

November 2001

Course 1

Mathematical Foundations of Actuarial Science

Society of Actuaries/Casualty Actuarial Society

1. An urn contains 10 balls: 4 red and 6 blue. A second urn contains 16 red balls and an unknown number of blue balls. A single ball is drawn from each urn. The probability that both balls are the same color is 0.44 .

Calculate the number of blue balls in the second urn.

- (A) 4
- (B) 20
- (C) 24
- (D) 44
- (E) 64

2. Let R be a region in the xy -plane with area 2 . Let $\iint_R f(x, y) dA = 6$.

Determine $\iint_R [4f(x, y) - 2] dA$.

- (A) 12
- (B) 18
- (C) 20
- (D) 22
- (E) 44

3. Sales, S , of a new insurance product are dependent upon the labor, L , of the sales force and the amount of advertising, A , for the product. The relationship can be modeled by

$$S = 175 L^{3/2} A^{4/5}.$$

Which of the following statements is true?

- (A) S increases at an increasing rate as L increases and increases at a decreasing rate as A increases.
- (B) S increases at an increasing rate as L increases and increases at an increasing rate as A increases.
- (C) S increases at a decreasing rate as L increases and increases at a decreasing rate as A increases.
- (D) S increases at a decreasing rate as L increases and increases at an increasing rate as A increases.
- (E) S increases at a constant rate as L increases and increases at a constant rate as A increases.

4. Upon arrival at a hospital's emergency room, patients are categorized according to their condition as critical, serious, or stable. In the past year:

- (i) 10% of the emergency room patients were critical;
- (ii) 30% of the emergency room patients were serious;
- (iii) the rest of the emergency room patients were stable;
- (iv) 40% of the critical patients died;
- (vi) 10% of the serious patients died; and
- (vii) 1% of the stable patients died.

Given that a patient survived, what is the probability that the patient was categorized as serious upon arrival?

- (A) 0.06
- (B) 0.29
- (C) 0.30
- (D) 0.39
- (E) 0.64

5. An insurance company sells a one-year automobile policy with a deductible of 2 . The probability that the insured will incur a loss is 0.05 . If there is a loss, the probability of a loss of amount N is $\frac{K}{N}$, for $N = 1, \dots, 5$ and K a constant. These are the only possible loss amounts and no more than one loss can occur.

Determine the net premium for this policy.

- (A) 0.031
- (B) 0.066
- (C) 0.072
- (D) 0.110
- (E) 0.150

6. Let C be the curve defined by the parametric equations $x = t^2 + t$, $y = t^2 - 1$.

Determine the value of t at which the line tangent to the graph of C is parallel to the line $5y - 4x = 3$.

- (A) $-\frac{1}{10}$
- (B) $\frac{2}{5}$
- (C) $\frac{5}{8}$
- (D) $\frac{5}{3}$
- (E) 2

7. Let X denote the size of a surgical claim and let Y denote the size of the associated hospital claim. An actuary is using a model in which $E(X) = 5$, $E(X^2) = 27.4$, $E(Y) = 7$, $E(Y^2) = 51.4$, and $\text{Var}(X+Y) = 8$.

Let $C_1 = X+Y$ denote the size of the combined claims before the application of a 20% surcharge on the hospital portion of the claim, and let C_2 denote the size of the combined claims after the application of that surcharge.

Calculate $\text{Cov}(C_1, C_2)$.

- (A) 8.80
- (B) 9.60
- (C) 9.76
- (D) 11.52
- (E) 12.32

8. A medical researcher conducts a ten-week study of patients infected with a chronic disease. Over the course of the study, the researcher finds that the fraction of patients exhibiting severe symptoms can be modeled as

$$F(t) = te^{-t}$$

where t is time elapsed, in weeks, since the study began.

What is the minimum fraction of patients exhibiting severe symptoms between the end of the first week and the end of the seventh week of the study?

- (A) 0.0000
- (B) 0.0004
- (C) 0.0027
- (D) 0.0064
- (E) 0.3679

9. Among a large group of patients recovering from shoulder injuries, it is found that 22% visit both a physical therapist and a chiropractor, whereas 12% visit neither of these. The probability that a patient visits a chiropractor exceeds by 0.14 the probability that a patient visits a physical therapist.

Determine the probability that a randomly chosen member of this group visits a physical therapist.

- (A) 0.26
- (B) 0.38
- (C) 0.40
- (D) 0.48
- (E) 0.62

10. Let $\{a_n\}$ be a sequence of real numbers.

For which of the following does the infinite series $\sum_{n=1}^{\infty} \left(a_n + \frac{1}{n} \right)$ converge?

(A) $a_n = 1$

(B) $a_n = \frac{1}{n}$

(C) $a_n = \frac{1}{n^2}$

(D) $a_n = \frac{(-1)^n}{n}$

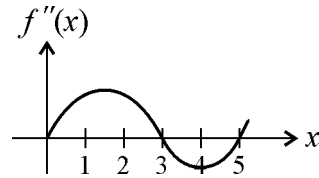
(E) $a_n = \frac{1-n}{n^2}$

11. A company takes out an insurance policy to cover accidents that occur at its manufacturing plant. The probability that one or more accidents will occur during any given month is $\frac{3}{5}$. The number of accidents that occur in any given month is independent of the number of accidents that occur in all other months.

Calculate the probability that there will be at least four months in which no accidents occur before the fourth month in which at least one accident occurs.

- (A) 0.01
- (B) 0.12
- (C) 0.23
- (D) 0.29
- (E) 0.41

12. Let f be a function such that $f'(0) = 0$. The graph of the second derivative f'' is shown below.



Determine the x -value on the interval $[0, 5]$ at which the maximum value of f occurs.

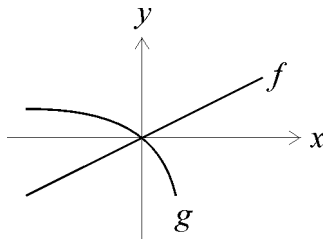
- (A) $x = 0$
- (B) At some x between 1 and 2
- (C) $x = 3$
- (D) At some x between 3 and 4
- (E) $x = 5$

13. An actuary models the lifetime of a device using the random variable $Y = 10X^{0.8}$, where X is an exponential random variable with mean 1 year.

Determine the probability density function $f(y)$, for $y > 0$, of the random variable Y .

- (A) $10y^{0.8}e^{-8y^{-0.2}}$
- (B) $8y^{-0.2}e^{-10y^{0.8}}$
- (C) $8y^{-0.2}e^{-(0.1y)^{1.25}}$
- (D) $(0.1y)^{1.25}e^{-0.125(0.1y)^{0.25}}$
- (E) $0.125(0.1y)^{0.25}e^{-(0.1y)^{1.25}}$

14. The graphs of differentiable functions f and g are shown in the diagram below.



Which of the following is true about $\lim_{x \rightarrow 0} \frac{f(x)}{g(x)}$?

- (A) The limit is less than 0 .
- (B) The limit is 0 .
- (C) The limit is 1 .
- (D) The limit is greater than 1 .
- (E) The limit does not exist.

15. A city has just added 100 new female recruits to its police force. The city will provide a pension to each new hire who remains with the force until retirement. In addition, if the new hire is married at the time of her retirement, a second pension will be provided for her husband. A consulting actuary makes the following assumptions:

- (i) Each new recruit has a 0.4 probability of remaining with the police force until retirement.
- (ii) Given that a new recruit reaches retirement with the police force, the probability that she is not married at the time of retirement is 0.25 .
- (iii) The number of pensions that the city will provide on behalf of each new hire is independent of the number of pensions it will provide on behalf of any other new hire.

Determine the probability that the city will provide at most 90 pensions to the 100 new hires and their husbands.

- (A) 0.60
- (B) 0.67
- (C) 0.75
- (D) 0.93
- (E) 0.99

16. You are given the following information about N , the annual number of claims for a randomly selected insured:

$$P(N = 0) = \frac{1}{2}$$

$$P(N = 1) = \frac{1}{3}$$

$$P(N > 1) = \frac{1}{6}$$

Let S denote the total annual claim amount for an insured. When $N = 1$, S is exponentially distributed with mean 5. When $N > 1$, S is exponentially distributed with mean 8.

Determine $P(4 < S < 8)$.

- (A) 0.04
- (B) 0.08
- (C) 0.12
- (D) 0.24
- (E) 0.25

17. The loss due to a fire in a commercial building is modeled by a random variable X with density function

$$f(x) = \begin{cases} 0.005(20-x) & \text{for } 0 < x < 20 \\ 0 & \text{otherwise.} \end{cases}$$

Given that a fire loss exceeds 8, what is the probability that it exceeds 16 ?

- (A) $\frac{1}{25}$
- (B) $\frac{1}{9}$
- (C) $\frac{1}{8}$
- (D) $\frac{1}{3}$
- (E) $\frac{3}{7}$

18. Given $f(x) = \begin{cases} x^2 & \text{for } x < 0, \\ \sqrt{x} & \text{for } 0 \leq x \leq 1, \\ x^2 & \text{for } x > 1, \end{cases}$

which statement is true?

- (A) f is differentiable everywhere.
- (B) f is differentiable everywhere except at $x = 0$, and continuous everywhere.
- (C) f is differentiable everywhere except at $x = 1$, and continuous everywhere.
- (D) f is differentiable everywhere except at $x = 0$ and $x = 1$, and continuous everywhere.
- (E) f is not continuous everywhere.

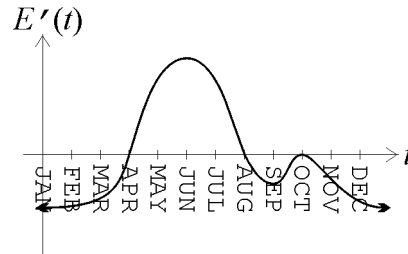
- 19.** A baseball team has scheduled its opening game for April 1. If it rains on April 1, the game is postponed and will be played on the next day that it does not rain. The team purchases insurance against rain. The policy will pay 1000 for each day, up to 2 days, that the opening game is postponed.

The insurance company determines that the number of consecutive days of rain beginning on April 1 is a Poisson random variable with mean 0.6 .

What is the standard deviation of the amount the insurance company will have to pay?

- (A) 668
- (B) 699
- (C) 775
- (D) 817
- (E) 904

20. Employment at a company can be approximated as a function of time t by the differentiable function E . The graph of its derivative is given below:



In which month is employment at the company minimized?

- (A) FEB
- (B) APR
- (C) AUG
- (D) SEP
- (E) OCT

- 21.** An insurance company determines that N , the number of claims received in a week, is a random variable with $P[N = n] = \frac{1}{2^{n+1}}$, where $n \geq 0$. The company also determines that the number of claims received in a given week is independent of the number of claims received in any other week.

Determine the probability that exactly seven claims will be received during a given two-week period.

(A) $\frac{1}{256}$

(B) $\frac{1}{128}$

(C) $\frac{7}{512}$

(D) $\frac{1}{64}$

(E) $\frac{1}{32}$

22. A dose of 250 of a certain drug is injected into a patient every day at noon. The amount of the drug that remains in the body from each injection is given by $r(t) = 250 e^{-t/6}$, where t is the time in days since the injection.

Calculate the least upper bound (to the nearest integer) for how much of the drug will be in a patient's body if the injections are given indefinitely.

- (A) 1352
- (B) 1378
- (C) 1402
- (D) 1628
- (E) 1652

23. Let R be the region bounded by the polar curve $r = \sin(\theta) + \sqrt{3} \cos(\theta)$.

Which of the following integrals represents the area of the subset of R to the

left of the line $\theta = \frac{\pi}{2}$?

(A) $\frac{1}{2} \int_0^{\pi/2} r^2 d\theta$

(B) $\frac{1}{2} \int_{\pi/2}^{\pi} r^2 d\theta$

(C) $\frac{1}{2} \int_{\pi/2}^{2\pi/3} r^2 d\theta$

(D) $\frac{1}{2} \int_{\pi/2}^{2\pi} r^2 d\theta$

(E) $\frac{1}{2} \int_{\pi/3}^{\pi/2} r^2 d\theta$

24. An insurance company has 150,000 to spend on the development and promotion of a new insurance policy for renters. If x is spent on the development and y is spent on the promotion, $100x^{1/4}y^{1/2}$ policies will be sold.

Calculate the maximum sales, in thousands, the company can attain.

- (A) 398
- (B) 435
- (C) 453
- (D) 473
- (E) 487

25. Once a fire is reported to a fire insurance company, the company makes an initial estimate, X , of the amount it will pay to the claimant for the fire loss. When the claim is finally settled, the company pays an amount, Y , to the claimant. The company has determined that X and Y have the joint density function

$$f(x, y) = \frac{2}{x^2(x-1)} y^{-(2x-1)/(x-1)} \quad x > 1, y > 1.$$

Given that the initial claim estimated by the company is 2, determine the probability that the final settlement amount is between 1 and 3 .

- (A) $\frac{1}{9}$
- (B) $\frac{2}{9}$
- (C) $\frac{1}{3}$
- (D) $\frac{2}{3}$
- (E) $\frac{8}{9}$

26. An insurance company introduces a new annuity at time $t = 0$, where t is in years. The company has found that, using its current marketing strategies, the instantaneous rate of change of sales of an annuity can be modeled by $s'(t) = t + \frac{5}{2}$. At time $t = 2$, a new advertising campaign is introduced. The instantaneous rate of sales increase changes to $t^2 + \frac{1}{2}$.

Calculate the difference in total sales from time $t = 2$ to time $t = 4$ over what total sales would have been without the new advertising campaign.

- (A) $\frac{16}{3}$
- (B) 7
- (C) $\frac{26}{3}$
- (D) 10
- (E) $\frac{59}{3}$

27. A company establishes a fund of 120 from which it wants to pay an amount, C , to any of its 20 employees who achieve a high performance level during the coming year. Each employee has a 2% chance of achieving a high performance level during the coming year, independent of any other employee.

Determine the maximum value of C for which the probability is less than 1% that the fund will be inadequate to cover all payments for high performance.

- (A) 24
- (B) 30
- (C) 40
- (D) 60
- (E) 120

- 28.** Two insurers provide bids on an insurance policy to a large company. The bids must be between 2000 and 2200 . The company decides to accept the lower bid if the two bids differ by 20 or more. Otherwise, the company will consider the two bids further. Assume that the two bids are independent and are both uniformly distributed on the interval from 2000 to 2200 .

Determine the probability that the company considers the two bids further.

- (A) 0.10
- (B) 0.19
- (C) 0.20
- (D) 0.41
- (E) 0.60

- 29.** The owner of an automobile insures it against damage by purchasing an insurance policy with a deductible of 250 . In the event that the automobile is damaged, repair costs can be modeled by a uniform random variable on the interval $(0, 1500)$.

Determine the standard deviation of the insurance payment in the event that the automobile is damaged.

- (A) 361
- (B) 403
- (C) 433
- (D) 464
- (E) 521

- 30.** Let T_1 be the time between a car accident and reporting a claim to the insurance company. Let T_2 be the time between the report of the claim and payment of the claim. The joint density function of T_1 and T_2 , $f(t_1, t_2)$, is constant over the region $0 < t_1 < 6$, $0 < t_2 < 6$, $t_1 + t_2 < 10$, and zero otherwise.

Determine $E[T_1 + T_2]$, the expected time between a car accident and payment of the claim.

- (A) 4.9
- (B) 5.0
- (C) 5.7
- (D) 6.0
- (E) 6.7

31. A town's annual birth rate and annual death rate are each proportional to its population, y , with constants of proportionality k_1 and k_2 , respectively. As a result, the net growth of the town can be modeled by the equation

$$\frac{dy}{dt} = (k_1 - k_2)y$$

where t is measured in years.

The town's population doubles every 24 years, but it would be halved in 8 years if there were no births.

Determine k_2 .

(A) $-\frac{\ln 2}{6}$

(B) $-\frac{\ln 2}{8}$

(C) $\frac{\ln 2}{24}$

(D) $\frac{\ln 2}{12}$

(E) $\frac{\ln 2}{8}$

32. The number of injury claims per month is modeled by a random variable N with

$$P[N = n] = \frac{1}{(n + 1)(n + 2)}, \text{ where } n \geq 0.$$

Determine the probability of at least one claim during a particular month, given that there have been at most four claims during that month.

(A) $\frac{1}{3}$

(B) $\frac{2}{5}$

(C) $\frac{1}{2}$

(D) $\frac{3}{5}$

(E) $\frac{5}{6}$

- 33.** An insurance policy on an electrical device pays a benefit of 4000 if the device fails during the first year. The amount of the benefit decreases by 1000 each successive year until it reaches 0 . If the device has not failed by the beginning of any given year, the probability of failure during that year is 0.4 .

What is the expected benefit under this policy?

- (A) 2234
- (B) 2400
- (C) 2500
- (D) 2667
- (E) 2694

34. An auto insurance policy will pay for damage to both the policyholder's car and the other driver's car in the event that the policyholder is responsible for an accident. The size of the payment for damage to the policyholder's car, X , has a marginal density function of 1 for $0 < x < 1$. Given $X = x$, the size of the payment for damage to the other driver's car, Y , has conditional density of 1 for $x < y < x + 1$.

If the policyholder is responsible for an accident, what is the probability that the payment for damage to the other driver's car will be greater than 0.500 ?

- (A) $\frac{3}{8}$
- (B) $\frac{1}{2}$
- (C) $\frac{3}{4}$
- (D) $\frac{7}{8}$
- (E) $\frac{15}{16}$

35. Auto claim amounts, in thousands, are modeled by a random variable with density function $f(x) = xe^{-x}$ for $x \geq 0$.

The company expects to pay 100 claims if there is no deductible.

How many claims does the company expect to pay if the company decides to introduce a deductible of 1000 ?

- (A) 26
- (B) 37
- (C) 50
- (D) 63
- (E) 74

- 36.** An insurance company sells health insurance policies to individuals. The company can sell 80 policies per month if it charges 60 per policy. Each increase of 1 in the price per policy the company charges reduces the number of policies the company can sell per month by 1 .

Calculate the maximum monthly revenue the company can attain.

- (A) 4500
- (B) 4800
- (C) 4900
- (D) 5100
- (E) 5200

37. A device containing two key components fails when, and only when, both components fail. The lifetimes, T_1 and T_2 , of these components are independent with common density function $f(t) = e^{-t}$, $t > 0$. The cost, X , of operating the device until failure is $2T_1 + T_2$.

Which of the following is the density function of X for $x > 0$?

- (A) $e^{-x/2} - e^{-x}$
- (B) $2(e^{-x/2} - e^{-x})$
- (C) $\frac{x^2 e^{-x}}{2}$
- (D) $\frac{e^{-x/2}}{2}$
- (E) $\frac{e^{-x/3}}{3}$

- 38.** In a small metropolitan area, annual losses due to storm, fire, and theft are assumed to be independent, exponentially distributed random variables with respective means 1.0, 1.5, and 2.4 .

Determine the probability that the maximum of these losses exceeds 3 .

- (A) 0.002
- (B) 0.050
- (C) 0.159
- (D) 0.287
- (E) 0.414

39. Let P be a profit function given by $P(x) = -x^2 + 50x - 25$ for $x \geq 0$, where x is the number of units sold. Due to production and labor problems, a new function $P^*(x)$ is used. The graph of P^* is formed by translating the graph of P to the right two units and down three units.

Determine the revised profit function, $P^*(x)$.

- (A) $P^*(x) = -x^2 + 46x + 68$
- (B) $P^*(x) = -x^2 + 46x + 74$
- (C) $P^*(x) = -x^2 + 54x - 126$
- (D) $P^*(x) = -x^2 + 54x - 132$
- (E) $P^*(x) = -x^2 + 56x - 186$

40. Let X and Y be the number of hours that a randomly selected person watches movies and sporting events, respectively, during a three-month period. The following information is known about X and Y :

$$E(X) = 50$$

$$E(Y) = 20$$

$$\text{Var}(X) = 50$$

$$\text{Var}(Y) = 30$$

$$\text{Cov}(X, Y) = 10$$

One hundred people are randomly selected and observed for these three months. Let T be the total number of hours that these one hundred people watch movies or sporting events during this three-month period.

Approximate the value of $P(T < 7100)$.

- (A) 0.62
- (B) 0.84
- (C) 0.87
- (D) 0.92
- (E) 0.97

Course 1
November 2001
Final Answer Key

1	A	21	D
2	C	22	D
3	A	23	C
4	B	24	D
5	A	25	E
6	E	26	C
7	A	27	D
8	D	28	B
9	D	29	B
10	E	30	C
11	D	31	E
12	E	32	B
13	E	33	E
14	A	34	D
15	E	35	E
16	C	36	C
17	B	37	A
18	D	38	E
19	B	39	D
20	B	40	B