35}|} = {}_{35}V$$

$$= \ddot{a}_{60}$$

(see above for remainder of solution)

This technique, for situations where the death benefit is a specified amount (here, 0) plus the benefit reserve is discussed in section 8.3 of Bowers. This specific problem is Example 8.3.1.

Question #31
Key: B

$$\mu_{\overline{xy}}(t) = \frac{{}_tq_y\ {}_tp_x\mu(x+t) + {}_tq_x\ {}_tp_y\ \mu(y+t)}{{}_tq_x \times {}_tp_y + {}_tp_x \times {}_tq_y + {}_tp_x \times {}_tp_y}$$

For (x) = (y) = (50)

$$\mu_{\overline{50:50}}(10.5) = \frac{({}_{10.5}q_{50})({}_{10}p_{50})q_{60}\cdot 2}{({}_{10.5}q_{50})({}_{10.5}p_{50})\cdot 2 + ({}_{10.5}p_{50})^2} = \frac{(0.09152)(0.91478)(0.01376)(2)}{(0.09152)(0.90848)(2)+(0.90848)^2} = 0.0023$$

where

$${}_{10.5}p_{50} = \frac{\tfrac{1}{2}(l_{60}+l_{61})}{l_{50}} = \frac{\tfrac{1}{2}(8,188,074+8,075,403)}{8,950,901} = 0.90848$$

$${}_{10.5}q_{50} = 1 - {}_{10.5}p_{50} = 0.09152$$

$${}_{10}p_{50} = \frac{8,188,074}{8,950,901} = 0.91478$$

${}_{10.5}p_{50}\ \mu(50+10.5) = ({}_{10}p_{50})q_{60}$ since UDD

Alternatively, ${}_{(10+t)}p_{50} = {}_{10}p_{50}\ {}_tp_{60}$

$${}_{(10+t)}p_{50:50} = ({}_{10}p_{50})^2\ ({}_tp_{60})^2$$

$${}_{(10+t)}p_{\overline{50:50}} = 2\ {}_{10}p_{50}\ {}_tp_{60} - ({}_{10}p_{50})^2\ ({}_tp_{60})^2$$

$$= 2\ {}_{10}p_{50}(1-tq_{60}) - ({}_{10}p_{50})^2\ (1-tq_{60})^2 \quad \text{since UDD}$$

Derivative $= -2\ {}_{10}p_{50}\ q_{60} + 2({}_{10}p_{50})^2\ (1-tq_{60})q_{60}$

Derivative at $10+t = 10.5$ is

$$-2(0.91478)(0.01376) + (0.91478)^2\ (1-(0.5)(0.01376))(0.01376) = -0.0023$$

$${}_{10.5}p_{\overline{50:50}} = 2\ {}_{10.5}p_{50} - ({}_{10.5}p_{50})^2$$

$$= 2(0.90848) - (0.90848)^2$$

$$= 0.99162$$

μ (for any sort of lifetime) $= \dfrac{-\dfrac{dp}{dt}}{p} = \dfrac{-(-0.0023)}{0.99162} = 0.0023$

Question #32
Key: E

$$E(W)=\frac{1}{4}\int_0^4\sum_{i=0}^{\infty}2^i\Pr\left(N=i\middle|\lambda\right)d\lambda \qquad \left[\frac{1}{4}\text{ is the density of }\lambda\text{ on }[0,4].\right]$$

$$=\frac{1}{4}\int_0^4 P\left(2\middle|\lambda\right)d\lambda \qquad [\text{see note}]$$

$$=\frac{1}{4}\int_0^4 e^{\lambda(2-1)}d\lambda \qquad [\text{using formula from tables for the pgf of the Poisson}]$$

$$=\frac{1}{4}e^{\lambda}\Big|_0^4=\frac{1}{4}\left(e^4-1\right)$$

$$=13.4$$

Note: the probability generating function (pgf) is $P(Z)=\sum_{k=0}^{\infty} p_k Z^k$ so the integrand is $P(2)$, or in this case $P(2|\lambda)$ since λ is not known.

Alternatively,

$$E(W)=\frac{1}{4}\int_0^4\sum_{i=0}^{\infty}2^i\Pr\left(N=i\middle|\lambda\right)d\lambda$$

$$=\frac{1}{4}\int_0^4\sum_{i=0}^{\infty}\frac{2^i e^{-\lambda}\lambda^i}{i!}d\lambda$$

$$=\frac{1}{4}\int_0^4\sum_{i=0}^{\infty}\frac{e^{-\lambda}(2\lambda)^i}{i!}d\lambda$$

We know $\sum_{i=0}^{\infty}\frac{e^{-2\lambda}(2\lambda)^i}{i!}=1$ since $\frac{e^{-2\lambda}(2\lambda)^i}{i!}$ is $f(i)$ for a Poisson with mean $Z\lambda$

so $\sum_{i=0}^{\infty}\frac{e^{-\lambda}(2\lambda)^i}{i!}=\frac{e^{-\lambda}}{e^{-2\lambda}}=e^{\lambda}$

Thus $E(W)=\frac{1}{4}\int_0^4 e^{\lambda}d\lambda$

$$=\frac{1}{4}e^{\lambda}\Big|_0^4=\frac{1}{4}\left(e^4-1\right)$$

$$=13.4$$

Question #33
Key: A or E

$E(S) = \lambda E[X] = 2/3(1/4 + 2/4 + 3/2) = 2/3 \times 9/4 = 3/2$

$\text{Var}(S) = \lambda E\left[X^2\right] = 2/3(1/4 + 4/4 + 9/2) = 23/6$

So cumulative premium to time 2 is $2\left(3/2 + 1.8\sqrt{23/6}\right) = 10$, where the expression in parentheses is the annual premium

Times between claims are determined by $-1/\lambda \log u$ and are 0.43, 0.77, 1.37, 2.41
So 2 claims before time 2 (second claim is at 1.20; third is at 2.57)

Sizes are 2, 3, 1, 3, where only the first two matter.

So gain to the insurer is 10-(2+3) = 5

Note: since the problem did not specify that we wanted the gain or loss from the insurer's viewpoint, we gave credit to answer A; a loss of 5 from the insured's viewpoint.

Question #34
Key: C

To get number of claims, set up cdf for Poisson:

x	*f*(*x*)	*F*(*x*)
0	0.135	0.135
1	0.271	0.406
2	0.271	0.677
3	0.180	0.857

0.80 simulates 3 claims.

$F(x) = 1 - (500/(x+500))^2 = u,$ so $x = (1-u)^{-1/2} 500 - 500$

0.6 simulates 290.57
0.25 simulates 77.35
0.7 simulates 412.87

So total losses equals 780.79

Insurer pays $(0.80)(750) + (780.79 - 750) = 631$

Question #35
Key: D

$$\mu_x^{(\tau)}(t) = \mu_x^{(1)}(t) + \mu_x^{(2)}(t) = 0.01 + 2.29 = 2.30$$

$$P = P\int_0^2 v^t \, {}_t p_x^{(\tau)} \mu_x^{(2)}(t) \; dt + 50,000\int_0^2 v^t \, {}_t p_x^{(\tau)} \mu_x^{(1)}(t) \; dt + 50,000\int_2^\infty v^t \, {}_t p_x^{(\tau)} \mu_x^{(\tau)}(t) \; dt$$

$$P = P\int_0^2 e^{-0.1t} e^{-2.3t} \times 2.29 dt + 50,000\int_0^2 e^{-0.1t} e^{-2.3t} \times 0.01 dt + 50,000\int_2^\infty e^{-0.1t} e^{-2.3t} \times 2.3 dt$$

$$P\left[1 - 2.29 \times \frac{1 - e^{-2(2.4)}}{2.4}\right] = 50000\left[0.01 \times \frac{1 - e^{-2(2.4)}}{2.4} + 2.3 \times \frac{e^{-2(2.4)}}{2.4}\right]$$

P = 11,194

Question #36
Key: D

$$\mu^{(accid)} = 0.001$$

$$\mu^{(total)} = 0.01$$

$$\mu^{(other)} = 0.01 - 0.001 = 0.009$$

$$\text{Actuarial present value} = \int_0^\infty 500,000 \, e^{-0.05t} \; e^{-0.01t} (0.009) dt$$

$$+10\int_0^\infty 50,000 \, e^{0.04t} \; e^{-0.05t} \; e^{-0.01t} (0.001) dt$$

$$= 500,000\left[\frac{0.009}{0.06} + \frac{0.001}{0.02}\right] = 100,000$$

Question #37
Key: B

Variance $= v^{30}\,{}_{15}p_x\,{}_{15}q_x$ Expected value $= v^{15}\,{}_{15}\,p_x$

$$v^{30}\,{}_{15}\,p_x\,{}_{15}q_x = 0.065 \quad v^{15}\,{}_{15}\,p_x$$

$$v^{15}\,{}_{15}q_x = 0.065 \Rightarrow {}_{15}q_x = 0.3157$$

Since μ is constant

$${}_{15}q_x = \left(1-(p_x)^{15}\right)$$

$$(p_x)^{15} = 0.6843$$

$$p_x = 0.975$$

$$q_x = 0.025$$

Question #38
Key: E

$$(1) \quad {}_{11}V^A = \left({}_{10}V^A + 0\right)\frac{(1+i)}{p_{x+10}} - \frac{q_{x+10}}{p_{x+10}} \times 1000$$

$$(2) \quad {}_{11}V^B = \left({}_{10}V^B + \pi^B\right)\frac{(1+i)}{p_{x+10}} - \frac{q_{x+10}}{p_{x+10}} \times 1000$$

$$(1)-(2) \quad {}_{11}V^A - {}_{11}V^B = \left({}_{10}V^A - {}_{10}V^B - \pi^B\right)\frac{(1+i)}{p_{x+10}}$$

$$= (101.35 - 8.36)\frac{(1.06)}{1-0.004}$$

$$= 98.97$$

Question #39
Key: A

$$\text{Actuarial present value Benefits } = \frac{(0.8)(0.1)(10,000)}{1.06^2} + \frac{(0.8)(0.9)(0.097)(9,000)}{1.06^3}$$
$$= 1,239.75$$

$$1,239.75 = P\left(1 + \frac{(0.8)}{1.06} + \frac{(0.8)(0.9)}{1.06^2}\right)$$
$$= P(2.3955)$$
$$P = 517.53 \quad \Rightarrow 518$$

Question #40
Key: C

Event	Prob	Present Value
$x = 0$	(0.05)	15
$x = 1$	$(0.95)(0.10) = 0.095$	$15 + 20/1.06 = 33.87$
$x \geq 2$	$(0.95)(0.90) = 0.855$	$15 + 20/1.06 + 25/1.06^2 = 56.12$

$$E[X] = (0.05)(15) + (0.095)(33.87) + (0.855)(56.12) = 51.95$$

$$E\left[X^2\right] = (0.05)(15)^2 + (0.095)(33.87)^2 + (0.855)(56.12)^2 = 2813.01$$

$$Var[X] = E\left(X^2\right) - E(X)^2 = 2813.01 - (51.95)^2 = 114.2$$