Course 1
May 2001
Answer Key

1	E	21	E
2	C	22	C
3	C	23	A
4	E	24	E
5	C	25	B
6	D	26	B
7	B	27	A
8	A	28	C
9	C	29	D
10	E	30	B
11	D	31	C
12	D	32	E
13	E	33	D
14	A	34	A
15	D	35	C
16	A	36	B
17	B	37	E
18	D	38	D
19	B	39	A
20	D	40	B

A	7
B	8
C	8
D	9
E	8

1. E

We are given that

$$5e^{-5b} = p_5 = p_6 = 6e^{-6b}$$

It follows that

$$\frac{5}{6} = \frac{e^{-6b}}{e^{-5b}} = e^{-6b}e^{5b} = e^{-6b+5b} = e^{-b}$$

$$\ln\left(\frac{5}{6}\right) = -b$$

$$b = -\ln\left(\frac{5}{6}\right) = \ln\left(\frac{6}{5}\right)$$

2. C

First, solve for m such that

$$500 = 8 + 8(1.07) + ... + 8(1.07)^{m-1} = 8\left[\frac{1-(1.07)^m}{1-1.07}\right] = 8\left[\frac{(1.07)^m - 1}{0.07}\right]$$

$$5.375 = (1.07)^m$$

$$\ln(5.375) = m \ \ln(1.07)$$

$$m = \frac{\ln(5.375)}{\ln(1.07)} = 24.86$$

We conclude that $m = 25$.

3. C

Observe that

$$\frac{dx}{dt} = 2 \cos \ (t/2) \quad \text{and} \quad \frac{dy}{dt} = 2 \ \text{cost} \ -2t \ \sin t$$

Therefore,

$$\left.\frac{dx}{dt}\right|_{t=\pi/2} = 2 \cos \ (\pi/4) = \frac{2}{\sqrt{2}} = \sqrt{2}$$

$$\left.\frac{dy}{dt}\right|_{t=\pi/2} = 2 \cos \ (\pi/2) - \pi \ \sin \ (\pi/2) = -\pi$$

It follows that the length of the velocity vector at time $t = \frac{\pi}{2}$ is given by

$$\sqrt{\left(\sqrt{2}\right)^2 + (-\pi)^2} = \sqrt{\pi^2 + 2} \ .$$

4. E

Let X_1, X_2, X_3, and X_4 denote the four independent bids with common distribution function F. Then if we define $Y = \max(X_1, X_2, X_3, X_4)$, the distribution function G of Y is given by

$$\begin{aligned}
G(y) &= \Pr[Y \le y] \\
&= \Pr[(X_1 \le y) \cap (X_2 \le y) \cap (X_3 \le y) \cap (X_4 \le y)] \\
&= \Pr[X_1 \le y]\Pr[X_2 \le y]\Pr[X_3 \le y]\Pr[X_4 \le y] \\
&= [F(y)]^4 \\
&= \frac{1}{16}(1+\sin\pi y)^4 \quad , \quad \frac{3}{2} \le y \le \frac{5}{2}
\end{aligned}$$

It then follows that the density function g of Y is given by

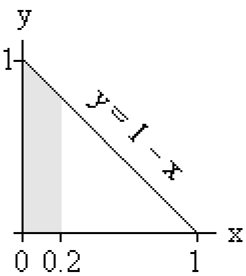
$$\begin{aligned}
g(y) &= G'(y) \\
&= \frac{1}{4}(1+\sin\pi y)^3(\pi\cos\pi y) \\
&= \frac{\pi}{4}\cos\pi y(1+\sin\pi y)^3 \quad , \quad \frac{3}{2} \le y \le \frac{5}{2}
\end{aligned}$$

Finally,

$$\begin{aligned}
E[Y] &= \int_{3/2}^{5/2} y g(y)\,dy \\
&= \int_{3/2}^{5/2} \frac{\pi}{4} y\cos\pi y(1+\sin\pi y)^3\,dy
\end{aligned}$$

5. C

The domain of X and Y is pictured below. The shaded region is the portion of the domain over which X<0.2 .



Now observe

$$\Pr[X<0.2]=\int_0^{0.2}\int_0^{1-x}6\left[1-(x+y)\right]dydx=6\int_0^{0.2}\left[y-xy-\frac{1}{2}y^2\right]_0^{1-x}dx$$

$$=6\int_0^{0.2}\left[1-x-x(1-x)-\frac{1}{2}(1-x)^2\right]dx=6\int_0^{0.2}\left[(1-x)^2-\frac{1}{2}(1-x)^2\right]dx$$

$$=6\int_0^{0.2}\frac{1}{2}(1-x)^2\,dx=-(1-x)^3\,\Big|_0^{0.2}=-(0.8)^3+1$$

$$=0.488$$

6. D

Let

S = Event of a standard policy
F = Event of a preferred policy
U = Event of an ultra-preferred policy
D = Event that a policyholder dies

Then

$$P[U\mid D]=\frac{P[D\mid U]P[U]}{P[D\mid S]P[S]+P[D\mid F]P[F]+P[D\mid U]P[U]}$$

$$=\frac{(0.001)(0.10)}{(0.01)(0.50)+(0.005)(0.40)+(0.001)(0.10)}$$

$$=0.0141$$

7. B

Let us first determine k:

$$1 = \int_0^1 \int_0^1 kx\,dx\,dy = \int_0^1 \frac{1}{2} kx^2 \Big|_0^1 dy = \int_0^1 \frac{k}{2} dy = \frac{k}{2}$$

$$k = 2$$

Then

$$E[X] = \int_0^1 \int_0^1 2x^2 dy\,dx = \int_0^1 2x^2 dx = \frac{2}{3} x^3 \Big|_0^1 = \frac{2}{3}$$

$$E[Y] = \int_0^1 \int_0^1 y \; 2x \; dx\,dy = \int_0^1 y\,dy = \frac{1}{2} y^2 \Big|_0^1 = \frac{1}{2}$$

$$E[XY] = \int_0^1 \int_0^1 2x^2 y\,dx\,dy = \int_0^1 \frac{2}{3} x^3 y \Big|_0^1 dy = \int_0^1 \frac{2}{3} y\,dy$$

$$= \frac{2}{6} y^2 \Big|_0^1 = \frac{2}{6} = \frac{1}{3}$$

$$\text{Cov}[X,Y] = E[XY] - E[X]E[Y] = \frac{1}{3} - \left(\frac{2}{3}\right)\left(\frac{1}{2}\right) = \frac{1}{3} - \frac{1}{3} = 0$$

(Alternative Solution)

Define $g(x) = kx$ and $h(y) = 1$. Then

$f(x,y) = g(x)h(x)$

In other words, $f(x,y)$ can be written as the product of a function of x alone and a function of y alone. It follows that X and Y are independent. Therefore, Cov[X, Y] = 0 .

8. A

By the chain rule,

$$\frac{dp}{dt} = \frac{d}{dt}\left[100\sqrt{xy}\right] = 50x^{\frac{-1}{2}} y^{\frac{1}{2}} \frac{dx}{dt} + 50x^{\frac{1}{2}} y^{\frac{-1}{2}} \frac{dy}{dt}$$

At the time t_0 in question, we are told that

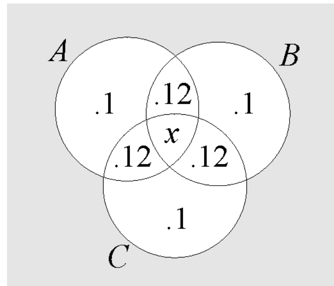
$$x = 2 \ , \ \frac{dx}{dt} = 1 \ , \ y = 3 \ , \text{ and } \ \frac{dy}{dt} = -\frac{1}{2}$$

Therefore,

$$\left.\frac{dp}{dt}\right|_{t=t_0} = 50\sqrt{\frac{3}{2}}(1) + 50\sqrt{\frac{2}{3}}\left(-\frac{1}{2}\right) = 40.8$$

9. C

The Venn diagram below summarizes the unconditional probabilities described in the problem.



In addition, we are told that

$$\frac{1}{3} = P[A \cap B \cap C \mid A \cap B] = \frac{P[A \cap B \cap C]}{P[A \cap B]} = \frac{x}{x + 0.12}$$

It follows that

$$x = \frac{1}{3}(x + 0.12) = \frac{1}{3}x + 0.04$$

$$\frac{2}{3}x = 0.04$$

$$x = 0.06$$

Now we want to find

$$P\left[(A \cup B \cup C)^c \mid A^c\right] = \frac{P\left[(A \cup B \cup C)^c\right]}{P\left[A^c\right]}$$

$$= \frac{1 - P[A \cup B \cup C]}{1 - P[A]}$$

$$= \frac{1 - 3(0.10) - 3(0.12) - 0.06}{1 - 0.10 - 2(0.12) - 0.06}$$

$$= \frac{0.28}{0.60} = 0.467$$

10. E

Let

W = event that wife survives at least 10 years
H = event that husband survives at least 10 years
B = benefit paid
P = profit from selling policies

Then

$$\Pr[H] = P[H \cap W] + \Pr\left[H \cap W^c\right] = 0.96 + 0.01 = 0.97$$

and

$$\Pr[W \mid H] = \frac{\Pr[W \cap H]}{\Pr[H]} = \frac{0.96}{0.97} = 0.9897$$

$$\Pr\left[W^c \mid H\right] = \frac{\Pr\left[H \cap W^c\right]}{\Pr[H]} = \frac{0.01}{0.97} = 0.0103$$

It follows that

$$\begin{aligned} E[P] &= E[1000 - B] \\ &= 1000 - E[B] \\ &= 1000 - \left\{(0)\Pr[W \mid H] + (10{,}000)\Pr\left[W^c \mid H\right]\right\} \\ &= 1000 - 10{,}000(0.0103) \\ &= 1000 - 103 \\ &= 897 \end{aligned}$$

11. D

Observe that x and y follow the constraint equation

$$x + y = 160,000$$

$$x = 160,000 - y \quad \text{where} \quad 0 \le y \le 160,000$$

Now this constraint equation can be used to express policy sales $g(x, y)$ as a function $f(y)$ of marketing y alone:

$$f(y) = g(160,000 - y, y) = 0.001(160,000 - y)^{1/4}\, y^{3/4}$$

We can then compute $f'(y)$ as follows:

$$\begin{aligned} f'(y) &= \left\{ -\frac{1}{4}(160,000 - y)^{-3/4}\, y^{3/4} + \frac{3}{4}(160,000 - y)^{1/4}\, y^{-1/4} \right\} / 1000 \\ &= \frac{-1}{4000}(160,000 - y)^{-3/4}\, y^{-1/4}\left[y - 3(160,000 - y) \right] \\ &= \frac{-1}{4000}(160,000 - y)^{-3/4}\, y^{-1/4}(4y - 480,000) \\ &= \frac{1}{1000}(160,000 - y)^{-3/4}\, y^{-1/4}(120,000 - y) \ , \ 0 \le y \le 160,000 \end{aligned}$$

Note that

$$f'(y) > 0 \quad \text{for} \quad 0 \le y < 120,000 \ ,$$

$$f'(y) = 0 \quad \text{for} \quad y = 120,000 \ , \text{ and}$$

$$f'(y) < 0 \quad \text{for} \quad 120,000 < y < 160,000$$

We conclude that sales are maximized when $y = 120,000$. Therefore,
$f(120,000) = 0.001(160,000 - 120,000)^{1/4}(120,000)^{3/4} = 91.2$ maximizes f.

11. Alternate solution using Lagrange multipliers:
Solve:

$$x + y - 160,000 = 0$$

$$\frac{\partial}{\partial x}\frac{x^{1/4}y^{3/4}}{1000} = \lambda \frac{\partial}{\partial x}(x + y - 160,000)$$

$$\frac{\partial}{\partial y}\frac{x^{1/4}y^{3/4}}{1000} = \lambda \frac{\partial}{\partial y}(x + y - 160,000)$$

From last two equations:

$$\frac{1}{4000}x^{-3/4}y^{3/4} = \lambda$$

$$\frac{3}{4000}x^{1/4}y^{-1/4} = \lambda$$

Eliminating λ:

$$3x^{1/4}y^{-1/4} = x^{-3/4}y^{3/4}$$

$$3x = y$$

Using first equation:

$$4x = 160,000$$

$$x = 40,000$$

$$y = 120,000$$

Extreme value (which must be a maximum) is $\dfrac{(40,000)^{1/4}(120,000)^{3/4}}{1000} = 91.2$

12. D
First note

$$P[A \cup B] = P[A] + P[B] - P[A \cap B]$$

$$P[A \cup B'] = P[A] + P[B'] - P[A \cap B']$$

Then add these two equations to get

$$P[A \cup B] + P[A \cup B'] = 2P[A] + \left(P[B] + P[B']\right) - \left(P[A \cap B] + P[A \cap B']\right)$$

$$0.7 + 0.9 = 2P[A] + 1 - P\left[(A \cap B) \cup (A \cap B')\right]$$

$$1.6 = 2P[A] + 1 - P[A]$$

$$P[A] = 0.6$$

13. E

Let

X = number of group 1 participants that complete the study.

Y = number of group 2 participants that complete the study.

Now we are given that X and Y are independent.

Therefore,

$$P\{[(X \geq 9)\cap(Y<9)]\cup[(X<9)\cap(Y\geq 9)]\}$$
$$= P[(X\geq 9)\cap(Y<9)] + P[(X<9)\cap(Y\geq 9)]$$
$$= 2P[(X\geq 9)\cap(Y<9)] \qquad \text{(due to symmetry)}$$
$$= 2P[X\geq 9]P[Y<9]$$
$$= 2P[X\geq 9]P[X<9] \qquad \text{(again due to symmetry)}$$
$$= 2P[X\geq 9](1-P[X\geq 9])$$
$$= 2\left[\tbinom{10}{9}(0.2)(0.8)^9 + \tbinom{10}{10}(0.8)^{10}\right]\left[1-\tbinom{10}{9}(0.2)(0.8)^9 - \tbinom{10}{10}(0.8)^{10}\right]$$
$$= 2[0.376][1-0.376] = 0.469$$

14. A

Let $f_1(x)$ denote the marginal density function of X. Then

$$f_1(x) = \int_x^{x+1} 2x\,dy = 2xy \,|_x^{x+1} = 2x(x+1-x) = 2x \quad , \quad 0<x<1$$

Consequently,

$$f(y\,|\,x) = \frac{f(x,y)}{f_1(x)} = \begin{cases} 1 & \text{if: } \; x<y<x+1 \\ 0 & \text{otherwise} \end{cases}$$

$$E[Y\,|\,X] = \int_x^{x+1} y\,dy = \frac{1}{2}y^2 \,|_x^{x+1} = \frac{1}{2}(x+1)^2 - \frac{1}{2}x^2 = \frac{1}{2}x^2 + x + \frac{1}{2} - \frac{1}{2}x^2 = x + \frac{1}{2}$$

$$E[Y^2\,|\,X] = \int_x^{x+1} y^2\,dy = \frac{1}{3}y^3 \,|_x^{x+1} = \frac{1}{3}(x+1)^3 - \frac{1}{3}x^3$$
$$= \frac{1}{3}x^3 + x^2 + x + \frac{1}{3} - \frac{1}{3}x^3 = x^2 + x + \frac{1}{3}$$

$$\operatorname{Var}[Y\,|\,X] = E[Y^2\,|\,X] - \{E[Y\,|\,X]\}^2 = x^2 + x + \frac{1}{3} - \left(x+\frac{1}{2}\right)^2$$
$$= x^2 + x + \frac{1}{3} - x^2 - x - \frac{1}{4} = \frac{1}{12}$$

15. D

At the point (0, 5), $0 = 2t^2 + t - 1 = (2t-1)(t+1)$

and $5 = t^2 - 3t + 1$

The first equation says $t = \frac{1}{2}$ or $t = -1$ and the second says $t = -1$.

The slope of the tangent line to C at (0, 5) is then

$$\left.\frac{dy}{dx}\right|_{(x,y)=(0,5)} = \left.\frac{dy}{dt}\right|_{t=-1} \Big/ \left.\frac{dx}{dt}\right|_{t=-1} = \left.\frac{2t-3}{4t+1}\right|_{t=-1}$$

$$= \left[2(-1)-3\right] \big/ \left[4(-1)+1\right] = \frac{5}{3}$$

16. A

We are given that

$$T(x) = \begin{cases} 0 & \text{for } 0 \le x \le 10 \\ \dfrac{0.02(x-10)}{x} & \text{for } 10 < x \le 20 \\ \dfrac{0.04(x-20)+0.02(10)}{x} & \text{for } x > 20 \end{cases} \qquad T(x) = \begin{cases} 0 & \text{for } 0 \le x \le 10 \\ 0.02 - \dfrac{1}{5x} & \text{for } 10 < x \le 20 \\ 0.04 - \dfrac{3}{5x} & \text{for } x > 20 \end{cases}$$

Therefore, $T'(x) = \begin{cases} 0 & \text{for } 0 < x < 10 \\ \dfrac{1}{5x^2} & \text{for } 10 < x < 20 \\ \dfrac{3}{5x^2} & \text{for } x > 20 \end{cases}$ and $T''(x) = \begin{cases} 0 & \text{for } 0 < x < 10 \\ -\dfrac{2}{5x^3} & \text{for } 10 < x < 20 \\ -\dfrac{6}{5x^3} & \text{for } x > 20 \end{cases}$

We can infer the following about $T(x)$:

i) $T(x) = 0$ for $0 < x \le 10$

ii) $T(x)$ is strictly increasing for $10 < x < 20$ and $x > 20$ since $T'(x) > 0$ on both of these intervals.

iii) $T(x)$ is concave down for $10 < x < 20$ and $x > 20$ since $T''(x) < 0$ on both of these intervals.

It follows that (A) is the only graph that satisfies conditions (i)-(iii).

17. B

Let Y denote the claim payment made by the insurance company.
Then

$$Y = \begin{cases} 0 & \text{with probability } 0.94 \\ \text{Max } (0, x-1) & \text{with probability } 0.04 \\ 14 & \text{with probability } 0.02 \end{cases}$$

and

$$\begin{aligned}
E[Y] &= (0.94)(0) + (0.04)(0.5003)\int_1^{15}(x-1)e^{-x/2}dx + (0.02)(14) \\
&= (0.020012)\left[\int_1^{15} xe^{-x/2}dx - \int_1^{15} e^{-x/2}dx\right] + 0.28 \\
&= 0.28 + (0.020012)\left[-2xe^{-x/2}\Big|_1^{15} + 2\int_1^{15} e^{-x/2}dx - \int_1^{15} e^{-x/2}dx\right] \\
&= 0.28 + (0.020012)\left[-30e^{-7.5} + 2e^{-0.5} + \int_1^{15} e^{-x/2}dx\right] \\
&= 0.28 + (0.020012)\left[-30e^{-7.5} + 2e^{-0.5} - 2e^{-x/2}\Big|_1^{15}\right] \\
&= 0.28 + (0.020012)\left(-30e^{-7.5} + 2e^{-0.5} - 2e^{-7.5} + 2e^{-0.5}\right) \\
&= 0.28 + (0.020012)\left(-32e^{-7.5} + 4e^{-0.5}\right) \\
&= 0.28 + (0.020012)(2.408) \\
&= 0.328 \quad \text{(in thousands)}
\end{aligned}$$

It follows that the expected claim payment is 328 .

18. D

By the chain rule,

$$\frac{\partial f}{\partial y} = ve^{uv}\frac{\partial u}{\partial y} + ue^{uv}\frac{\partial v}{\partial y} = ve^{uv}2x + ue^{uv}2y$$

Therefore,

$$\left.\frac{\partial f}{\partial y}\right|_{(x,y)=(2,1)} = 5e^{(4)(5)}(2)(2) + 4e^{(4)(5)}(2)(1) = 28e^{20}$$

19. B

Let $X_1,\ldots, X_n$ denote the life spans of the n light bulbs purchased. Since these random variables are independent and normally distributed with mean 3 and variance 1, the random variable $S = X_1 + \ldots + X_n$ is also normally distributed with mean

$$\mu = 3n$$

and standard deviation

$$\sigma = \sqrt{n}$$

Now we want to choose the smallest value for n such that

$$0.9772 \le \Pr\left[S > 40\right] = \Pr\left[\frac{S-3n}{\sqrt{n}} > \frac{40-3n}{\sqrt{n}}\right]$$

This implies that n should satisfy the following inequality:

$$-2 \ge \frac{40-3n}{\sqrt{n}}$$

To find such an n, let's solve the corresponding equation for n:

$$-2 = \frac{40-3n}{\sqrt{n}}$$

$$-2\sqrt{n} = 40-3n$$

$$3n - 2\sqrt{n} - 40 = 0$$

$$\left(3\sqrt{n}+10\right)\left(\sqrt{n}-4\right) = 0$$

$$\sqrt{n} = 4$$

$$n = 16$$

20. D

The density function of T is

$$f\left(t\right) = \frac{1}{3}e^{-t/3} \quad , \quad 0 < t < \infty$$

Therefore,

$$\begin{aligned} E\left[X\right] &= E\left[\max\left(T,2\right)\right] \\ &= \int_0^2 \frac{2}{3}e^{-t/3}dt + \int_2^\infty \frac{t}{3}e^{-t/3}dt \\ &= -2e^{-t/3}\,\Big|_0^2 - te^{-t/3}\,\Big|_2^\infty + \int_2^\infty e^{-t/3}dt \\ &= -2e^{-2/3} + 2 + 2e^{-2/3} - 3e^{-t/3}\,\Big|_2^\infty \\ &= 2 + 3e^{-2/3} \end{aligned}$$

21. E

The differential equation that we are given is separable. As a result, the general solution is given by

$$\int \frac{1}{Q(N-Q)} dQ = \int dt = t + C$$

where C is a constant. Now in order to calculate the integral on the lefthand side of this equation, we first need to determine the partial fractions of the integrand. In other words, we need to find constants A and B such that

$$\frac{1}{Q(N-Q)} = \frac{A}{Q} + \frac{B}{N-Q}$$

$$1 = A(N-Q) + BQ$$

$$1 = AN + (B-A)Q$$

It follows that

$$AN = 1$$

$$B - A = 0$$

$$B = A = \frac{1}{N}$$

so $\dfrac{1}{Q(N-Q)} = \dfrac{1}{NQ} + \dfrac{1}{N(N-Q)}$ and

$$\int \frac{1}{Q(N-Q)} dQ = \frac{1}{N}\int \frac{1}{Q} dQ + \frac{1}{N}\int \frac{1}{N-Q} dQ = \frac{1}{N} 1nQ - \frac{1}{N} 1n(N-Q) + K = \frac{1}{N} 1n\left[\frac{Q}{N-Q}\right] + K$$

where K is a constant. Consequently,

$$\frac{1}{N} ln\left[\frac{Q}{N-Q}\right] + K = t + C$$

$$\left(\frac{Q}{N-Q}\right)^{1/N} e^K = e^t e^C$$

$$\left(\frac{Q}{N-Q}\right)^{1/N} = e^t e^{C-K}$$

$$\frac{Q}{N-Q} = e^{Nt} e^{N(C-K)}$$

$Q = ae^{Nt}(N-Q) = aNe^{Nt} - ae^{Nt}Q$ where $a = e^{N(C-K)}$ is a constant

$$\left(1 + ae^{Nt}\right)Q = aNe^{Nt}$$

$$Q(t) = \frac{aNe^{Nt}}{1 + ae^{Nt}}$$

22. C

Let X denote the waiting time for a first claim from a good driver, and let Y denote the waiting time for a first claim from a bad driver. The problem statement implies that the respective distribution functions for X and Y are

$$F(x) = 1 - e^{-x/6} \quad , \ x > 0 \qquad \text{and}$$

$$G(y) = 1 - e^{-y/3} \quad , \ y > 0$$

Therefore,

$$\begin{aligned}\Pr\left[(X \le 3) \cap (Y \le 2)\right] &= \Pr[X \le 3]\Pr[Y \le 2] \\ &= F(3)G(2) \\ &= \left(1 - e^{-1/2}\right)\left(1 - e^{-2/3}\right) \\ &= 1 - e^{-2/3} - e^{-1/2} + e^{-7/6}\end{aligned}$$

23. A

Let

C = Event that shipment came from Company X

I_1 = Event that one of the vaccine vials tested is ineffective

Then by Bayes' Formula, $P[C \mid I_1] = \dfrac{P[I_1 \mid C]P[C]}{P[I_1 \mid C]P[C] + P\left[I_1 \mid C^c\right]P\left[C^c\right]}$

Now

$$P[C] = \frac{1}{5}$$

$$P\left[C^c\right] = 1 - P[C] = 1 - \frac{1}{5} = \frac{4}{5}$$

$$P[I_1 \mid C] = \binom{30}{1}(0.10)(0.90)^{29} = 0.141$$

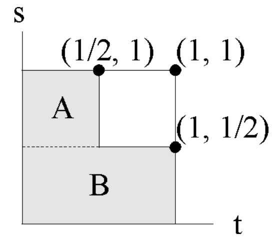
$$P\left[I_1 \mid C^c\right] = \binom{30}{1}(0.02)(0.98)^{29} = 0.334$$

Therefore,

$$P[C \mid I_1] = \frac{(0.141)(1/5)}{(0.141)(1/5) + (0.334)(4/5)} = 0.096$$

24. E

The domain of s and t is pictured below.



Note that the shaded region is the portion of the domain of s and t over which the device fails sometime during the first half hour. Therefore,

$$\Pr\left[\left(S \le \frac{1}{2}\right) \cup \left(T \le \frac{1}{2}\right)\right] = \int_0^{1/2} \int_{1/2}^{1} f(s,t)\, ds dt + \int_0^1 \int_0^{1/2} f(s,t)\, ds dt$$

(where the first integral covers A and the second integral covers B).

25. B

Note that V, S and r are all functions of time t. Therefore,

$$\frac{dV}{dt} = 4\pi r^2 \frac{dr}{dt}$$

and

$$\frac{dS}{dt} = 8\pi r \frac{dr}{dt}$$

We are given that

$$\frac{dV}{dt} = 60 \quad \text{when} \quad r = \frac{6}{2} = 3.$$

It follows that

$$60 = 4\pi (3)^2 \frac{dr}{dt}$$

$$\frac{dr}{dt} = \frac{5}{3\pi}$$

$$\frac{dS}{dt} = 8\pi (3)\left(\frac{5}{3\pi}\right) = 40$$

26. B

Let

u be annual claims,
v be annual premiums,
$g(u, v)$ be the joint density function of U and V,
$f(x)$ be the density function of X, and
$F(x)$ be the distribution function of X.

Then since U and V are independent,

$$g(u,v) = \left(e^{-u}\right)\left(\frac{1}{2}e^{-v/2}\right) = \frac{1}{2}e^{-u}e^{-v/2} \quad , \quad 0 < u < \infty \quad , \quad 0 < v < \infty$$

and

$$F(x) = \Pr[X \le x] = \Pr\left[\frac{u}{v} \le x\right] = \Pr[U \le Vx]$$

$$= \int_0^\infty \int_0^{vx} g(u,v)\,dudv = \int_0^\infty \int_0^{vx} \frac{1}{2}e^{-u}e^{-v/2}\,dudv$$

$$= \int_0^\infty -\frac{1}{2}e^{-u}e^{-v/2}\Big|_0^{vx}\,dv = \int_0^\infty \left(-\frac{1}{2}e^{-vx}e^{-v/2} + \frac{1}{2}e^{-v/2}\right)dv$$

$$= \int_0^\infty \left(-\frac{1}{2}e^{-v(x+1/2)} + \frac{1}{2}e^{-v/2}\right)dv$$

$$= \left[\frac{1}{2x+1}e^{-v(x+1/2)} - e^{-v/2}\right]_0^\infty$$

$$= -\frac{1}{2x+1} + 1$$

Finally,

$$f(x) = F'(x) = \frac{2}{(2x+1)^2}$$

27. A

First, observe that the distribution function of X is given by

$$F(x) = \int_1^x \frac{3}{t^4}\,dt = -\frac{1}{t^3}\Big|_1^x = 1 - \frac{1}{x^3} \quad , \quad x > 1$$

Next, let X_1, X_2, and X_3 denote the three claims made that have this distribution. Then if Y denotes the largest of these three claims, it follows that the distribution function of Y is given by

$$G(y) = \Pr[X_1 \le y]\Pr[X_2 \le y]\Pr[X_3 \le y]$$

$$= \left(1 - \frac{1}{y^3}\right)^3 \quad , \quad y > 1$$

while the density function of Y is given by

$$g(y) = G'(y) = 3\left(1 - \frac{1}{y^3}\right)^2\left(\frac{3}{y^4}\right) = \left(\frac{9}{y^4}\right)\left(1 - \frac{1}{y^3}\right)^2 \quad , \quad y > 1$$

Therefore,

$$E[Y] = \int_1^\infty \frac{9}{y^3}\left(1 - \frac{1}{y^3}\right)^2 dy = \int_1^\infty \frac{9}{y^3}\left(1 - \frac{2}{y^3} + \frac{1}{y^6}\right)dy$$

$$= \int_1^\infty \left(\frac{9}{y^3} - \frac{18}{y^6} + \frac{9}{y^9}\right)dy = \left[-\frac{9}{2y^2} + \frac{18}{5y^5} - \frac{9}{8y^8}\right]_1^\infty$$

$$= 9\left[\frac{1}{2} - \frac{2}{5} + \frac{1}{8}\right] = 2.025 \text{ (in thousands)}$$

28. C

Since $f(t) > 0$ and $f'(t) < 0$ for $t \ge 0$, the following inequalities hold:

(i) $f(t_0) > f(t_1)$ if $0 \le t_0 < t_1$

(ii) $f(k) < \int_{k-1}^{k} f(t)dt$ if $k \ge 1$

(iii) $f(k) > \int_{k}^{k+1} f(t)dt$ if $k \ge 0$

Applying these inequalities, we see that

$$\boxed{f(0) + f(1) + \int_2^{20} f(t)\,dt} > f(0) + \int_1^2 f(t)\,dt + \int_2^{20} f(t)\,dt$$

$$= \boxed{f(0) + \int_1^{20} f(t)\,dt} > \int_0^1 f(t)\,dt + \int_1^{20} f(t)\,dt$$

$$= \boxed{\int_0^{20} f(t)\,dt} = \int_0^1 f(t)\,dt + \int_1^{20} f(t)\,dt$$

$$> \boxed{f(1) + \int_1^{20} f(t)\,dt} = f(1) + \sum_{k=1}^{19} \int_k^{k+1} f(t)\,dt$$

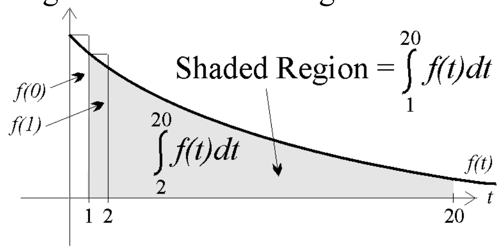
$$> f(1) + \sum_{k=1}^{19} f(k+1) = f(1) + \sum_{k=2}^{20} f(k)$$

$$= \boxed{\sum_{k=1}^{20} f(k)} > \sum_{k=1}^{20} \int_k^{k+1} f(t)\,dt = \int_1^{21} f(t)dt$$

$$> \boxed{\int_1^{20} f(t)\,dt}$$

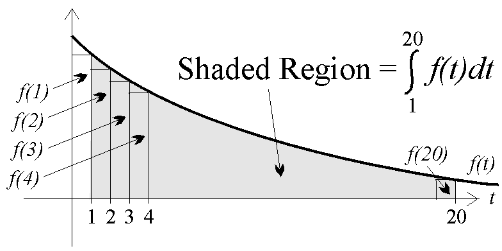
We conclude that $f(1) + \int_1^{20} f(t)dt$ produces the smallest number that exceeds $N = \sum_{k=1}^{20} f(k)$.

28. Note a more heuristic approach to the result that (E) > (B) > (A) > (C) > $\sum_{k=1}^{20} f(k)$ > (D) can be obtained from diagrams of the following sort:



gives $(E) = f(0) + f(1) + \int_2^{20} f(t)\,dt > f(0) + \int_1^{20} f(t)\,dt = (B)$

and



gives $f(1) + \int_1^{20} f(t)\,dt > \sum_{k=1}^{20} f(t)$

29. D

Let

X = number of low-risk drivers insured
Y = number of moderate-risk drivers insured
Z = number of high-risk drivers insured
$f(x, y, z)$ = probability function of X, Y, and Z

Then f is a trinomial probability function, so

$$\Pr[z \geq x + 2] = f(0,0,4) + f(1,0,3) + f(0,1,3) + f(0,2,2)$$

$$= (0.20)^4 + 4(0.50)(0.20)^3 + 4(0.30)(0.20)^3 + \frac{4!}{2!2!}(0.30)^2(0.20)^2$$

$$= 0.0488$$

30. B

Let

x = number of ice cream cones sold

$p(x)$ = price of x ice cream cones

$C(x)$ = cost of selling x ice cream cones

$R(x)$ = revenue from selling x ice cream cones

$P(x)$ = profit from selling x ice cream cones

We are told that $p(x)$ satisfies the following relationship:

$$x = 500 - 5\left[\frac{p(x)-2}{0.01}\right] = 500 - 500p(x) + 1000 = 1500 - 500p(x)$$

$$500p(x) = 1500 - x$$

$$p(x) = 3 - \frac{x}{500}$$

Therefore,

$$R(x) = xp(x) = 3x - \frac{x^2}{500}$$

$$C(x) = 0.10x + 75$$

$$P(x) = R(x) - C(x) = 3x - \frac{x^2}{500} - 0.10x - 75 = 2.9x - \frac{x^2}{500} - 75$$

Now, since $P(x)$ is quadratic, it is clear that $P(x)$ will be maximized for x such that

$$0 = P'(x) = 2.9 - \frac{x}{250}$$

$$\frac{x}{250} = 2.9$$

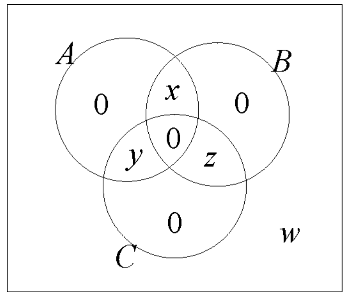
$$x = 725$$

The profit maximizing price is thus

$$p(725) = 3 - \frac{725}{500} = 1.55$$

31. C

A Venn diagram for this situation looks like:



We want to find $w = 1 - (x + y + z)$

We have $x + y = \frac{1}{4}, \quad x + z = \frac{1}{3}, \quad y + z = \frac{5}{12}$

Adding these three equations gives

$$(x+y)+(x+z)+(y+z) = \frac{1}{4}+\frac{1}{3}+\frac{5}{12}$$

$$2(x+y+z) = 1$$

$$x+y+z = \frac{1}{2}$$

$$w = 1-(x+y+z) = 1-\frac{1}{2} = \frac{1}{2}$$

Alternatively the three equations can be solved to give $x = 1/12$, $y = 1/6$, $z = 1/4$

again leading to $w = 1 - \left(\frac{1}{12}+\frac{1}{6}+\frac{1}{4}\right) = \frac{1}{2}$

32. E

Let X and Y denote the times that the two backup generators can operate. Now the variance of an exponential random variable with mean β is β^2. Therefore,

$$\text{Var}[X] = \text{Var}[Y] = 10^2 = 100$$

Then assuming that X and Y are independent, we see

$$\text{Var}[\text{X+Y}] = \text{Var}[\text{X}] + \text{Var}[\text{Y}] = 100 + 100 = 200$$

33. D

Let

I_A = Event that Company A makes a claim
I_B = Event that Company B makes a claim
X_A = Expense paid to Company A if claims are made
X_B = Expense paid to Company B if claims are made

Then we want to find

$$\begin{aligned}
&\Pr\left\{\left[I_A^C \cap I_B\right] \cup \left[\left(I_A \cap I_B\right) \cap \left(X_A < X_B\right)\right]\right\} \\
&\quad = \Pr\left[I_A^C \cap I_B\right] + \Pr\left[\left(I_A \cap I_B\right) \cap \left(X_A < X_B\right)\right] \\
&\quad = \Pr\left[I_A^C\right]\Pr\left[I_B\right] + \Pr\left[I_A\right]\Pr\left[I_B\right]\Pr\left[X_A < X_B\right] \qquad \text{(independence)} \\
&\quad = (0.60)(0.30) + (0.40)(0.30)\Pr\left[X_B - X_A \geq 0\right] \\
&\quad = 0.18 + 0.12\Pr\left[X_B - X_A \geq 0\right]
\end{aligned}$$

Now $X_B - X_A$ is a linear combination of independent normal random variables. Therefore, $X_B - X_A$ is also a normal random variable with mean

$$M = E\left[X_B - X_A\right] = E\left[X_B\right] - E\left[X_A\right] = 9{,}000 - 10{,}000 = -1{,}000$$

and standard deviation $\sigma = \sqrt{\operatorname{Var}(X_B) + \operatorname{Var}(X_A)} = \sqrt{(2000)^2 + (2000)^2} = 2000\sqrt{2}$

It follows that

$$\begin{aligned}
\Pr\left[X_B - X_A \geq 0\right] &= \Pr\left[Z \geq \frac{1000}{2000\sqrt{2}}\right] \qquad (Z \text{ is standard normal}) \\
&= \Pr\left[Z \geq \frac{1}{2\sqrt{2}}\right] \\
&= 1 - \Pr\left[Z < \frac{1}{2\sqrt{2}}\right] \\
&= 1 - \Pr\left[Z < 0.354\right] \\
&= 1 - 0.638 = 0.362
\end{aligned}$$

Finally,

$$\begin{aligned}
\Pr\left\{\left[I_A^C \cap I_B\right] \cup \left[\left(I_A \cap I_B\right) \cap \left(X_A < X_B\right)\right]\right\} &= 0.18 + (0.12)(0.362) \\
&= 0.223
\end{aligned}$$

34. A

The graph (A) contains the curves $y = x - 1$ and $y = 1 = \frac{d}{dx}[x-1]$.

(Note graph (D) can be eliminated because both curves have non-zero slopes where the other crosses the x-axis.)

35. C

Note $Y = \begin{cases} X & \text{if} \quad 0 \le X \le 4 \\ 4 & \text{if} \quad 4 < X \le 5 \end{cases}$

Therefore,

$$E[Y] = \int_0^4 \frac{1}{5} x\,dx + \int_4^5 \frac{4}{5}\,dx = \frac{1}{10} x^2 \Big|_0^4 + \frac{4}{5} x \Big|_4^5$$

$$= \frac{16}{10} + \frac{20}{5} - \frac{16}{5} = \frac{8}{5} + \frac{4}{5} = \frac{12}{5}$$

$$E\left[Y^2\right] = \int_0^4 \frac{1}{5} x^2\,dx + \int_4^5 \frac{16}{5}\,dx = \frac{1}{15} x^3 \Big|_0^4 + \frac{16}{5} x \Big|_4^5$$

$$= \frac{64}{15} + \frac{80}{5} - \frac{64}{5} = \frac{64}{15} + \frac{16}{5} = \frac{64}{15} + \frac{48}{15} = \frac{112}{15}$$

$$\text{Var}[Y] = E\left[Y^2\right] - \left(E[Y]\right)^2 = \frac{112}{15} - \left(\frac{12}{5}\right)^2 = 1.71$$

36. B

Let T denote the total concentration of pollutants over the town. Then due to symmetry,

$$T = 4\int_0^2 \int_0^2 22{,}500\left(8 - x^2 - y^2\right) dx\,dy$$

$$= (4)(7500)\int_0^2 \left[24x - x^3 - 3xy^2\right]_0^2 dy$$

$$= 30{,}000\int_0^2 \left(48 - 8 - 6y^2\right) dy$$

$$= 30{,}000\int_0^2 \left(40 - 6y^2\right) dy$$

$$= 30{,}000\left[40y - 2y^3\right]_0^2 = 30{,}000(80 - 16)$$

$$= 30{,}000(64) = 1{,}920{,}000$$

And since the town covers 16 square miles, it follows that the average pollution concentration A is $A = T/16 = 1{,}920{,}000/16 = 120{,}000$

37. E

Observe that the bus driver collect 21x50 = 1050 for the 21 tickets he sells. However, he may be required to refund 100 to one passenger if all 21 ticket holders show up. Since passengers show up or do not show up independently of one another, the probability that all 21 passengers will show up is $(1 - 0.02)^{21} = (0.98)^{21} = 0.65$. Therefore, the tour operator's expected revenue is $1050 - (100)(0.65) = 985$.

38. D

From f', observe that

$$f(x)=\begin{cases} 4x+c_1 & \text{for} \quad 0<x<10 \\ kx+c_2 & \text{for} \quad 10<x<30 \\ 3x+c_3 & \text{for} \quad x>30 \end{cases}$$

As a result, 200 = f(50) = 3(50) + c_3 = 150 + c_3 implies $c_3 = 50$

And $0 = f(0) = 4(0) + c_1 = c_1$,

Then due to the continuity requirement,

$$10k + c_2 = f(10) = 4(10) + c_1 = 40 + 0 = 40, \text{ and}$$

$$30k + c_2 = f(30) = 3(30) + c_3 = 90 + 50 = 140$$

Solving these last two equations simultaneously, we see that

$$20k = 100 \quad \text{or} \quad k = 5 \ .$$

39. A

Let F denote the distribution function of f. Then

$$F(x) = \Pr[X \le x] = \int_1^x 3t^{-4}\,dt = -t^{-3}\Big|_1^x = 1 - x^{-3}$$

Using this result, we see

$$\Pr[X<2 \mid X \ge 1.5] = \frac{\Pr[(X<2)\cap(X\ge 1.5)]}{\Pr[X \ge 1.5]} = \frac{\Pr[X<2] - \Pr[X \le 1.5]}{\Pr[X \ge 1.5]}$$

$$= \frac{F(2)-F(1.5)}{1-F(1.5)} = \frac{(1.5)^{-3}-(2)^{-3}}{(1.5)^{-3}} \quad = 1-\left(\frac{3}{4}\right)^3 \quad = 0.578$$

40. B

Let

H = event that a death is due to heart disease

F = event that at least one parent suffered from heart disease

Then based on the medical records,

$$P[H \cap F^c] = \frac{210-102}{937} = \frac{108}{937}$$

$$P[F^c] = \frac{937-312}{937} = \frac{625}{937}$$

and $P[H \mid F^c] = \dfrac{P[H\cap F^c]}{P[F^c]} = \dfrac{108}{937}\Big/\dfrac{625}{937} = \dfrac{108}{625} = 0.173$