Course 1
May 2003
Answer Key

1.	D	21.	D
2.	A	22.	B
3.	E	23.	A
4.	D	24.	C
5.	B	25.	B
6.	A	26.	A
7.	C	27.	D
8.	B	28.	C
9.	D	29.	E
10.	E	30.	E
11.	B	31.	D
12.	D	32.	C
13.	C	33.	B
14.	C	34.	C
15.	C	35.	E
16.	B	36.	D
17.	C	37.	D
18.	D	38.	C
19.	A	39.	E
20.	C	40.	B

1. Solution: D

Let

$G =$ event that a viewer watched gymnastics

$B =$ event that a viewer watched baseball

$S =$ event that a viewer watched soccer

Then we want to find

$$\Pr\left[(G \cup B \cup S)^c\right] = 1 - \Pr(G \cup B \cup S)$$

$$= 1 - \left[\Pr(G) + \Pr(B) + \Pr(S) - \Pr(G \cap B) - \Pr(G \cap S) - \Pr(B \cap S) + \Pr(G \cap B \cap S)\right]$$

$$= 1 - (0.28 + 0.29 + 0.19 - 0.14 - 0.10 - 0.12 + 0.08) = 1 - 0.48 = 0.52$$

2. Solution: A

The graph in A contains the graphs of the functions $f(x) = x^3$ and $f''(x) = 6x$.

More generally (without making any assumptions regarding the exact definition of the functions $f(x)$), one can reason as follows:

(E) is out because the second derivative of a linear function is identically 0.

(B) and (D) are out because the curve which is non-linear and would have to be $f(x)$ is increasing at an increasing rate in the first quadrant. This says $f'(x)$ is positive and increasing which means $f''(x)$ must be positive for $x > 0$.

(C) is out because the curve which would have to be $f(x)$ is decreasing at a decreasing rate in the second quadrant. Therefore $f'(x)$ would have to be negative and increasing which implies $f''(x)$ must be positive when $x < 0$.

3. Solution: E

Observe

$$\lim_{x\to 0} \frac{c\,f(x) - d\,g(x)}{f(x) - g(x)} = \frac{c \lim_{x\to 0} f(x) - d \lim_{x\to 0} g(x)}{\lim_{x\to 0} f(x) - \lim_{x\to 0} g(x)} = \frac{c^2 - d^2}{c - d} = \frac{(c-d)(c+d)}{c-d} = c + d$$

(Note L'Hôspital's Rule does not apply in this problem because the limit in the denominator is not 0.)

4. Solution: D

The distribution function of an exponential random variable T with parameter θ is given by $F(t) = 1 - e^{-t/\theta},\ t > 0$

Since we are told that T has a median of four hours, we may determine θ as follows:

$$\frac{1}{2} = F(4) = 1 - e^{-4/\theta}$$

$$\frac{1}{2} = e^{-4/\theta}$$

$$-\ln(2) = -\frac{4}{\theta}$$

$$\theta = \frac{4}{\ln(2)}$$

Therefore, $\Pr(T \geq 5) = 1 - F(5) = e^{-5/\theta} = e^{-\frac{5\ln(2)}{4}} = 2^{-5/4} = 0.42$

5. Solution: B

Let

M = event that customer insures more than one car

S = event that customer insures a sports car

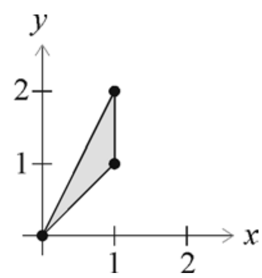
Then applying DeMorgan's Law, we may compute the desired probability as follows:

$$\Pr(M^c \cap S^c) = \Pr\left[(M \cup S)^c\right] = 1 - \Pr(M \cup S) = 1 - \left[\Pr(M) + \Pr(S) - \Pr(M \cap S)\right]$$

$$= 1 - \Pr(M) - \Pr(S) + \Pr(S|M)\Pr(M) = 1 - 0.70 - 0.20 + (0.15)(0.70) = 0.205$$

6. Solution: A

The calculation requires integrating over the indicated region.



$$E(X) = \int_0^1 \int_x^{2x} \frac{8}{3} x^2 y \, dy \, dx = \int_0^1 \frac{4}{3} x^2 y^2 \Big|_x^{2x} dx = \int_0^1 \frac{4}{3} x^2 \left(4x^2 - x^2\right) dx = \int_0^1 4x^4 dx = \frac{4}{5} x^5 \Big|_0^1 = \frac{4}{5}$$

$$E(Y) = \int_0^1 \int_x^{2x} \frac{8}{3} x y^2 dy \, dx = \int_0^1 \frac{8}{9} x y^3 \Big|_x^{2x} dy \, dx = \int_0^1 \frac{8}{9} x \left(8x^3 - x^3\right) dx = \int_0^1 \frac{56}{9} x^4 dx = \frac{56}{45} x^5 \Big|_0^1 = \frac{56}{45}$$

$$E(XY) = \int_0^1 \int_x^{2x} \frac{8}{3} x^2 y^2 dy \, dx = \int_0^1 \frac{8}{9} x^2 y^3 \Big|_x^{2x} dx = \int_0^1 \frac{8}{9} x^2 \left(8x^3 - x^3\right) dx = \int_0^1 \frac{56}{9} x^5 dx = \frac{56}{54} = \frac{28}{27}$$

$$\text{Cov}(X,Y) = E(XY) - E(X)E(Y) = \frac{28}{27} - \left(\frac{56}{45}\right)\left(\frac{4}{5}\right) = 0.04$$

7. Solution: C

Let $u = 2x$. Then $\int_0^2 f(2x)\,dx = \int_0^4 f(u)\frac{1}{2}du = \frac{1}{2}\left(\int_0^2 f(u)\,du + \int_2^4 f(u)\,du\right)$

$$= \frac{1}{2}(3+5) = 4$$

8. Solution: B

Apply Bayes' Formula. Let

A = Event of an accident

B_1 = Event the driver's age is in the range 16-20

B_2 = Event the driver's age is in the range 21-30

B_3 = Event the driver's age is in the range 30-65

B_4 = Event the driver's age is in the range 66-99

Then

$$\Pr(B_1|A) = \frac{\Pr(A|B_1)\Pr(B_1)}{\Pr(A|B_1)\Pr(B_1)+\Pr(A|B_2)\Pr(B_2)+\Pr(A|B_3)\Pr(B_3)+\Pr(A|B_4)\Pr(B_4)}$$

$$= \frac{(0.06)(0.08)}{(0.06)(0.08)+(0.03)(0.15)+(0.02)(0.49)+(0.04)(0.28)} = 0.1584$$

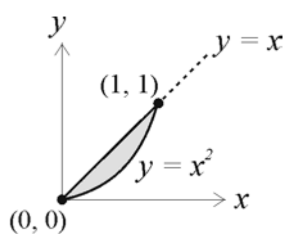
9. Solution: D

Total Medical Malpractice payments that the company makes after it stops selling the coverage may be represented by the geometric series

$$60+60(0.8)+60(0.8)^2+\ldots = 60\sum_{k=0}^{\infty}(0.8)^k = \frac{60}{1-0.8} = \frac{60}{0.2} = 300$$

10. Solution: E

The shaded portion of the graph below shows the region over which $f(x,y)$ is nonzero:



We can infer from the graph that the marginal density function of Y is given by

$$g(y) = \int_y^{\sqrt{y}} 15y\,dx = 15xy\Big|_y^{\sqrt{y}} = 15y\left(\sqrt{y}-y\right) = 15y^{3/2}\left(1-y^{1/2}\right),\ 0<y<1$$

or more precisely, $g(y) = \begin{cases} 15y^{3/2}(1-y)^{1/2}, & 0<y<1 \\ 0 & \text{otherwise} \end{cases}$

11. Solution: B

$$S(t) = 5000e^{0.1e^{0.25t}}$$

$$S'(t) = 5000e^{0.1e^{0.25t}}\left(0.1e^{0.25t}\right)(0.25) = 125e^{0.1e^{0.25t}}e^{0.25t}$$

$$S'(8) = 125e^{0.1e^{2}}e^{2} = 1934$$

12. Solution: D

Note that $E(X) = \int_{-2}^{0} -\frac{x^2}{10}dx + \int_{0}^{4} \frac{x^2}{10}dx = -\left.\frac{x^3}{30}\right|_{-2}^{0} + \left.\frac{x^3}{30}\right|_{0}^{4} = -\frac{8}{30} + \frac{64}{30} = \frac{56}{30} = \frac{28}{15}$

13. Solution: C

By the central limit theorem, the total contributions are approximately normally distributed with mean $n\mu = (2025)(3125) = 6,328,125$ and standard deviation $\sigma\sqrt{n} = 250\sqrt{2025} = 11,250$. From the tables, the 90th percentile for a standard normal random variable is 1.282 . Letting p be the 90th percentile for total contributions, $\frac{p - n\mu}{\sigma\sqrt{n}} = 1.282$, and so $p = n\mu + 1.282\sigma\sqrt{n} = 6,328,125 + (1.282)(11,250) = 6,342,548$.

14. Solution: C

Note that $f'(x) = \lim_{h\to 0}\frac{f(x+h) - f(x)}{h} = \lim_{h\to 0}\left(3x^2 + 3xh + h^2 + 2\right) = 3x^2 + 2$

Moreover, $f(x) = \int f'(x)dx = \int\left(3x^2 + 2\right)dx = x^3 + 2x + C$

where C is some constant. And since $f(0) = 1$, we see that $1 = f(0) = C$

In summary,

$i)\ f(x) = x^3 + 2x + 1$, and

$ii)\ f'(x) = 3x^2 + 2$

Finally, for $g(x) = e^{-x}f(x)$,

we have

$$g'(x) = -e^{-x}f(x) + e^{-x}f'(x) = e^{-x}\left[f'(x) - f(x)\right]$$

$$= e^{-x}\left(3x^2 + 2 - x^3 - 2x - 1\right)$$

and $g'(3) = e^{-3}(27 + 2 - 27 - 6 - 1) = -5e^{-3}$

15. Solution: C

We use the relationships $\text{Var}(aX+b)=a^2\ \text{Var}(X)$, $\text{Cov}(aX,bY)=ab\ \text{Cov}(X,Y)$, and $\text{Var}(X+Y)=\text{Var}(X)+\text{Var}(Y)+2\ \text{Cov}(X,Y)$. First we observe

$17{,}000=\text{Var}(X+Y)=5000+10{,}000+2\ \text{Cov}(X,Y)$, and so $\text{Cov}(X,Y)=1000.$

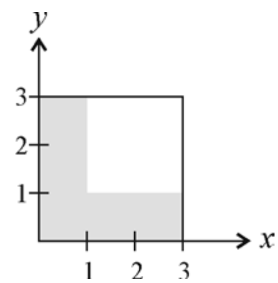
We want to find $\text{Var}\left[(X+100)+1.1Y\right]=\ \text{Var}\left[(X+1.1Y)+100\right]$

$=\text{Var}\left[X+1.1Y\right]=\text{Var}\,X+\text{Var}\left[(1.1)Y\right]+2\ \text{Cov}(X,1.1Y)$

$=\text{Var}\,X+(1.1)^2\ \text{Var}\,Y+2(1.1)\text{Cov}(X,Y)=5000+12{,}100+2200=19{,}300.$

16. Solution: B

That the device fails within the first hour means the joint density function must be integrated over the shaded region shown below.



This evaluation is more easily performed by integrating over the unshaded region and subtracting from 1.

$\Pr\left[(X<1)\cup(Y<1)\right]$

$$=1-\int_1^3\int_1^3\frac{x+y}{27}\,dx\,dy=1-\int_1^3\left.\frac{x^2+2xy}{54}\right|_1^3 dy=1-\frac{1}{54}\int_1^3(9+6y-1-2y)\,dy$$

$$=1-\frac{1}{54}\int_1^3(8+4y)\,dy=1-\frac{1}{54}\left.\left(8y+2y^2\right)\right|_1^3=1-\frac{1}{54}(24+18-8-2)=1-\frac{32}{54}=\frac{11}{27}=0.41$$

17. Solution: C

From the integral as the limit of Riemann Sums

$$\lim_{n\to\infty}\frac{1}{n}\left(e^{\frac{1}{n}}+e^{\frac{2}{n}}+\cdots+e^{\frac{n}{n}}\right)=\int_0^1 e^x dx=e^1-e^0=e-1.$$

Alternate Solution:

$$\lim_{n\to\infty}\frac{1}{n}\left(e^{1/n}+e^{2/n}+\ldots+e^{n/n}\right)=\lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n}\left(e^{1/n}\right)^k=\lim_{n\to\infty}\frac{1}{n}e^{1/n}\frac{1-e^{n/n}}{1-e^{1/n}}=\left(1-e\right)\lim_{n\to\infty}\frac{\frac{1}{n}e^{1/n}}{1-e^{1/n}}$$

$$=\left(1-e\right)\lim_{n\to\infty}\frac{1/n}{e^{-1/n}-1}=\left(1-e\right)\lim_{n\to\infty}\frac{d/dn\left[1/n\right]}{d/dn\left[e^{-1/n}-1\right]}\quad\text{(L'Hôspital's Rule)}$$

$$=\left(1-e\right)\lim_{n\to\infty}\frac{-1/n^2}{1/n^2e^{-1/n}}=\left(1-e\right)\lim_{n\to\infty}\left(-e^{1/n}\right)=e-1$$

18. Solution: D

Let

A = event that a policyholder has an auto policy

H = event that a policyholder has a homeowners policy

Then based on the information given,

$$\Pr\left(A\cap H\right)=0.15$$

$$\Pr\left(A\cap H^c\right)=\Pr\left(A\right)-\Pr\left(A\cap H\right)=0.65-0.15=0.50$$

$$\Pr\left(A^c\cap H\right)=\Pr\left(H\right)-\Pr\left(A\cap H\right)=0.50-0.15=0.35$$

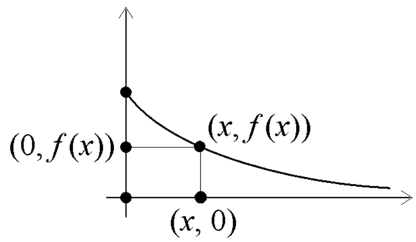
and the portion of policyholders that will renew at least one policy is given by

$$0.4\ \Pr\left(A\cap H^c\right)+0.6\ \Pr\left(A^c\cap H\right)+0.8\ \Pr\left(A\cap H\right)$$

$$=\left(0.4\right)\left(0.5\right)+\left(0.6\right)\left(0.35\right)+\left(0.8\right)\left(0.15\right)=0.53\qquad\left(=53\%\right)$$

19. Solution: A

The diagram shows a typical rectangle.



The rectangle's perimeter is given by $P(x) = 2x + 2e^{-2x}$, $x > 0$.

Therefore,

$$P'(x) = 2 - 4e^{-2x}$$

$$P''(x) = 8e^{-2x} > 0 \quad \text{for} \quad \text{all } x > 0$$

Now the critical points of P are determined by the condition

$$0 = P'(x) = 2 - 4e^{-2x}$$

$$4e^{-2x} = 2$$

$$e^{-2x} = \frac{1}{2}$$

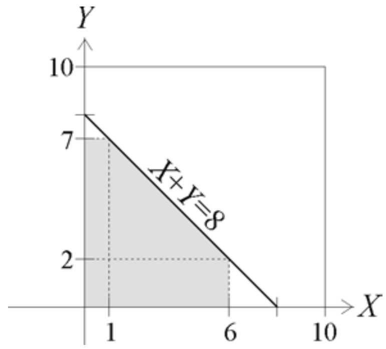
$$-2x = \ln\left(\frac{1}{2}\right) = -\ln(2)$$

$$x = \frac{1}{2}\ln(2)$$

Moreover, the critical point at $x = \frac{1}{2}\ln(2)$ is an absolute minimum because $P''(x) > 0$ for all $x > 0$. Note that actually determining the value of the critical point is not essential to the solution. $P''(x) > 0$ for all $x > 0$ also tells us that P has no absolute maximum and no points of inflection.

20. Solution: C

Define X and Y to be loss amounts covered by the policies having deductibles of 1 and 2, respectively. The shaded portion of the graph below shows the region over which the total benefit paid to the family does not exceed 5:



We can also infer from the graph that the uniform random variables X and Y have joint density function $f(x,y) = \dfrac{1}{100}$, $0 < x < 10$, $0 < y < 10$

We could integrate f over the shaded region in order to determine the desired probability. However, since X and Y are uniform random variables, it is simpler to determine the portion of the 10 x 10 square that is shaded in the graph above. That is,

$$\Pr(\text{Total Benefit Paid Does not Exceed 5})$$
$$= \Pr(0 < X < 6,\ 0 < Y < 2) + \Pr(0 < X < 1,\ 2 < Y < 7) + \Pr(1 < X < 6,\ 2 < Y < 8 - X)$$
$$= \frac{(6)(2)}{100} + \frac{(1)(5)}{100} + \frac{(1/2)(5)(5)}{100} = \frac{12}{100} + \frac{5}{100} + \frac{12.5}{100} = 0.295$$

21. Solution: D

$$\frac{dP}{dt} = 3.5\left(\frac{6}{5}L^{1/5}\frac{dL}{dt}C^{1/2} + L^{6/5}\frac{1}{2}C^{-1/2}\frac{dC}{dt}\right)$$

At the time, t_0, when $L = 12$ and $C = 4$

$$\left.\frac{dP}{dt}\right|_{t=t_0} = 3.5\left[\frac{6}{5}\left(12^{1/5}\right)(2.5)\left(4^{1/2}\right) + 12^{6/5}\left(\frac{1}{2}\right)\left(4^{-1/2}\right)(0.5)\right] = 43.148$$

22. Solution: B

The distribution function of X is given by

$$F(x) = \int_{200}^{x} \frac{2.5(200)^{2.5}}{t^{3.5}} dt = \left. \frac{-(200)^{2.5}}{t^{2.5}} \right|_{200}^{x} = 1 - \frac{(200)^{2.5}}{x^{2.5}} \quad , \quad x > 200$$

Therefore, the p^{th} percentile x_p of X is given by

$$\frac{p}{100} = F(x_p) = 1 - \frac{(200)^{2.5}}{x_p^{\ 2.5}}$$

$$1 - 0.01p = \frac{(200)^{2.5}}{x_p^{\ 2.5}}$$

$$(1 - 0.01p)^{2/5} = \frac{200}{x_p}$$

$$x_p = \frac{200}{(1 - 0.01p)^{2/5}}$$

It follows that $x_{70} - x_{30} = \dfrac{200}{(0.30)^{2/5}} - \dfrac{200}{(0.70)^{2/5}} = 93.06$

23. Solution: A

The distribution function of Y is given by

$G(y) = \Pr(T^2 \le y) = \Pr(T \le \sqrt{y}) = F(\sqrt{y}) = 1 - 4/y$

for $y > 4$. Differentiate to obtain the density function $g(y) = 4y^{-2}$

Alternate solution:

Differentiate $F(t)$ to obtain $f(t) = 8t^{-3}$ and set $y = t^2$. Then $t = \sqrt{y}$ and

$$g(y) = f(t(y))|dt/dy| = f(\sqrt{y})\left|\frac{d}{dt}(\sqrt{y})\right| = 8y^{-3/2}\left(\frac{1}{2}y^{-1/2}\right) = 4y^{-2}$$

24. Solution: C

The marginal density of X is given by

$$f_x(x) = \int_0^1 \frac{1}{64}\left(10 - xy^2\right) dy = \frac{1}{64}\left(10y - \frac{xy^3}{3}\right)\Bigg|_0^1 = \frac{1}{64}\left(10 - \frac{x}{3}\right)$$

Then $E(X) = \int_2^{10} x\,\mathrm{f}_x(x)dx = \int_2^{10} \frac{1}{64}\left(10x - \frac{x^2}{3}\right)dx = \frac{1}{64}\left(5x^2 - \frac{x^3}{9}\right)\Bigg|_2^{10}$

$$= \frac{1}{64}\left[\left(500 - \frac{1000}{9}\right) - \left(20 - \frac{8}{9}\right)\right] = 5.778$$

25. Solution: B

Denote the insurance payment by the random variable Y. Then

$$Y = \begin{cases} 0 & \text{if} \quad 0 < X \le C \\ X - C & \text{if} \quad C < X < 1 \end{cases}$$

Now we are given that

$$0.64 = \Pr\left(Y < 0.5\right) = \Pr\left(0 < X < 0.5 + C\right) = \int_0^{0.5+C} 2x\ dx = x^2\Big|_0^{0.5+C} = \left(0.5 + C\right)^2$$

Therefore, solving for C, we find $C = \pm 0.8 - 0.5$

Finally, since $0 < C < 1$, we conclude that $C = 0.3$

26. Solution: A

Note that $g\left(x\right)$ is discontinuous at x such that

$$0 = x^2 + 2x - 8 = \left(x+4\right)\left(x-2\right)$$

It follows that $g\left(x\right)$ is discontinuous if $x = -4$ or $x = 2$.

Since $g\left(x\right) = \dfrac{x+4}{\left(x+4\right)\left(x-2\right)} = \dfrac{1}{x-2}$ for all $x \ne -4$, the discontinuity at $x = -4$ can be removed by defining $g\left(4\right) = \dfrac{1}{-4-2} = -\dfrac{1}{6}$. But the discontinuity at $x = 2$ cannot be removed because $\lim\limits_{x \to 2} g\left(x\right)$ does not exist.

27. Solution: D

We begin by solving the differential equation $\frac{dA}{dt} = Ai, \quad A(0) = 5000$.

$$\int \frac{1}{A} dA = \int i \, dt$$
$$\ln A = it + C$$
$$A = e^{it+C} = e^C e^{it}$$
$$5000 = A(0) = e^C$$
$$A(t) = 5000e^{it}$$

Now we need to find i such that

$$20,000 = A(24) = 5000e^{24i}$$
$$e^{24i} = 4$$
$$24i = \ln(4)$$
$$i = \frac{1}{24}\ln(2^2) = \frac{1}{12}\ln(2)$$

28. Solution: C

Note that the conditional density function

$$f\left(y \middle| x = \frac{1}{3}\right) = \frac{f(1/3, y)}{f_x(1/3)}, \quad 0 < y < \frac{2}{3},$$
$$f_x\left(\frac{1}{3}\right) = \int_0^{2/3} 24(1/3)\, y \, dy = \int_0^{2/3} 8y\, dy = 4y^2\Big|_0^{2/3} = \frac{16}{9}$$

It follows that $f\left(y \middle| x = \frac{1}{3}\right) = \frac{9}{16} f(1/3, y) = \frac{9}{2} y \quad , \quad 0 < y < \frac{2}{3}$

Consequently,

$$\Pr\left[Y < X \middle| X = 1/3\right] = \int_0^{1/3} \frac{9}{2} y \, dy = \frac{9}{4} y^2 \Big|_0^{1/3} = \frac{1}{4}$$

29. Solution: E

We are given that R is uniform on the interval $(0.04, 0.08)$ and $V = 10,000e^R$

Therefore, the distribution function of V is given by

$$F(v) = \Pr[V \le v] = \Pr\left[10,000e^R \le v\right] = \Pr\left[R \le \ln(v) - \ln(10,000)\right]$$

$$= \frac{1}{0.04}\int_{0.04}^{\ln(v)-\ln(10,000)} dr = \frac{1}{0.04}\, r\Big|_{0.04}^{\ln(v)-\ln(10,000)} = 25\ln(v) - 25\ln(10,000) - 1$$

$$= 25\left[\ln\left(\frac{v}{10,000}\right) - 0.04\right]$$

30. Solution: E

The given power series is $1 - x - x^2 + x^3 + x^4 - x^5 - x^6 + \cdots$, where two positive terms are followed by two negative terms. By regrouping terms, this can be written as the sum of two geometric series:

$$\left(1 - x^2 + x^4 - x^6 + \cdots\right) - \left(x - x^3 + x^5 \cdots\right) = \frac{1}{1+x^2} - \frac{x}{1+x^2} = \frac{1-x}{1+x^2}$$

The solution can also be obtained by the process of elimination:

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + x^4 + \cdots$$

$$\frac{1}{1+x} = 1 - x + x^2 - x^3 + x^4 \cdots$$

$$\frac{1-2x}{1-x} = (1-2x)\left(1 + x + x^2 + x^3 + x^4 + \cdots\right) = 1 - x - x^2 - x^3 - x^4 - \cdots$$

$$\frac{x^2+x}{1+x^2} = \left(x^2 + x\right)\left(1 - x^2 + x^4 - x^6 + x^8 - \cdots\right) = x + x^2 - x^3 - x^4 + x^5 + x^6 - \cdots$$

31. Solution: D

Let

H = Event of a heavy smoker

L = Event of a light smoker

N = Event of a non-smoker

D = Event of a death within five-year period

Now we are given that $\Pr\left[D|L\right] = 2\ \Pr\left[D|N\right]$ and $\Pr\left[D|L\right] = \frac{1}{2}\Pr\left[D|H\right]$

Therefore, upon applying Bayes' Formula, we find that

$$\Pr\left[H|D\right] = \frac{\Pr\left[D|H\right]\Pr\left[H\right]}{\Pr\left[D|N\right]\Pr\left[N\right] + \Pr\left[D|L\right]\Pr\left[L\right] + \Pr\left[D|H\right]\Pr\left[H\right]}$$

$$= \frac{2\Pr\left[D|L\right](0.2)}{\frac{1}{2}\Pr\left[D|L\right](0.5) + \Pr\left[D|L\right](0.3) + 2\Pr\left[D|L\right](0.2)} = \frac{0.4}{0.25 + 0.3 + 0.4} = 0.42$$

32. Solution: C

First note that the density function of X is given by

$$f(x) = \begin{cases} \frac{1}{2} & \text{if} \quad x = 1 \\ x - 1 & \text{if} \quad 1 < x < 2 \\ 0 & \text{otherwise} \end{cases}$$

Then

$$E(X) = \frac{1}{2} + \int_1^2 x(x-1)\,dx = \frac{1}{2} + \int_1^2 \left(x^2 - x\right)dx = \frac{1}{2} + \left(\frac{1}{3}x^3 - \frac{1}{2}x^2\right)\Big|_1^2$$

$$= \frac{1}{2} + \frac{8}{3} - \frac{4}{2} - \frac{1}{3} + \frac{1}{2} = \frac{7}{3} - 1 = \frac{4}{3}$$

$$E\left(X^2\right) = \frac{1}{2} + \int_1^2 x^2(x-1)\,dx = \frac{1}{2} + \int_1^2 \left(x^3 - x^2\right)dx = \frac{1}{2} + \left(\frac{1}{4}x^4 - \frac{1}{3}x^3\right)\Big|_1^2$$

$$= \frac{1}{2} + \frac{16}{4} - \frac{8}{3} - \frac{1}{4} + \frac{1}{3} = \frac{17}{4} - \frac{7}{3} = \frac{23}{12}$$

$$Var(X) = E\left(X^2\right) - \left[E(X)\right]^2 = \frac{23}{12} - \left(\frac{4}{3}\right)^2 = \frac{23}{12} - \frac{16}{9} = \frac{5}{36}$$

33. Solution: B

Observe that

$$f(x) = \frac{2x}{x+1}$$

$$f^2(x) = 2 \cdot \left[\frac{2x}{x+1}\right] \bigg/ \left[\frac{2x}{x+1} + 1\right] = \frac{4x}{3x+1}$$

$$f^3(x) = 2 \cdot \left[\frac{4x}{3x+1}\right] \bigg/ \left[\frac{4x}{3x+1} + 1\right] = \frac{8x}{7x+1}$$

$$f^4(x) = 2 \cdot \left[\frac{8x}{7x+1}\right] \bigg/ \left[\frac{8x}{7x+1} + 1\right] = \frac{16x}{15x+1}$$

$$\vdots$$

$$f^n(x) = \frac{2^n x}{\left(2^n - 1\right)x + 1}$$

Therefore, for $x \neq 0$, $\lim_{n\to\infty} f^n(x) = \lim_{n\to\infty} \frac{2^n x}{\left(2^n - 1\right)x + 1} = \lim_{n\to\infty} \frac{x}{x - \left(1/2^n\right)x + 1/2^n} = \frac{x}{x} = 1$

34. Solution: C

We know the density has the form $C\left(10 + x\right)^{-2}$ for $0 < x < 40$ (equals zero otherwise).

First, determine the proportionality constant *C* from the condition $\int_0^{40} f(x)dx = 1$:

$$1 = \int_0^{40} C\left(10 + x\right)^{-2} dx = -C(10 + x)^{-1}\Big|_0^{40} = \frac{C}{10} - \frac{C}{50} = \frac{2}{25}C$$

so $C = 25/2$, or 12.5. Then, calculate the probability over the interval (0, 6):

$$12.5\int_0^6 \left(10 + x\right)^{-2} dx = -\left(10 + x\right)^{-1}\Big|_0^6 = \left(\frac{1}{10} - \frac{1}{16}\right)(12.5) = 0.47 \ .$$

35. Solution: E

We have

$f(x,y) = y^2 - 2x^2 y + 4x^3 + 20x^2$

$f_x(x,y) = -4xy + 12x^2 + 40x$

$f_{xx}(x,y) = -4y + 24x + 40$

$f_{xy}(x,y) = -4x$

$f_y(x,y) = 2y - 2x^2$

$f_{yy}(x,y) = 2$

$D(x,y) = f_{xx}(x,y) f_{yy}(x,y) - f_{xy}^2(x,y) = -8y + 48x + 80 - 16x^2 = -8\left(y - 6x - 10 + 2x^2\right)$

Applying the "Second-Partials Test," we then infer the following about each critical point:

$i)$ $(0,0)$

$D(0,0) = 80 > 0$

$f_{xx}(0,0) = 40 > 0$

f has a local minimum at $(0,0)$

$ii)$ $(5,25)$

$D(5,25) = -8(25 - 30 - 10 + 50) = -280 < 0$

f has a saddle point at $(5,25)$.

$iii)$ $(-2,4)$

$D(-2,4) = -8(4 + 12 - 10 + 8) = -112$

f has a saddle point at $(-2,4)$.

36. Solution: D

Define $f(X)$ to be hospitalization payments made by the insurance policy. Then

$$f(X) = \begin{cases} 100X & \text{if } X = 1,2,3 \\ 300 + 25(X-3) & \text{if } X = 4,5 \end{cases}$$

and

$$E\left[f(X)\right] = \sum_{k=1}^{5} f(k)\Pr\left[X = k\right]$$

$$= 100\left(\frac{5}{15}\right) + 200\left(\frac{4}{15}\right) + 300\left(\frac{3}{15}\right) + 325\left(\frac{2}{15}\right) + 350\left(\frac{1}{15}\right)$$

$$= \frac{1}{3}\left[100 + 160 + 180 + 130 + 70\right] = \frac{640}{3} = 213.33$$

37. Solution: D

Let

$O =$ Event of operating room charges

$E =$ Event of emergency room charges

Then

$$0.85 = \Pr(O \cup E) = \Pr(O) + \Pr(E) - \Pr(O \cap E)$$

$$= \Pr(O) + \Pr(E) - \Pr(O)\Pr(E) \quad \text{(Independence)}$$

Since $\Pr(E^c) = 0.25 = 1 - \Pr(E)$, it follows $\Pr(E) = 0.75$.

So $0.85 = \Pr(O) + 0.75 - \Pr(O)(0.75)$

$$\Pr(O)(1 - 0.75) = 0.10$$

$$\Pr(O) = 0.40$$

38. Solution: C

The time T at which the inventory must be replenished is determined by

$$19 - S(T) = 1$$

$$S(T) = 18 = e^{3T} - 1$$

$$e^{3T} = 19$$

$$T = \frac{1}{3}\ln(19)$$

Denote inventory carrying cost incurred through time t by $C(t)$. Then

$$C(T) = 15\int_0^T \left[19 - S(t)\right]dt = 15\int_0^T \left[19 - \left(e^{3t} - 1\right)\right]dt = 15\int_0^T \left(20 - e^{3t}\right)dt = 15\left[20t - \frac{1}{3}e^{3t}\right]\Bigg|_0^T$$

$$= 15\left[20T - \frac{1}{3}e^{3T} + \frac{1}{3}\right] = 15\left[\frac{20}{3}\ln(19) - \frac{19}{3} + \frac{1}{3}\right] = 204.44$$

39. Solution: E

$$M(t_1, t_2) = E\left[e^{t_1 W + t_2 Z}\right] = E\left[e^{t_1(X+Y) + t_2(Y-X)}\right] = E\left[e^{(t_1 - t_2)X} e^{(t_1 + t_2)Y}\right]$$

$$= E\left[e^{(t_1 - t_2)X}\right]E\left[e^{(t_1 + t_2)Y}\right] = e^{\frac{1}{2}(t_1 - t_2)^2} e^{\frac{1}{2}(t_1 + t_2)^2} = e^{\frac{1}{2}\left(t_1^2 - 2t_1t_2 + t_2^2\right)} e^{\frac{1}{2}\left(t_1^2 + 2t_1t_2 + t_2^2\right)} = e^{t_1^2 + t_2^2}$$

40. Solution: B

The speed of the particle at time t is given by

$$v(t) = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} = \sqrt{(2t-7)^2 + \left(\frac{1}{2}t-6\right)^2} \quad , \quad t \geq 0$$

In order to simplify the calculations involved in determining the time at which the minimum speed occurs, however, we instead seek the time t that minimizes

$$h(t) = \left[v(t)\right]^2 = (2t-7)^2 + \left(\frac{1}{2}t-6\right)^2$$

since $h(t)$ and $v(t)$ are minimized at the same time t.

To this end, note that

$$h'(t) = (2)(2)(2t-7) + (2)(1/2)\left(\frac{1}{2}t-6\right) = 8t - 28 + \frac{1}{2}t - 6 = \frac{17}{2}t - 34$$

$$h''(t) = \frac{17}{2} > 0$$

Since $h''(t) > 0$ for all t, $h(t)$ is minimized at any critical point in the domain $t \geq 0$. To find the location of the desired critical point, solve

$$0 = h'(t) = \frac{17}{2}t - 34$$

$$\frac{17}{2}t = 34$$

$$t = 4$$