EDUCATION AND EXAMINATION COMMITTEE OF THE SOCIETY OF ACTUARIES (SOA)

SYLLABUS & EXAMINATION COMMITTEES OF THE CASUALTY ACTUARIAL SOCIETY (CAS)

SPRING 2007
EXAM C/EXAM 4

CONSTRUCTION AND EVALUATION OF ACTUARIAL MODELS

INTRODUCTORY STUDY NOTE

1. The Construction and Evaluation of Actuarial Models examination for Spring 2007 will be given on **Wednesday, May 16, from 8:30 a.m. to 12:30 p.m**. The examination will consist of 40 multiple-choice questions.

   The score for the examination is determined solely on the basis of correct answers. Therefore, candidates should answer every question to maximize their scores.

2. Any changes in the Course of Reading for this exam since the publication of the *Spring 2007 Basic Education Catalog* of the SOA or the *2007 Syllabus of Examinations* of the CAS are reflected in this Introductory Study Note and will also be posted on our Web sites. **If any difference exists between information contained in this Introductory Study Note and that contained in the *Spring 2007 Basic Education Catalog* or the *2007 Syllabus of Examinations,* this Introductory Study Note will govern.**

3. The following list contains all study notes for this exam in Spring 2007. Candidates who have ordered the complete set of study notes should verify immediately that they have copies of the listed items. Items marked with a # are new/updated for this session.

C-05-07 #	Introductory Study Note (this study note)
	Sample Problem Mapping for Exam C
C-10-05	May 2005 Exam C
C-12-05	November 2005 Exam C Questions and Solutions
C-12-06 #	November 2006 Exam C Questions and Solutions
C-21-01	Credibility (to be used with Option B only)
C-24-05	Topics in Credibility Theory (to be used with Option B only)
C-25-07 #	An Introduction to Risk Measurers for Actuarial Applications

4. The study notes for this exam include sample questions and solutions. The sample questions provide the candidate with the opportunity to practice on the types of questions that are likely to appear on the examination. New sample examinations will be released periodically or whenever the nature of the examination changes substantially.

5. Enclosed are errata for the following texts:
   Loss Models: From Data to Decisions, 2004, Second Edition, by Klugman, Panjer and Willmot.

   Derivative Markets, 2006 Second Edition, first printing, by R. McDonald
   http://www.kellogg.northwestern.edu/faculty/mcdonald/htm/typos2e_01.html

6 A copy of the mathematical tables to be included with this examination are enclosed, to enable candidates to familiarize themselves with their formats and contents in advance. A copy will be distributed at the examination. Note that candidates will not be allowed to bring copies of the tables into the examination room.

7 A survey for examinations FM, MLC, MFE and C will be available on the SOA and CAS web sites after the examinations have been administered. Candidates are encouraged to provide feedback on the course readings and the examinations that they have taken.

8 Several book distributors carry some or all of the textbooks for the Society of Actuaries and Casualty Actuarial Society exams A list of distributors appears in the *Spring 2007 Basic Education Catalog* and *2007 Syllabus of Examinations* A set of order forms from these distributors is included with the study note package.

The order forms contain information about prices, shipping charges, mailing policy and credit card acceptance. Any book distributor who carries books for SOA or CAS exams may have their order form included in this set unless the SOA office receives substantial complaints about service. Candidates should notify the Publication Orders Department of the SOA in writing if they encounter serious problems with any distributor

9. The examination questions for this exam will be based on the required readings for this exam. If a conflict exists (in definitions, terminology, etc.) between the readings for this exam and the readings for other exams, the questions should be answered on the basis of the readings for this exam.

10 Candidates may use the battery or solar-powered Texas Instruments BA-35 model calculator, BA II Plus*, BA II Plus Professional*, TI-30X, TI-30Xa, or TI-30X II*(IIS solar or IIB battery) Candidates may use more than one of the approved calculators during the examinations.

Calculator instructions cannot be brought into the exam room. During the exam, the calculator must be removed from its carrying case so the supervisor can confirm it is an approved model. Candidates using a calculator other than the approved models will have their examinations disqualified

Candidates can purchase calculators directly from: Texas Instruments, Attn: Order Entry, PO Box 650311, Mail Station 3962, Dallas, TX 75265, phone 800/842-2737 or http://epsstore.ti.com.

The memory of **TI-30X II, **BA II Plus** and **BA II Plus Professional** will need to be cleared by the examination supervisor upon the candidates' entrance to the examination room.*

11. Order forms for various seminars/workshops and study manuals are included with this set of study notes. These seminars/workshops and study manuals do not reflect any official interpretation, opinion, or endorsement of the Society of Actuaries or the Casualty Actuarial Society

12. A candidate planning to seek admission to the SOA should submit the Application for Admission as Associate *before* completing the education requirements for Associateship as detailed in the *Spring 2007 Basic Education Catalog*.

13 In addition to the examination requirements, all prospective SOA and CAS Associates will be required to attend and successfully complete a seminar on professionalism prior to admission as a member. See the *SOA Spring 2007 Basic Education Catalog* or the *CAS 2007 Syllabus of*

Examinations for more information.

14 The Society of Actuaries and the Casualty Actuarial Society provide study notes to persons preparing for this examination. They are intended to acquaint candidates with some of the theoretical and practical considerations involved in the various subjects. While varying opinions are presented where appropriate, limits on the length of the material and other considerations sometimes prevent the inclusion of all possible opinions. These study notes do not, however, represent any official opinion, interpretation or endorsement of the Casualty Actuarial Society or the Society of Actuaries. The Societies are grateful to the authors for their contributions in preparing study notes.

The Casualty Actuarial Society and the Society of Actuaries jointly sponsor and jointly administer Exams P, FM and C.

The American Academy of Actuaries (AAA) and the Conference of Consulting Actuaries (CCA) jointly sponsor the Associateship and Fellowship examinations with the SOA.

Errata for Loss Models, 2nd ed., July 6, 2006

Items marked with a * have been corrected in the second printing. Items marked ** have been corrected in the third printing. Items marked *** have been corrected in the fourth printing. All other items have not appeared in a later printing.

***Page 35**: In Figure 3.6 the caption should refer to Model 3.

***Page 49**: In the development of μ'_k, the word "integral" in the last line should be "integer."

***Page 62**: In Example 4.31, the resulting distribution is inverse Burr, not Burr.

Page 77: In lines two to four from the bottom change "long" to "heavy" and "short" to "light". While heavy/light and long/short are both used to describe tails, these changes bring a consistent use of heavy/light throughout the text.

***Page 102**: In Example 4.61, delete the expression "$|\theta$" in the two places it appears.

***Pages 103-104**: In the solution to Example 4.64, the sentence at the bottom of page 103 should be: This means that about 94% of drivers were "good" with a risk of $\lambda_1 = 0.11$ expected accidents per year and 6% were "bad" with a risk of $\lambda_2 = 0.70$ expected accidents per year.

****Page 106**: Change Example 4.69 to read "Use the above results and (4.31) to demonstrate ..."

*****Page 123**: In Exercise 5.13 change "that" to "given it"

*****Page 128**: In Exercise 5.16 change Y to Y^P both times it appears.

****Page 139**: At the end of the paragraph following (6.2) change "Such distributions are said to be infinitely divisible" to "Such distributions can be shown to be infinitely divisible"

***Page 142**: Equation (6.6) requires a second power on the final term of the second equation. The correct equation is

$$\mathrm{Var}(S) = \mu_{S2} = \mu'_{N1}\mu_{X2} + \mu_{N2}(\mu'_{X1})^2$$

*****Page 170**: In Exercise 6.36 replace f_{i+1} with f_i and set $i = 1, 2, \ldots$ not $i = 0, 1, 2, \ldots$

Page 171: In Exercise 6.41(a) the left hand side should be $f_S(x)$, not $f_X(x)$.

Page 192: In line 1 change "long" to "heavy". See comment on Page 77 errata.

*****Page 275**: In Definition 9.16 add "random" between "a pair of" and "values".

****Page 276**: In Example 9.17, the first line of the solution, it should state $\mathrm{Var}(\hat{\theta}) = \sigma^2/n$.

*****Page 277**: In Exercise 9.11, in part (b) add "as in equation (9.4)" and in part (c) add "as in Example 9.18".

****Page 292**: In Exercise 10.2, eliminate part (b) and change (c) to (b).

*****Page 293**: Add a footnote to the sentence just before Definition 10.12 - Technically, for the interval from c_{j-1} to c_j, $x = c_j$ should be included and $x = c_{j-1}$ excluded in order for $F_n(c_j)$ to be the empirical distribution function.

****Page 295**: In Exercise 10.5, in part (b) the reference should be to part (a).

Page 299: In equation (11.2) change the middle line to (number of x_is equal to y_{j-1}). The current version is correct, this reflects the fact that by definition xs are always equal to ys.

Page 303: The second integral at the top of the page should be $\int_0^t h(u)du$.

Page 316: In Exercise 11.20 change /two2 to two.

*****Page 325**: At the end of the sentence that ends on the third line add ", noting that policy 33 is assumed to enter at mid-duration, 1.5."

****Page 359**: In Exercise 12.64, use the data from Exercise 12.44, but do not assume that α is known.

Page 360: In Exercise 12.66(c) the expression should be $V\hat{a}r(\hat{\mu}) = \frac{\hat{\mu}^2}{n\tau^2}$.

Page 366: In the last paragraph of Example 12.37 change "The solid vertical bars" to "The thinner vertical bars"

***Page 379**: In Exercise 12.75(a) the pdf should be

$$f_{X_j}(x_j) = \frac{\Gamma(\alpha + \frac{1}{2})}{\sqrt{2\pi\beta}\ \Gamma(\alpha)}\left[1 + \frac{1}{2\beta}(x_j - \mu)^2\right]^{-\alpha-1/2}, \quad -\infty < x_j < \infty.$$

Note that the two appearances of β have have been moved.

***Page 380**: In Exercise 12.79 replace "also a parameter" with"a known parameter."

*****Page 417**: In Exercise 12.103 rewrite the last sentence as "Determine a 95% linear confidence interval for $\beta_1 - \beta_2$ and then use the result to obtain a confidence interval for the relative risk of a male child compared to a female adult.

***Page 494**: In the definition of the curvature-adjusted cubic spline, the requirement in parentheses should be (m_0 and m_n fixed). There is no requirement that they also be equal.

*****Page 508**: In line 2 the reference should be to equation (15.15). In the equation just after the phrase "after dividing the derivative by 2" the right hand side should be $\mathbf{0}^T$ instead of $\mathbf{0}$.

*****Page 520**: In the second line following (16.5) the integral should be $\int E(X|Y = y)f_Y(y)dy$.

*****Page 530**: In line 2 of Section 16.3 change "part of the century" to "1900s".

Page 534: In the second line of Case 2 change $\mathrm{Var}(Y_j)$ to $\mathrm{Var}(X_j)$.

*****Page 540**: In Exercise 16.10 the reference should be to Example 16.9, not 16.10.

***Page 541**: In Exercise 16.18 add an assumption that the number of claims has the Poisson distribution.

Page 552: In the second to last line $\mu = \alpha\beta$ should be $\mu_{n+1} = \alpha\beta$.

***Page 567**: About one-third way down the page replace "Suppose, for example," with "We parameterize such".

***Page 616**: In the table near the top of the page, in the column "the simulated value is" the entries should be 0, 1, 2, 3, and 4 rather than all zeros.

*****Page 617**: The middle part of the sentence about two-thirds the way down the page should be .... a is the greatest integer less than or equal to $0.9n + 0.5 - 1.96\sqrt{0.9(0.1)n}$, b is the smallest integer greater than or equal to $0.9n + 0.5 + 1.96\sqrt{0.9(0.1)n}$, and the process terminates when both ...

***Page 618**: In Exercise 17.4, the term 1% in the last line should be 2%.

*****Page 624**: In the first equation on the page, delete the 27 from the denominator and multiply the last term in the numerator by (1/27). The correct expression is

$(2-4)^2(1/27)+(\frac{7}{3}-4)^2(3/27)+\cdots+(7-4)^2(1/27) = 14/9.$

***Page 640**: The formula for E$[(X \wedge x)^k]$ for the single-parameter Pareto distribution only applies for $x \geq \theta$.

Errata for the Solutions Manual to Accompany Loss Models, 2nd ed., July 6, 2006 All remain uncorrected in published versions. Those marked with a * are new since the November 2005 errata list.

Page 20, Exercise 4.10: The first sentence should state that the density if the sum of five, not six, function.

Page 23, Exercise 4.14: The gamma and lognormal densities are equal at 2,221, not 2,617. The gamma density is $0.62851x^{-0.8}e^{-0.002x}$.

Page 27, Exercise 4.25: The differential $d\lambda$ is missing from the end of the first line.

Page 30, Exercise 4.35: The last entry should read $1 - F_Y(2.2) = (2.2/1.1)^{-3} = 0.125$

Page 37, Exercise 4.50: In the last line replace $\sum_{i=1}^{n} m\lambda_i$ with $\sum_{i=1}^{n} \lambda_i$.

***Page 62, Exercise 6.16**: $E(A) = 50k^{-1} - 50 + 12.5k$. When set equal to $50k$ the solution of $k = 2/3$ is correct.

***Page 66, Exercise 6.31**: The correct calculation, using μ for the mean is $E(S) = E[E(S|\mu)] = E(\mu) = 300,000$.

***Page 72, Exercise 6.46**: With $f_2 = 11/29$ the expected number of claims is $29(11/29) = 11$.

Page 115: Exercise 9.10: Because the problem did not specify which MSE should be in the numerator, $0.32/0.2 = 1.6$ is also a correct answer.

Page 119, Exercise 10.2: Delete the solutions to (b) and (c) and reletter (d) to (b).

Page 121, Exercise 10.5: Delete part (a) and reletter part (b) to (a) and part (c) to (b).

Page 136, Exercise 11.25: The last column in Table 11.7 should be headed $q_j'^{(w)}$ rather than $q_j'^{(d)}$.

Page 136, Exercise 11.26: In Table 11.8 the second entry in the first column should be 45.4 instead of 45.6. This changes the second r value to 7. The five probabilities in the last column should be 0.875, 0.750, 0.656, 0.549, and 0.438. The answer is then $\hat{q}_{45} = 0.250$ and $\hat{q}_{46} = 0.416$.

***Page 142, Exercise 12.13**: Delete part (a) and remove the lable (b) from the second part.

Page 152, Exercise 12.43: Delete part (b).

***Page 159, Exercise 12.60(b)**: Referece should be to Exercise 12.43.

Page 160, Exercise 12.64: Delete the reference to Exercise 12.14.

***Page 161, Exercise 12.66(b)**: The last term in the expression for $\ln f(x)$ should be $+(\tau - 1)\ln x$ and not $-(\tau - 1)\ln x$. Make the same change in the expression for $l(\mu)$.

Page 163, Exercise 12.67: Change $W = \ln Y - 100$ to $W = \ln Y - \ln 100$.

Page 166, Exercise 12.75(a): In line 2, the extra e should be deleted; in line 3 the a in the superscript should be α; and in line 4 the β should be under the square root along with the 2π.

Page 167, Exercise 12.78: In the second to last line, the variable is X, not x, so write $E(X) = \mu(\theta)$.

Page 169, Exercise 12.80: The last line should be - But $\pi(\theta|s) \propto f(s|\theta)\pi(\theta) \propto \frac{e^{-\theta s}}{[q(\theta)]^n}\pi(\theta)$.

***Page 175, Exercise 12.98(e)**: $\hat{q} = 0.166/7 = 0.0237143$.

Page 206, Exercise 15 5: For part (c), the fifth line should be $f_0''(0) = 0, f_1''(0) = 4$.

Page 212, Exercise 16 2(d): Using fractions, the exact variance of 0.6 can be obtained.

Page 216, Exercise 16.11: There is no error here, but there may be confusion because λ is defined not as the expected number of claims per policy, but rather as the expected total number of claims. In this problem λ and λ_0 are identical.

Page 218, Exercise 16.22(h): The $\frac{1}{6}$ should be outside the brackets and the first term inside the brackets should be $\frac{25(1)}{900}$. The answer is correct.

Page 227, Exercise 16.30: In the last line, add a minus sign to produce $p(m,x) = \left(\frac{m}{2\pi x^3}\right)^{1/2} \exp(-\frac{m}{2x})$

Page 227, Exercise 16.31(d): Replace the given text with "This is the usual Bühlmann-Straub credibility premium, updated with inflation.

Page 228, Exercise 16.32: In the first and second lines, remove the negation in the exponents of the pgfs. Then $P_{X_j}(z|\theta) = e^{\theta(z-1)}$ and $P_S(z|\theta) = e^{n\theta(z-1)}$.

Typos and errors in *Derivatives Markets*, Second Edition, first printing

Page	Item
198	Last line: "stock hedge" should be "stack hedge".
200	The definition of "Heating Degree Day" should be "the maximum of zero and the difference between 65 degrees farenheit and the average daily temperature."
264	In the box about the P&G swap, bottom of the first column: the spread evaluated to -.17, which is -17 *percent*, not basis points.
309	Problems 9.17 and 9.18. Table 9.1 was updated without the problem having been updated. The appropriately revised problems are here.
316	Footnote 2: the superscipt "Sh" should be a "delta h"
344	The early exercise condition rK>delta S is correct only for infinitely-lived options (The demonstration of this condition is on pp. 566-567.)
380-381 Ex 12.3	The example is correct as stated, however, at the top of p. 381 there should be some discussion of the appropriate volatility being that of the prepaid forward price, S-PV(Div), rather than of S (this issue is discussed on p. 365.)
384-386	Figures 12.1-4: The phrase "at-the-money" should be deleted from the figure captions.
431	Equation (13.11) applies for a delta-hedged *long* call. To be consistent with the text there should be a minus sign in foint of equation (13.11)
431	Footnote 5: "Variance" should be "standard deviation".
456	Example 14.2: There are two corrections. 1. The CallOnPut calculation should use the prepaid forward price ($95.0987) as the stock price, instead of $100. This gives a compound option value of $1.7552. The correct value for the American option is therefore $13.5325. 2. The Black-Scholes value should be $11.764. For more details, see an expanded discussion of this section and example 14.2.
466	Appendix 14.A: The text should say that 1) The pricing formulas for barrier options are covered in Section 22.3 and 2) All of the options discussed in this chapter have pricing formulas (and VBA code) available in the spreadsheet accompanying the book.
493	Figure 15.1 "prepair" should be "prepaid"

502	Problem 15.22: the bullet list should be enumerated (i.e., as parts "a", "b", and "c")
508	Fourth line of text, "of the equity increases by $0.735 and ..."
518	Figure 16.4, Panel F. The value 5361.58 should be italicized since the firm calls the bond.
560	The formula for "sigma hat squared sub t" right below equation (17.3) should not have an H in it. (By analogy, if you were pricing an option to exchange one share of S for k shares of Q, the option price would depend on the volatility of the return difference between S and Q, and not depend on k.)
612	Figure 18.6. In the bottom two panels, the plotted distribution is normal with mean 3 and standard deviation 5, not standard normal.
655	Equation (20.9) is an Ornstein-Uhlenbeck process even when alpha is not equal to zero.
779	Sixth line from bottom: Duration is defined in Section 7.3, not 7.8.
787	Line below equation 24.27: "variance" should be "standard deviation"
787	Line above equation 24.28: "risk premium" should be "Sharpe ratio"
789	Figure 24.1. In the bottom panel, the fourth tick mark should be "20", not "0".
795	Example 24.3, first line: "Figure 24.2" should be "Figure 24.3"
799	Table 24.2, caption: "Volatility refers to the volatility of the bond **yield**" (not "price").
800	Fourth line: "time-t" should be "time-h".
800	Fifth line: It would be clearer to say "The *annualized* yield of the bond is"
800	Equation 24.48: The right-hand side should be divided by sqrt (h) in order to annualize the volatility. (This doesn't affect any of the calculations, since h=1 throughout the example.)
800	Figure 24.4. In Period 1, at the upper node an equals sign is missing (should be R_u=R_h etc.)
804	The sixth line of text should say "Both yield volatilities match ..."
805	In the caplets and caps example: 1) the loan should be referred to as a 4-year loan, since (in Fig 24.9) the final payment is made four years from the initiation date. 2) the reference to "2-year caplet" and "year-2 cap payment" should be changed to "3-year caplet" and "year-3 cap payment" for consistency with the caplet definition on p. 792.
910, 912	The definitions of "heating degree day" and "cooling degree day" are reversed.

Tables for Exam C/4

The reading material for Exam C/4 includes a variety of textbooks. Each text has a set of probability distributions that are used in its readings. For those distributions used in more than one text, the choices of parameterization may not be the same in all of the books. This may be of educational value while you study, but could add a layer of uncertainty in the examination. For this latter reason, we have adopted one set of parameterizations to be used in examinations. This set will be based on Appendices A & B of *Loss Models: From Data to Decisions* by Klugman, Panjer and Willmot. A slightly revised version of these appendices is included in this note. A copy of this note will also be distributed to each candidate at the examination.

Each text also has its own system of dedicated notation and terminology. Sometimes these may conflict. If alternative meanings could apply in an examination question, the symbols will be defined.

For Exam C/4, in addition to the abridged table from *Loss Models*, sets of values from the standard normal and chi-square distributions will be available for use in examinations. These are also included in this note.

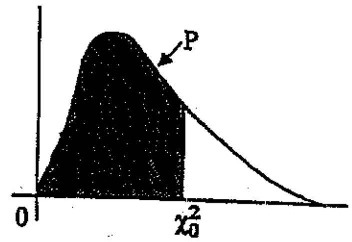
NORMAL DISTRIBUTION TABLE

Entries represent the area under the standardized normal distribution from -∞ to z, Pr(Z<z).
The value of z to the first decimal is given in the left column. The second decimal place is given in the top row.

z	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
0.0	0.5000	0.5040	0.5080	0.5120	0.5160	0.5199	0.5239	0.5279	0.5319	0.5359
0.1	0.5398	0.5438	0.5478	0.5517	0.5557	0.5596	0.5636	0.5675	0.5714	0.5753
0.2	0.5793	0.5832	0.5871	0.5910	0.5948	0.5987	0.6026	0.6064	0.6103	0.6141
0.3	0.6179	0.6217	0.6255	0.6293	0.6331	0.6368	0.6406	0.6443	0.6480	0.6517
0.4	0.6554	0.6591	0.6628	0.6664	0.6700	0.6736	0.6772	0.6808	0.6844	0.6879
0.5	0.6915	0.6950	0.6985	0.7019	0.7054	0.7088	0.7123	0.7157	0.7190	0.7224
0.6	0.7257	0.7291	0.7324	0.7357	0.7389	0.7422	0.7454	0.7486	0.7517	0.7549
0.7	0.7580	0.7611	0.7642	0.7673	0.7704	0.7734	0.7764	0.7794	0.7823	0.7852
0.8	0.7881	0.7910	0.7939	0.7967	0.7995	0.8023	0.8051	0.8078	0.8106	0.8133
0.9	0.8159	0.8186	0.8212	0.8238	0.8264	0.8289	0.8315	0.8340	0.8365	0.8389
1.0	0.8413	0.8438	0.8461	0.8485	0.8508	0.8531	0.8554	0.8577	0.8599	0.8621
1.1	0.8643	0.8665	0.8686	0.8708	0.8729	0.8749	0.8770	0.8790	0.8810	0.8830
1.2	0.8849	0.8869	0.8888	0.8907	0.8925	0.8944	0.8962	0.8980	0.8997	0.9015
1.3	0.9032	0.9049	0.9066	0.9082	0.9099	0.9115	0.9131	0.9147	0.9162	0.9177
1.4	0.9192	0.9207	0.9222	0.9236	0.9251	0.9265	0.9279	0.9292	0.9306	0.9319
1.5	0.9332	0.9345	0.9357	0.9370	0.9382	0.9394	0.9406	0.9418	0.9429	0.9441
1.6	0.9452	0.9463	0.9474	0.9484	0.9495	0.9505	0.9515	0.9525	0.9535	0.9545
1.7	0.9554	0.9564	0.9573	0.9582	0.9591	0.9599	0.9608	0.9616	0.9625	0.9633
1.8	0.9641	0.9649	0.9656	0.9664	0.9671	0.9678	0.9686	0.9693	0.9699	0.9706
1.9	0.9713	0.9719	0.9726	0.9732	0.9738	0.9744	0.9750	0.9756	0.9761	0.9767
2.0	0.9772	0.9778	0.9783	0.9788	0.9793	0.9798	0.9803	0.9808	0.9812	0.9817
2.1	0.9821	0.9826	0.9830	0.9834	0.9838	0.9842	0.9846	0.9850	0.9854	0.9857
2.2	0.9861	0.9864	0.9868	0.9871	0.9875	0.9878	0.9881	0.9884	0.9887	0.9890
2.3	0.9893	0.9896	0.9898	0.9901	0.9904	0.9906	0.9909	0.9911	0.9913	0.9916
2.4	0.9918	0.9920	0.9922	0.9925	0.9927	0.9929	0.9931	0.9932	0.9934	0.9936
2.5	0.9938	0.9940	0.9941	0.9943	0.9945	0.9946	0.9948	0.9949	0.9951	0.9952
2.6	0.9953	0.9955	0.9956	0.9957	0.9959	0.9960	0.9961	0.9962	0.9963	0.9964
2.7	0.9965	0.9966	0.9967	0.9968	0.9969	0.9970	0.9971	0.9972	0.9973	0.9974
2.8	0.9974	0.9975	0.9976	0.9977	0.9977	0.9978	0.9979	0.9979	0.9980	0.9981
2.9	0.9981	0.9982	0.9982	0.9983	0.9984	0.9984	0.9985	0.9985	0.9986	0.9986
3.0	0.9987	0.9987	0.9987	0.9988	0.9988	0.9989	0.9989	0.9989	0.9990	0.9990
3.1	0.9990	0.9991	0.9991	0.9991	0.9992	0.9992	0.9992	0.9992	0.9993	0.9993
3.2	0.9993	0.9993	0.9994	0.9994	0.9994	0.9994	0.9994	0.9995	0.9995	0.9995
3.3	0.9995	0.9995	0.9995	0.9996	0.9996	0.9996	0.9996	0.9996	0.9996	0.9997
3.4	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9998
3.5	0.9998	0.9998	0.9998	0.9998	0.9998	0.9998	0.9998	0.9998	0.9998	0.9998
3.6	0.9998	0.9998	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999
3.7	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999
3.8	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999
3.9	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000

Values of z for selected values of Pr(Z<z)							
z	0.842	1.036	1.282	1.645	1.960	2.326	2.576
Pr(Z<z)	0.800	0.850	0.900	0.950	0.975	0.990	0.995

χ^2 Distribution



The table below gives the value x_0^2 for which $P[x^2 < x_0^2] = P$ for a given number of degrees of freedom and a given value of P.

Degrees of Freedom	Values of P									
	0.005	**0.010**	**0.025**	**0.050**	**0.100**	**0.900**	**0.950**	**0.975**	**0.990**	**0.995**
1	---	---	0.001	0.004	0.016	2.706	3.841	5.024	6.635	7.879
2	0.01	0.020	0.051	0.103	0.211	4.605	5.991	7.378	9.210	10.597
3	0.072	0.115	0.216	0.352	0.584	6.251	7.815	9.348	11.345	12.838
4	0.207	0.297	0.484	0.711	1.064	7.779	9.488	11.143	13.277	14.860
5	0.412	0.554	0.831	1.145	1.610	9.236	11.070	12.833	15.086	16.750
6	0.676	0.872	1.237	1.635	2.204	10.645	12.592	14.449	16.812	18.548
7	0.989	1.239	1.690	2.167	2.833	12.017	14.067	16.013	18.475	20.278
8	1.344	1.646	2.180	2.733	3.490	13.362	15.507	17.535	20.090	21.955
9	1.735	2.088	2.700	3.325	4.168	14.684	16.919	19.023	21.666	23.589
10	2.156	2.558	3.247	3.940	4.865	15.987	18.307	20.483	23.209	25.188
11	2.603	3.053	3.816	4.575	5.578	17.275	19.675	21.920	24.725	26.757
12	3.074	3.571	4.404	5.226	6.304	18.549	21.026	23.337	26.217	28.300
13	3.565	4.107	5.009	5.892	7.042	19.812	22.362	24.736	27.688	29.819
14	4.075	4.660	5.629	6.571	7.790	21.064	23.685	26.119	29.141	31.319
15	4.601	5.229	6.262	7.261	8.547	22.307	24.996	27.488	30.578	32.801
16	5.142	5.812	6.908	7.962	9.312	23.542	26.296	28.845	32.000	34.267
17	5.697	6.408	7.564	8.672	10.085	24.769	27.587	30.191	33.409	35.718
18	6.265	7.015	8.231	9.390	10.865	25.989	28.869	31.526	34.805	37.156
19	6.844	7.633	8.907	10.117	11.651	27.204	30.144	32.852	36.191	38.582
20	7.434	8.260	9.591	10.851	12.443	28.412	31.410	34.170	37.566	39.997

APPENDICES A & B FROM

LOSS MODELS: FROM DATA TO DECISIONS, SECOND EDITION

by

Stuart A. Klugman
Harry H. Panjer
and
Gordon E. Willmot

Appendix A
An inventory of continuous distributions

A.1 INTRODUCTION

Descriptions of the models are given below. First a few mathematical preliminaries are presented that indicate how the various quantities can be computed.

The incomplete gamma function[1] is given by

$$\Gamma(\alpha; x) = \frac{1}{\Gamma(\alpha)} \int_0^x t^{\alpha-1} e^{-t}\, dt, \quad \alpha > 0,\ x > 0$$

$$\text{with } \Gamma(\alpha) = \int_0^\infty t^{\alpha-1} e^{-t}\, dt, \quad \alpha > 0.$$

[1]Some references, such as [3], denote this integral $P(\alpha, x)$ and define $\Gamma(\alpha, x) = \int_x^\infty t^{\alpha-1}e^{-t}\, dt$. Note that this definition does not normalize by dividing by $\Gamma(\alpha)$. When using software to evaluate the incomplete gamma function, be sure to note how it is defined.

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Also, define

$$G(\alpha;x)=\int_x^\infty t^{\alpha-1}e^{-t}\,dt,\quad x>0.$$

At times we will need this integral for nonpositive values of α. Integration by parts produces the relationship

$$G(\alpha;x)=-\frac{x^\alpha e^{-x}}{\alpha}+\frac{1}{\alpha}G(\alpha+1;x).$$

This can be repeated until the first argument of G is $\alpha+k$, a positive number. Then it can be evaluated from

$$G(\alpha+k;x)=\Gamma(\alpha+k)[1-\Gamma(\alpha+k;x)].$$

However, if α is a negative integer or zero, the value of $G(0;x)$ is needed. It is

$$G(0;x)=\int_x^\infty t^{-1}e^{-t}\,dt=E_1(x),$$

which is called the **exponential integral**. A series expansion for this integral is

$$E_1(x)=-0.57721566490153-\ln x-\sum_{n=1}^{\infty}\frac{(-1)^n x^n}{n(n!)}.$$

When α is a positive integer, the incomplete gamma function can be evaluated exactly as given in the following theorem.

Theorem A.1 *For integer* α,

$$\Gamma(\alpha;x)=1-\sum_{j=0}^{\alpha-1}\frac{x^j e^{-x}}{j!}.$$

Proof: For $\alpha=1$, $\Gamma(1;x)=\int_0^x e^{-t}\,dt=1-e^{-x}$, and so the theorem is true for this case. The proof is completed by induction. Assume it is true for $\alpha=1,\ldots,n$. Then

$$\begin{aligned}\Gamma(n+1;x)&=\frac{1}{n!}\int_0^x t^n e^{-t}\,dt\\&=\frac{1}{n!}\left(-t^n e^{-t}\Big|_0^x+\int_0^x nt^{n-1}e^{-t}\,dt\right)\\&=\frac{1}{n!}\left(-x^n e^{-x}\right)+\Gamma(n;x)\\&=-\frac{x^n e^{-x}}{n!}+1-\sum_{j=0}^{n-1}\frac{x^j e^{-x}}{j!}\\&=1-\sum_{j=0}^{n}\frac{x^j e^{-x}}{j!}.\end{aligned}$$

The incomplete beta function is given by

$$\beta(a,b;x)=\frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)}\int_0^x t^{a-1}(1-t)^{b-1}\,dt,\quad a>0,\ b>0,\ 0<x<1,$$

and when $b<0$ (but $a>1+\lfloor -b\rfloor$), repeated integration by parts produces

$$\begin{aligned}\Gamma(a)\Gamma(b)\beta(a,b;x) &= -\Gamma(a+b)\left[\frac{x^{a-1}(1-x)^b}{b}\right.\\ &\quad+\frac{(a-1)x^{a-2}(1-x)^{b+1}}{b(b+1)}+\cdots\\ &\quad\left.+\frac{(a-1)\cdots(a-r)x^{a-r-1}(1-x)^{b+r}}{b(b+1)\cdots(b+r)}\right]\\ &\quad+\frac{(a-1)\cdots(a-r-1)}{b(b+1)\cdots(b+r)}\Gamma(a-r-1)\\ &\quad\times\Gamma(b+r+1)\beta(a-r-1,b+r+1;x),\end{aligned}$$

where r is the smallest integer such that $b+r+1>0$. The first argument must be positive (that is, $a-r-1>0$).

Numerical approximations for both the incomplete gamma and the incomplete beta function are available in many statistical computing packages as well as in many spreadsheets because they are just the distribution functions of the gamma and beta distributions. The following approximations are taken from [3]. The suggestion regarding using different formulas for small and large x when evaluating the incomplete gamma function is from [107]. That reference also contains computer subroutines for evaluating these expressions. In particular, it provides an effective way of evaluating continued fractions.

For $x\le\alpha+1$ use the series expansion

$$\Gamma(\alpha;x)=\frac{x^\alpha e^{-x}}{\Gamma(\alpha)}\sum_{n=0}^{\infty}\frac{x^n}{\alpha(\alpha+1)\cdots(\alpha+n)}$$

while for $x>\alpha+1$ use the continued-fraction expansion

$$1-\Gamma(\alpha;x)=\frac{x^\alpha e^{-x}}{\Gamma(\alpha)}\cfrac{1}{x+\cfrac{1-\alpha}{1+\cfrac{1}{x+\cfrac{2-\alpha}{1+\cfrac{2}{x+\cdots}}}}}$$

The incomplete gamma function can also be used to produce cumulative probabilities from the standard normal distribution. Let $\Phi(z)=\Pr(Z\le z)$, where Z has the standard normal distribution. Then, for $z \geq 0$, $\Phi(z) = 0.5 + \Gamma(0.5; z^2/2)/2$ while, for $z < 0$, $\Phi(z) = 1 - \Phi(-z)$.

The incomplete beta function can be evaluated by the series expansion

$$\beta(a, b; x) = \frac{\Gamma(a+b)x^a(1-x)^b}{a\Gamma(a)\Gamma(b)} \times \left[1 + \sum_{n=0}^{\infty} \frac{(a+b)(a+b+1)\cdots(a+b+n)}{(a+1)(a+2)\cdots(a+n+1)} x^{n+1}\right].$$

The gamma function itself can be found from

$$\begin{aligned}\ln \Gamma(\alpha) \doteq\ & (\alpha - \tfrac{1}{2}) \ln \alpha - \alpha + \frac{\ln(2\pi)}{2} \\ & + \frac{1}{12\alpha} - \frac{1}{360\alpha^3} + \frac{1}{1{,}260\alpha^5} - \frac{1}{1{,}680\alpha^7} + \frac{1}{1{,}188\alpha^9} - \frac{691}{360{,}360\alpha^{11}} \\ & + \frac{1}{156\alpha^{13}} - \frac{3{,}617}{122{,}400\alpha^{15}} + \frac{43{,}867}{244{,}188\alpha^{17}} - \frac{174{,}611}{125{,}400\alpha^{19}}.\end{aligned}$$

For values of α above 10 the error is less than 10^{-19}. For values below 10 use the relationship

$$\ln \Gamma(\alpha) = \ln \Gamma(\alpha + 1) - \ln \alpha.$$

The distributions are presented in the following way. First the name is given along with the parameters. Many of the distributions have other names, which are noted in parentheses. Next the density function $f(x)$ and distribution function $F(x)$ are given. For some distributions, formulas for starting values are given. Within each family the distributions are presented in decreasing order with regard to the number of parameters. The Greek letters used are selected to be consistent. Any Greek letter that is not used in the distribution means that that distribution is a special case of one with more parameters but with the missing parameters set equal to 1. Unless specifically indicated, all parameters must be positive.

Except for two distributions, inflation can be recognized by simply inflating the scale parameter θ. That is, if X has a particular distribution, then cX has the same distribution type, with all parameters unchanged except θ is changed to $c\theta$. For the lognormal distribution, μ changes to $\mu + \ln(c)$ with σ unchanged, while for the inverse Gaussian both μ and θ are multiplied by c.

For several of the distributions, starting values are suggested. They are not necessarily good estimators, just places from which to start an iterative procedure to maximize the likelihood or other objective function. These are found by either the methods of moments or percentile matching. The quantities used are:

$$\text{Moments:} \quad m = \frac{1}{n}\sum_{i=1}^{n} x_i, \quad t = \frac{1}{n}\sum_{i=1}^{n} x_i^2,$$

Percentile matching: $p =$ 25th percentile, $q =$ 75th percentile

For grouped data or data that have been truncated or censored, these quantities may have to be approximated. Because the purpose is to obtain starting values and not a useful estimate, it is often sufficient to just ignore modifications. For three- and four-parameter distributions, starting values can be obtained by using estimates from a special case, then making the new parameters equal to 1. An all-purpose starting value rule (for when all else fails) is to set the scale parameter (θ) equal to the mean and set all other parameters equal to 2.

All the distributions listed here (and many more) are discussed in great detail in [73]. In many cases, alternatives to maximum likelihood estimators are presented.

A.2 TRANSFORMED BETA FAMILY

A.2.1 Four-parameter distribution

A.2.1.1 Transformed beta—$\alpha, \theta, \gamma, \tau$ (generalized beta of the second kind, Pearson Type VI)

$$\begin{aligned}
f(x) &= \frac{\Gamma(\alpha+\tau)}{\Gamma(\alpha)\Gamma(\tau)} \frac{\gamma(x/\theta)^{\gamma\tau}}{x[1+(x/\theta)^{\gamma}]^{\alpha+\tau}}, \\
F(x) &= \beta(\tau,\alpha;u), \quad u = \frac{(x/\theta)^{\gamma}}{1+(x/\theta)^{\gamma}}, \\
\mathrm{E}[X^k] &= \frac{\theta^k\Gamma(\tau+k/\gamma)\Gamma(\alpha-k/\gamma)}{\Gamma(\alpha)\Gamma(\tau)}, \quad -\tau\gamma < k < \alpha\gamma, \\
\mathrm{E}[(X\wedge x)^k] &= \frac{\theta^k\Gamma(\tau+k/\gamma)\Gamma(\alpha-k/\gamma)}{\Gamma(\alpha)\Gamma(\tau)}\beta(\tau+k/\gamma,\alpha-k/\gamma;u) \\
&\quad + x^k[1-F(x)], \quad k > -\tau\gamma, \\
\text{Mode} &= \theta\left(\frac{\tau\gamma-1}{\alpha\gamma+1}\right)^{1/\gamma}, \quad \tau\gamma > 1, \text{ else } 0.
\end{aligned}$$

A.2.2 Three-parameter distributions

A.2.2.1 Generalized Pareto—α, θ, τ (beta of the second kind)

$$\begin{aligned}
f(x) &= \frac{\Gamma(\alpha+\tau)}{\Gamma(\alpha)\Gamma(\tau)} \frac{\theta^{\alpha}x^{\tau-1}}{(x+\theta)^{\alpha+\tau}}, \\
F(x) &= \beta(\tau,\alpha;u), \quad u = \frac{x}{x+\theta},
\end{aligned}$$

$$\mathrm{E}[X^k] = \frac{\theta^k\Gamma(\tau+k)\Gamma(\alpha-k)}{\Gamma(\alpha)\Gamma(\tau)}, \quad -\tau < k < \alpha,$$

$$\mathrm{E}[X^k] = \frac{\theta^k\tau(\tau+1)\cdots(\tau+k-1)}{(\alpha-1)\cdots(\alpha-k)} \quad \text{if } k \text{ is an integer},$$

$$\mathrm{E}[(X\wedge x)^k] = \frac{\theta^k\Gamma(\tau+k)\Gamma(\alpha-k)}{\Gamma(\alpha)\Gamma(\tau)}\beta(\tau+k,\alpha-k;u), + x^k[1-F(x)], \quad k > -\tau,$$

$$\text{Mode} = \theta\frac{\tau-1}{\alpha+1}, \quad \tau > 1, \text{ else } 0.$$

A.2.2.2 *Burr*—α, θ, γ (Burr Type XII, Singh–Maddala)

$$f(x) = \frac{\alpha\gamma(x/\theta)^\gamma}{x[1+(x/\theta)^\gamma]^{\alpha+1}},$$

$$F(x) = 1-u^\alpha, \quad u = \frac{1}{1+(x/\theta)^\gamma},$$

$$\mathrm{E}[X^k] = \frac{\theta^k\Gamma(1+k/\gamma)\Gamma(\alpha-k/\gamma)}{\Gamma(\alpha)}, \quad -\gamma < k < \alpha\gamma,$$

$$\mathrm{E}[(X\wedge x)^k] = \frac{\theta^k\Gamma(1+k/\gamma)\Gamma(\alpha-k/\gamma)}{\Gamma(\alpha)}\beta(1+k/\gamma,\alpha-k/\gamma;1-u) + x^k u^\alpha, \quad k > -\gamma,$$

$$\text{Mode} = \theta\left(\frac{\gamma-1}{\alpha\gamma+1}\right)^{1/\gamma}, \quad \gamma > 1, \text{ else } 0.$$

A.2.2.3 *Inverse Burr*—τ, θ, γ (Dagum)

$$f(x) = \frac{\tau\gamma(x/\theta)^{\gamma\tau}}{x[1+(x/\theta)^\gamma]^{\tau+1}},$$

$$F(x) = u^\tau, \quad u = \frac{(x/\theta)^\gamma}{1+(x/\theta)^\gamma},$$

$$\mathrm{E}[X^k] = \frac{\theta^k\Gamma(\tau+k/\gamma)\Gamma(1-k/\gamma)}{\Gamma(\tau)}, \quad -\tau\gamma < k < \gamma,$$

$$\mathrm{E}[(X\wedge x)^k] = \frac{\theta^k\Gamma(\tau+k/\gamma)\Gamma(1-k/\gamma)}{\Gamma(\tau)}\beta(\tau+k/\gamma,1-k/\gamma;u) + x^k[1-u^\tau], \quad k > -\tau\gamma,$$

$$\text{Mode} = \theta\left(\frac{\tau\gamma-1}{\gamma+1}\right)^{1/\gamma}, \quad \tau\gamma > 1, \text{ else } 0.$$

A.2.3 Two-parameter distributions

A.2.3.1 Pareto—α, θ (Pareto Type II, Lomax)

$$
\begin{aligned}
f(x) &= \frac{\alpha\theta^\alpha}{(x+\theta)^{\alpha+1}}, \\
F(x) &= 1-\left(\frac{\theta}{x+\theta}\right)^\alpha, \\
\mathrm{E}[X^k] &= \frac{\theta^k\Gamma(k+1)\Gamma(\alpha-k)}{\Gamma(\alpha)}, \quad -1<k<\alpha, \\
\mathrm{E}[X^k] &= \frac{\theta^k k!}{(\alpha-1)\cdots(\alpha-k)} \quad \text{if } k \text{ is an integer} \\
\mathrm{E}[X\wedge x] &= \frac{\theta}{\alpha-1}\left[1-\left(\frac{\theta}{x+\theta}\right)^{\alpha-1}\right], \quad \alpha\neq 1, \\
\mathrm{E}[X\wedge x] &= -\theta\ln\left(\frac{\theta}{x+\theta}\right), \quad \alpha=1, \\
\mathrm{E}[(X\wedge x)^k] &= \frac{\theta^k\Gamma(k+1)\Gamma(\alpha-k)}{\Gamma(\alpha)}\beta[k+1,\alpha-k;x/(x+\theta)] \\
&\quad +x^k\left(\frac{\theta}{x+\theta}\right)^\alpha, \quad \text{all } k, \\
\text{Mode} &= 0, \\
\hat{\alpha} &= 2\frac{t-m^2}{t-2m^2}, \quad \hat{\theta}=\frac{mt}{t-2m^2}
\end{aligned}
$$

A.2.3.2 Inverse Pareto—τ, θ

$$
\begin{aligned}
f(x) &= \frac{\tau\theta x^{\tau-1}}{(x+\theta)^{\tau+1}}, \\
F(x) &= \left(\frac{x}{x+\theta}\right)^\tau, \\
\mathrm{E}[X^k] &= \frac{\theta^k\Gamma(\tau+k)\Gamma(1-k)}{\Gamma(\tau)}, \quad -\tau<k<1, \\
\mathrm{E}[X^k] &= \frac{\theta^k(-k)!}{(\tau-1)\cdots(\tau+k)} \quad \text{if } k \text{ is a negative integer}, \\
\mathrm{E}[(X\wedge x)^k] &= \theta^k\tau\int_0^{x/(x+\theta)} y^{\tau+k-1}(1-y)^{-k}dy \\
&\quad +x^k\left[1-\left(\frac{x}{x+\theta}\right)^\tau\right], \quad k>-\tau, \\
\text{Mode} &= \theta\frac{\tau-1}{2}, \quad \tau>1, \text{ else } 0
\end{aligned}
$$

A.2.3.3 Loglogistic—γ, θ (Fisk)

$$
\begin{aligned}
f(x) &= \frac{\gamma(x/\theta)^\gamma}{x[1+(x/\theta)^\gamma]^2}, \\
F(x) &= u, \quad u = \frac{(x/\theta)^\gamma}{1+(x/\theta)^\gamma}, \\
\mathrm{E}[X^k] &= \theta^k\Gamma(1+k/\gamma)\Gamma(1-k/\gamma), \quad -\gamma < k < \gamma, \\
\mathrm{E}[(X \wedge x)^k] &= \theta^k\Gamma(1+k/\gamma)\Gamma(1-k/\gamma)\beta(1+k/\gamma, 1-k/\gamma; u) \\
&\quad + x^k(1-u), \quad k > -\gamma, \\
\text{Mode} &= \theta\left(\frac{\gamma-1}{\gamma+1}\right)^{1/\gamma}, \quad \gamma > 1, \text{ else } 0, \\
\hat{\gamma} &= \frac{2\ln(3)}{\ln(q)-\ln(p)}, \quad \hat{\theta} = \exp\left(\frac{\ln(q)+\ln(p)}{2}\right).
\end{aligned}
$$

A.2.3.4 Paralogistic—α, θ This is a Burr distribution with $\gamma = \alpha$

$$
\begin{aligned}
f(x) &= \frac{\alpha^2(x/\theta)^\alpha}{x[1+(x/\theta)^\alpha]^{\alpha+1}}, \\
F(x) &= 1-u^\alpha, \quad u = \frac{1}{1+(x/\theta)^\alpha}, \\
\mathrm{E}[X^k] &= \frac{\theta^k\Gamma(1+k/\alpha)\Gamma(\alpha-k/\alpha)}{\Gamma(\alpha)}, \quad -\alpha < k < \alpha^2, \\
\mathrm{E}[(X \wedge x)^k] &= \frac{\theta^k\Gamma(1+k/\alpha)\Gamma(\alpha-k/\alpha)}{\Gamma(\alpha)}\beta(1+k/\alpha, \alpha-k/\alpha; 1-u) \\
&\quad + x^k u^\alpha, \quad k > -\alpha, \\
\text{Mode} &= \theta\left(\frac{\alpha-1}{\alpha^2+1}\right)^{1/\alpha}, \quad \alpha > 1, \text{ else } 0
\end{aligned}
$$

Starting values can use estimates from the loglogistic (use γ for α) or Pareto (use α) distributions.

A.2.3.5 Inverse paralogistic—τ, θ This is an inverse Burr distribution with $\gamma = \tau$.

$$
\begin{aligned}
f(x) &= \frac{\tau^2 (x/\theta)^{\tau^2}}{x[1+(x/\theta)^\tau]^{\tau+1}}, \\
F(x) &= u^\tau, \quad u = \frac{(x/\theta)^\tau}{1+(x/\theta)^\tau}, \\
\mathrm{E}[X^k] &= \frac{\theta^k \Gamma(\tau + k/\tau)\Gamma(1-k/\tau)}{\Gamma(\tau)}, \quad -\tau^2 < k < \tau, \\
\mathrm{E}[(X \wedge x)^k] &= \frac{\theta^k \Gamma(\tau + k/\tau)\Gamma(1-k/\tau)}{\Gamma(\tau)} \beta(\tau + k/\tau, 1-k/\tau; u) \\
&\quad + x^k[1-u^\tau], \quad k > -\tau^2, \\
\text{Mode} &= \theta\,(\tau-1)^{1/\tau}, \quad \tau > 1, \text{ else } 0
\end{aligned}
$$

Starting values can use estimates from the loglogistic (use γ for τ) or inverse Pareto (use τ) distributions

A.3 TRANSFORMED GAMMA FAMILY

A.3.1 Three-parameter distributions

A.3.1.1 Transformed gamma—$\boldsymbol{\alpha, \theta, \tau}$ (generalized gamma)

$$
\begin{aligned}
f(x) &= \frac{\tau u^\alpha e^{-u}}{x\Gamma(\alpha)}, \quad u = (x/\theta)^\tau, \\
F(x) &= \Gamma(\alpha; u), \\
\mathrm{E}[X^k] &= \frac{\theta^k \Gamma(\alpha + k/\tau)}{\Gamma(\alpha)}, \quad k > -\alpha\tau, \\
\mathrm{E}[(X \wedge x)^k] &= \frac{\theta^k \Gamma(\alpha + k/\tau)}{\Gamma(\alpha)} \Gamma(\alpha + k/\tau; u) \\
&\quad + x^k[1-\Gamma(\alpha; u)], \quad k > -\alpha\tau, \\
\text{Mode} &= \theta\left(\frac{\alpha\tau - 1}{\tau}\right)^{1/\tau}, \quad \alpha\tau > 1, \text{ else } 0.
\end{aligned}
$$

A.3.1.2 Inverse transformed gamma—α, θ, τ (inverse generalized gamma)

$$
\begin{aligned}
f(x) &= \frac{\tau u^\alpha e^{-u}}{x\Gamma(\alpha)}, \quad u = (\theta/x)^\tau, \\
F(x) &= 1 - \Gamma(\alpha; u), \\
\mathrm{E}[X^k] &= \frac{\theta^k \Gamma(\alpha - k/\tau)}{\Gamma(\alpha)}, \quad k < \alpha\tau, \\
\mathrm{E}[(X \wedge x)^k] &= \frac{\theta^k \Gamma(\alpha - k/\tau)}{\Gamma(\alpha)}[1 - \Gamma(\alpha - k/\tau; u)] + x^k \Gamma(\alpha; u) \\
&= \frac{\theta^k G(\alpha - k/\tau; u)}{\Gamma(\alpha)} + x^k \Gamma(\alpha; u), \quad \text{all } k, \\
\text{Mode} &= \theta \left(\frac{\tau}{\alpha\tau + 1} \right)^{1/\tau}
\end{aligned}
$$

A.3.2 Two-parameter distributions

A.3.2.1 Gamma—α, θ

$$
\begin{aligned}
f(x) &= \frac{(x/\theta)^\alpha e^{-x/\theta}}{x\Gamma(\alpha)}, \\
F(x) &= \Gamma(\alpha; x/\theta), \\
\mathrm{E}[X^k] &= \frac{\theta^k \Gamma(\alpha + k)}{\Gamma(\alpha)}, \quad k > -\alpha, \\
\mathrm{E}[X^k] &= \theta^k (\alpha + k - 1) \cdots \alpha \quad \text{if } k \text{ is an integer}
\end{aligned}
$$

$$
\begin{aligned}
\mathrm{E}[(X \wedge x)^k] &= \frac{\theta^k \Gamma(\alpha + k)}{\Gamma(\alpha)} \Gamma(\alpha + k; x/\theta) + x^k [1 - \Gamma(\alpha; x/\theta)], \quad k > -\alpha \\
\mathrm{E}[(X \wedge x)^k] &= \alpha(\alpha + 1) \cdots (\alpha + k - 1)\theta^k \Gamma(\alpha + k; x/\theta) \\
&\quad + x^k [1 - \Gamma(\alpha; x/\theta)] \quad \text{if } k \text{ is an integer}, \\
M(t) &= (1 - \theta t)^{-\alpha}, \quad t < 1/\theta, \\
\text{Mode} &= \theta(\alpha - 1), \quad \alpha > 1, \text{ else } 0, \\
\hat{\alpha} &= \frac{m^2}{t - m^2}, \quad \hat{\theta} = \frac{t - m^2}{m}
\end{aligned}
$$

A.3.2.2 Inverse gamma—α, θ (Vinci)

$$
\begin{aligned}
f(x) &= \frac{(\theta/x)^\alpha e^{-\theta/x}}{x\Gamma(\alpha)}, , \\
F(x) &= 1 - \Gamma(\alpha; \theta/x)
\end{aligned}
$$

$$\begin{aligned}
\mathrm{E}[X^k] &= \frac{\theta^k\Gamma(\alpha-k)}{\Gamma(\alpha)}, \quad k<\alpha, \\
\mathrm{E}[X^k] &= \frac{\theta^k}{(\alpha-1)\cdots(\alpha-k)} \quad \text{if } k \text{ is an integer}, \\
\mathrm{E}[(X\wedge x)^k] &= \frac{\theta^k\Gamma(\alpha-k)}{\Gamma(\alpha)}[1-\Gamma(\alpha-k;\theta/x)]+x^k\Gamma(\alpha;\theta/x) \\
&= \frac{\theta^k G(\alpha-k;\theta/x)}{\Gamma(\alpha)}+x^k\Gamma(\alpha;\theta/x), \text{ all } k, \\
\text{Mode} &= \theta/(\alpha+1), \\
\hat{\alpha} &= \frac{2t-m^2}{t-m^2}, \quad \hat{\theta}=\frac{mt}{t-m^2}
\end{aligned}$$

A.3.2.3 Weibull—θ, τ

$$\begin{aligned}
f(x) &= \frac{\tau(x/\theta)^\tau e^{-(x/\theta)^\tau}}{x}, \\
F(x) &= 1-e^{-(x/\theta)^\tau}, \\
\mathrm{E}[X^k] &= \theta^k\Gamma(1+k/\tau), \quad k>-\tau, \\
\mathrm{E}[(X\wedge x)^k] &= \theta^k\Gamma(1+k/\tau)\Gamma[1+k/\tau;(x/\theta)^\tau]+x^k e^{-(x/\theta)^\tau}, \quad k>-\tau, \\
\text{Mode} &= \theta\left(\frac{\tau-1}{\tau}\right)^{1/\tau}, \quad \tau>1, \text{ else } 0, \\
\hat{\theta} &= \exp\left(\frac{g\ln(p)-\ln(q)}{g-1}\right), \quad g=\frac{\ln(\ln(4))}{\ln(\ln(4/3))}, \\
\hat{\tau} &= \frac{\ln(\ln(4))}{\ln(q)-\ln(\hat{\theta})}.
\end{aligned}$$

A.3.2.4 Inverse Weibull—θ, τ (log-Gompertz)

$$\begin{aligned}
f(x) &= \frac{\tau(\theta/x)^\tau e^{-(\theta/x)^\tau}}{x}, \\
F(x) &= e^{-(\theta/x)^\tau}, \\
\mathrm{E}[X^k] &= \theta^k\Gamma(1-k/\tau), \quad k<\tau, \\
\mathrm{E}[(X\wedge x)^k] &= \theta^k\Gamma(1-k/\tau)\{1-\Gamma[1-k/\tau;(\theta/x)^\tau]\} \\
&\quad +x^k\left[1-e^{-(\theta/x)^\tau}\right], \\
&= \theta^k G[1-k/\tau;(\theta/x)^\tau]+x^k\left[1-e^{-(\theta/x)^\tau}\right], \quad \text{all } k,
\end{aligned}$$

$$
\begin{aligned}
\text{Mode} &= \theta\left(\frac{\tau}{\tau+1}\right)^{1/\tau}, \\
\hat{\theta} &= \exp\left(\frac{g\ln(q)-\ln(p)}{g-1}\right), \quad g=\frac{\ln(\ln(4))}{\ln(\ln(4/3))}, \\
\hat{\tau} &= \frac{\ln(\ln(4))}{\ln(\hat{\theta})-\ln(p)}
\end{aligned}
$$

A.3.3 One-parameter distributions

A.3.3.1 Exponential—θ

$$
\begin{aligned}
f(x) &= \frac{e^{-x/\theta}}{\theta}, \\
F(x) &= 1-e^{-x/\theta}, \\
\mathrm{E}[X^k] &= \theta^k\Gamma(k+1), \quad k>-1, \\
\mathrm{E}[X^k] &= \theta^k k! \quad \text{if } k \text{ is an integer}, \\
\mathrm{E}[X\wedge x] &= \theta(1-e^{-x/\theta}), \\
\mathrm{E}[(X\wedge x)^k] &= \theta^k\Gamma(k+1)\Gamma(k+1;x/\theta)+x^k e^{-x/\theta}, \quad k>-1, \\
\mathrm{E}[(X\wedge x)^k] &= \theta^k k!\Gamma(k+1;x/\theta)+x^k e^{-x/\theta} \quad \text{if } k>-1 \text{ is an integer}, \\
M(t) &= (1-\theta t)^{-1}, \quad t<1/\theta, \\
\text{Mode} &= 0, \\
\hat{\theta} &= m.
\end{aligned}
$$

A.3.3.2 Inverse exponential—θ

$$
\begin{aligned}
f(x) &= \frac{\theta e^{-\theta/x}}{x^2}, \\
F(x) &= e^{-\theta/x}, \\
\mathrm{E}[X^k] &= \theta^k\Gamma(1-k), \quad k<1, \\
\mathrm{E}[(X\wedge x)^k] &= \theta^k G(1-k;\theta/x)+x^k(1-e^{-\theta/x}), \quad \text{all } k, \\
\text{Mode} &= \theta/2, \\
\hat{\theta} &= -q\ln(3/4)
\end{aligned}
$$

A.4 OTHER DISTRIBUTIONS

A.4.1.1 Lognormal—μ,σ (μ can be negative)

$$
\begin{aligned}
f(x) &= \frac{1}{x\sigma\sqrt{2\pi}}\exp(-z^2/2)=\phi(z)/(\sigma x), \quad z=\frac{\ln x-\mu}{\sigma}, \\
F(x) &= \Phi(z),
\end{aligned}
$$

$$\begin{aligned}
\mathrm{E}[X^k] &= \exp\left(k\mu + \tfrac{1}{2}k^2\sigma^2\right), \\
\mathrm{E}[(X \wedge x)^k] &= \exp\left(k\mu + \tfrac{1}{2}k^2\sigma^2\right)\Phi\left(\frac{\ln x - \mu - k\sigma^2}{\sigma}\right) + x^k[1 - F(x)], \\
\text{Mode} &= \exp(\mu - \sigma^2), \\
\hat{\sigma} &= \sqrt{\ln(t) - 2\ln(m)}, \quad \hat{\mu} = \ln(m) - \tfrac{1}{2}\hat{\sigma}^2
\end{aligned}$$

A 4 1 2 Inverse Gaussian—μ, θ

$$\begin{aligned}
f(x) &= \left(\frac{\theta}{2\pi x^3}\right)^{1/2} \exp\left(-\frac{\theta z^2}{2x}\right), \quad z = \frac{x-\mu}{\mu}, \\
F(x) &= \Phi\left[z\left(\frac{\theta}{x}\right)^{1/2}\right] + \exp\left(\frac{2\theta}{\mu}\right)\Phi\left[-y\left(\frac{\theta}{x}\right)^{1/2}\right], \quad y = \frac{x+\mu}{\mu}, \\
\mathrm{E}[X] &= \mu, \quad Var[X] = \mu^3/\theta, \\
\mathrm{E}[X \wedge x] &= x - \mu z \Phi\left[z\left(\frac{\theta}{x}\right)^{1/2}\right] - \mu y \exp(2\theta/\mu)\Phi\left[-y\left(\frac{\theta}{x}\right)^{1/2}\right], \\
M(t) &= \exp\left[\frac{\theta}{\mu}\left(1 - \sqrt{1 - \frac{2\mu^2}{\theta}t}\right)\right], \quad t < \frac{\theta}{2\mu^2}, \\
\hat{\mu} &= m, \quad \hat{\theta} = \frac{m^3}{t - m^2}.
\end{aligned}$$

A.4.1.3 log-t—r, μ, σ **(μ can be negative)** Let Y have a t distribution with r degrees of freedom. Then $X = \exp(\sigma Y + \mu)$ has the log-t distribution. Positive moments do not exist for this distribution Just as the t distribution has a heavier tail than the normal distribution, this distribution has a heavier tail than the lognormal distribution.

$$\begin{aligned}
f(x) &= \frac{\Gamma\left(\frac{r+1}{2}\right)}{x\sigma\sqrt{\pi r}\,\Gamma\left(\frac{r}{2}\right)\left[1 + \frac{1}{r}\left(\frac{\ln x - \mu}{\sigma}\right)^2\right]^{(r+1)/2}}, \\
F(x) &= F_r\left(\frac{\ln x - \mu}{\sigma}\right) \text{ with } F_r(t) \text{ the cdf of a } t \text{ distribution with } r \text{ d f},
\end{aligned}$$

$$F(x) = \begin{cases} \frac{1}{2}\beta\left[\frac{r}{2}, \frac{1}{2}; \frac{r}{r + \left(\frac{\ln x - \mu}{\sigma}\right)^2}\right], & 0 < x \le e^{\mu}, \\ 1 - \frac{1}{2}\beta\left[\frac{r}{2}, \frac{1}{2}; \frac{r}{r + \left(\frac{\ln x - \mu}{\sigma}\right)^2}\right], & x \ge e^{\mu} \end{cases}$$

A.4.1.4 Single-parameter Pareto—$\boldsymbol{\alpha}, \boldsymbol{\theta}$

$$\begin{aligned} f(x) &= \frac{\alpha\theta^{\alpha}}{x^{\alpha+1}}, \quad x > \theta, \\ F(x) &= 1 - \left(\frac{\theta}{x}\right)^{\alpha}, \quad x > \theta, \end{aligned}$$

$$\begin{aligned} \mathrm{E}[X^k] &= \frac{\alpha\theta^k}{\alpha - k}, \quad k < \alpha, \\ \mathrm{E}[(X \wedge x)^k] &= \frac{\alpha\theta^k}{\alpha - k} - \frac{k\theta^{\alpha}}{(\alpha - k)x^{\alpha-k}}, \\ \text{Mode} &= \theta, \\ \hat{\alpha} &= \frac{m}{m - \theta}. \end{aligned}$$

Note: Although there appears to be two parameters, only α is a true parameter. The value of θ must be set in advance

A.5 DISTRIBUTIONS WITH FINITE SUPPORT

For these two distributions, the scale parameter θ is assumed known.

A.5.1.1 Generalized beta—$\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{\theta}, \boldsymbol{\tau}$

$$\begin{aligned} f(x) &= \frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)} u^a (1-u)^{b-1} \frac{\tau}{x}, \quad 0 < x < \theta, \quad u = (x/\theta)^{\tau}, \\ F(x) &= \beta(a, b; u), \\ \mathrm{E}[X^k] &= \frac{\theta^k \Gamma(a+b)\Gamma(a+k/\tau)}{\Gamma(a)\Gamma(a+b+k/\tau)}, \quad k > -a\tau, \\ \mathrm{E}[(X \wedge x)^k] &= \frac{\theta^k \Gamma(a+b)\Gamma(a+k/\tau)}{\Gamma(a)\Gamma(a+b+k/\tau)} \beta(a + k/\tau, b; u) + x^k[1 - \beta(a, b; u)]. \end{aligned}$$

A 5.1.2 beta—a, b, θ

$$
\begin{aligned}
f(x) &= \frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)} u^a (1-u)^{b-1} \frac{1}{x}, \quad 0 < x < \theta, \quad u = x/\theta, \\
F(x) &= \beta(a, b; u), \\
\mathrm{E}[X^k] &= \frac{\theta^k \Gamma(a+b)\Gamma(a+k)}{\Gamma(a)\Gamma(a+b+k)}, \quad k > -a, \\
\mathrm{E}[X^k] &= \frac{\theta^k a(a+1)\cdots(a+k-1)}{(a+b)(a+b+1)\cdots(a+b+k-1)} \quad \text{if } k \text{ is an integer}, \\
\mathrm{E}[(X \wedge x)^k] &= \frac{\theta^k a(a+1)\cdots(a+k-1)}{(a+b)(a+b+1)\cdots(a+b+k-1)} \beta(a+k, b; u) \\
&\quad + x^k[1 - \beta(a, b; u)], \\
\hat{a} &= \frac{\theta m^2 - mt}{\theta t - \theta m^2}, \quad \hat{b} = \frac{(\theta m - t)(\theta - m)}{\theta t - \theta m^2}.
\end{aligned}
$$

Appendix B
An inventory of discrete distributions

B.1 INTRODUCTION

The 16 models fall into three classes. The divisions are based on the algorithm by which the probabilities are computed. For some of the more familiar distributions these formulas will look different from the ones you may have learned, but they produce the same probabilities. After each name, the parameters are given. All parameters are positive unless otherwise indicated In all cases, p_k is the probability of observing k losses

For finding moments, the most convenient form is to give the factorial moments. The jth factorial moment is $\mu_{(j)} = \mathrm{E}[N(N-1)\cdots(N-j+1)]$. We have $\mathrm{E}[N] = \mu_{(1)}$ and $\mathrm{Var}(N) = \mu_{(2)} + \mu_{(1)} - \mu_{(1)}^2$.

The estimators which are presented are not intended to be useful estimators but rather for providing starting values for maximizing the likelihood (or other) function. For determining starting values, the following quantities are

Loss Models: From Data to Decisions, Second Edition.
By Stuart A. Klugman, Harry H. Panjer, and Gordon E. Willmot
ISBN 0-471-21577-5

used [where n_k is the observed frequency at k (if, for the last entry, n_k represents the number of observations at k or more, assume it was at exactly k) and n is the sample size]:

$$\hat{\mu} = \frac{1}{n}\sum_{k=1}^{\infty} kn_k, \quad \hat{\sigma}^2 = \frac{1}{n}\sum_{k=1}^{\infty} k^2 n_k - \hat{\mu}^2$$

When the method of moments is used to determine the starting value, a circumflex (e.g., $\hat{\lambda}$) is used. For any other method, a tilde (e.g., $\tilde{\lambda}$) is used. When the starting value formulas do not provide admissible parameter values, a truly crude guess is to set the product of all λ and β parameters equal to the sample mean and set all other parameters equal to 1. If there are two λ and/or β parameters, an easy choice is to set each to the square root of the sample mean.

The last item presented is the probability generating function,

$$P(z) = \mathrm{E}[z^N].$$

B.2 THE $(a, b, 0)$ CLASS

The distributions in this class have support on $0, 1, \ldots$. For this class, a particular distribution is specified by setting p_0 and then using $p_k = (a + b/k)p_{k-1}$. Specific members are created by setting p_0, a, and b. For any member, $\mu_{(1)} = (a+b)/(1-a)$, and for higher j, $\mu_{(j)} = (aj+b)\mu_{(j-1)}/(1-a)$. The variance is $(a+b)/(1-a)^2$.

B.2.1.1 *Poisson—λ*

$$\begin{aligned} p_0 &= e^{-\lambda}, \quad a = 0, \quad b = \lambda, \\ p_k &= \frac{e^{-\lambda}\lambda^k}{k!}, \\ \mathrm{E}[N] &= \lambda, \quad \mathrm{Var}[N] = \lambda, \\ \hat{\lambda} &= \hat{\mu}, \\ P(z) &= e^{\lambda(z-1)} \end{aligned}$$

B.2.1.2 *Geometric—β*

$$\begin{aligned} p_0 &= \frac{1}{1+\beta}, \quad a = \frac{\beta}{1+\beta}, \quad b = 0, \\ p_k &= \frac{\beta^k}{(1+\beta)^{k+1}}, \\ \mathrm{E}[N] &= \beta, \quad \mathrm{Var}[N] = \beta(1+\beta), \\ \hat{\beta} &= \hat{\mu}, \\ P(z) &= [1 - \beta(z-1)]^{-1} \end{aligned}$$

This is a special case of the negative binomial with $r = 1$.

B.2.1.3 *Binomial*—$q, m, (0 < q < 1, m$ **an integer**)

$$\begin{aligned} p_0 &= (1-q)^m, \quad a = -\frac{q}{1-q}, \quad b = \frac{(m+1)q}{1-q}, \\ p_k &= \binom{m}{k} q^k (1-q)^{m-k}, \quad k = 0, 1, \ldots, m, \\ \mathrm{E}[N] &= mq, \quad \mathrm{Var}[N] = mq(1-q), \\ \hat{q} &= \hat{\mu}/m, \\ P(z) &= [1 + q(z-1)]^m \end{aligned}$$

B.2.1.4 *Negative binomial*—β, r

$$\begin{aligned} p_0 &= (1+\beta)^{-r}, \quad a = \frac{\beta}{1+\beta}, \quad b = \frac{(r-1)\beta}{1+\beta}, \\ p_k &= \frac{r(r+1)\cdots(r+k-1)\beta^k}{k!(1+\beta)^{r+k}}, \\ \mathrm{E}[N] &= r\beta, \quad \mathrm{Var}[N] = r\beta(1+\beta), \\ \hat{\beta} &= \frac{\hat{\sigma}^2}{\hat{\mu}} - 1, \quad \hat{r} = \frac{\hat{\mu}^2}{\hat{\sigma}^2 - \hat{\mu}}, \\ P(z) &= [1 - \beta(z-1)]^{-r}. \end{aligned}$$

B.3 THE $(a, b, 1)$ CLASS

To distinguish this class from the $(a, b, 0)$ class, the probabilities are denoted $\Pr(N = k) = p_k^M$ or $\Pr(N = k) = p_k^T$ depending on which subclass is being represented. For this class, p_0^M is arbitrary (that is, it is a parameter) and then p_1^M or p_1^T is a specified function of the parameters a and b. Subsequent probabilities are obtained recursively as in the $(a, b, 0)$ class: $p_k^M = (a + b/k)p_{k-1}^M$, $k = 2, 3, \ldots$, with the same recursion for p_k^T. There are two subclasses of this class. When discussing their members, we often refer to the "corresponding" member of the $(a, b, 0)$ class. This refers to the member of that class with the same values for a and b. The notation p_k will continue to be used for probabilities for the corresponding $(a, b, 0)$ distribution.

B.3.1 The zero-truncated subclass

The members of this class have $p_0^T = 0$ and therefore it need not be estimated. These distributions should only be used when a value of zero is impossible. The first factorial moment is $\mu_{(1)} = (a+b)/[(1-a)(1-p_0)]$, where p_0 is the value for the corresponding member of the $(a, b, 0)$ class. For the logarithmic distribution (which has no corresponding member), $\mu_{(1)} = \beta/\ln(1+\beta)$. Higher factorial moments are obtained recursively with the same formula as with the $(a,b,0)$ class. The variance is $(a+b)[1-(a+b+1)p_0]/[(1-a)(1-p_0)]^2$. For those members of the subclass which have corresponding $(a,b,0)$ distributions, $p_k^T = p_k/(1-p_0)$.

B.3.1.1 Zero-truncated Poisson—$\boldsymbol{\lambda}$

$$\begin{aligned} p_1^T &= \frac{\lambda}{e^\lambda - 1}, \quad a = 0, \quad b = \lambda, \\ p_k^T &= \frac{\lambda^k}{k!(e^\lambda - 1)}, \\ \mathrm{E}[N] &= \lambda/(1-e^{-\lambda}), \quad \mathrm{Var}[N] = \lambda[1-(\lambda+1)e^{-\lambda}]/(1-e^{-\lambda})^2, \\ \tilde{\lambda} &= \ln(n\hat{\mu}/n_1), \\ P(z) &= \frac{e^{\lambda z} - 1}{e^\lambda - 1}. \end{aligned}$$

B.3.1.2 Zero-truncated geometric—$\boldsymbol{\beta}$

$$\begin{aligned} p_1^T &= \frac{1}{1+\beta}, \quad a = \frac{\beta}{1+\beta}, \quad b = 0, \\ p_k^T &= \frac{\beta^{k-1}}{(1+\beta)^k}, \\ \mathrm{E}[N] &= 1+\beta, \quad \mathrm{Var}[N] = \beta(1+\beta), \\ \hat{\beta} &= \hat{\mu} - 1, \\ P(z) &= \frac{[1-\beta(z-1)]^{-1} - (1+\beta)^{-1}}{1-(1+\beta)^{-1}}. \end{aligned}$$

This is a special case of the zero-truncated negative binomial with $r = 1$.

B.3.1.3 Logarithmic—$\boldsymbol{\beta}$

$$\begin{aligned} p_1^T &= \frac{\beta}{(1+\beta)\ln(1+\beta)}, \quad a = \frac{\beta}{1+\beta}, \quad b = -\frac{\beta}{1+\beta}, \\ p_k^T &= \frac{\beta^k}{k(1+\beta)^k \ln(1+\beta)}, \\ \mathrm{E}[N] &= \beta/\ln(1+\beta), \quad \mathrm{Var}[N] = \frac{\beta[1+\beta-\beta/\ln(1+\beta)]}{\ln(1+\beta)}, \\ \tilde{\beta} &= \frac{n\hat{\mu}}{n_1} - 1 \quad \text{or} \quad \frac{2(\hat{\mu}-1)}{\hat{\mu}}, \\ P(z) &= 1 - \frac{\ln[1-\beta(z-1)]}{\ln(1+\beta)}. \end{aligned}$$

This is a limiting case of the zero-truncated negative binomial as $r \to 0$.

B.3.1.4 Zero-truncated binomial—$q, m, (0 < q < 1,\ m$ *an integer*)

$$
\begin{aligned}
p_1^T &= \frac{m(1-q)^{m-1}q}{1-(1-q)^m}, \quad a = -\frac{q}{1-q}, \quad b = \frac{(m+1)q}{1-q}, \\
p_k^T &= \frac{\binom{m}{k}q^k(1-q)^{m-k}}{1-(1-q)^m}, \quad k = 1,2,\ldots,m, \\
\mathrm{E}[N] &= \frac{mq}{1-(1-q)^m}, \\
\mathrm{Var}[N] &= \frac{mq[(1-q)-(1-q+mq)(1-q)^m]}{[1-(1-q)^m]^2}, \\
\tilde{q} &= \frac{\hat{\mu}}{m}, \\
P(z) &= \frac{[1+q(z-1)]^m-(1-q)^m}{1-(1-q)^m}.
\end{aligned}
$$

B.3.1.5 Zero-truncated negative binomial—$\beta, r, (r > -1, r \neq 0)$

$$
\begin{aligned}
p_1^T &= \frac{r\beta}{(1+\beta)^{r+1}-(1+\beta)}, \quad a = \frac{\beta}{1+\beta}, \quad b = \frac{(r-1)\beta}{1+\beta}, \\
p_k^T &= \frac{r(r+1)\cdots(r+k-1)}{k![(1+\beta)^r-1]}\left(\frac{\beta}{1+\beta}\right)^k, \\
\mathrm{E}[N] &= \frac{r\beta}{1-(1+\beta)^{-r}}, \\
Var[N] &= \frac{r\beta[(1+\beta)-(1+\beta+r\beta)(1+\beta)^{-r}]}{[1-(1+\beta)^{-r}]^2}, \\
\tilde{\beta} &= \frac{\hat{\sigma}^2}{\hat{\mu}}-1, \quad \tilde{r} = \frac{\hat{\mu}^2}{\hat{\sigma}^2-\hat{\mu}}, \\
P(z) &= \frac{[1-\beta(z-1)]^{-r}-(1+\beta)^{-r}}{1-(1+\beta)^{-r}}.
\end{aligned}
$$

This distribution is sometimes called the extended truncated negative binomial distribution because the parameter r can extend below 0.

B.3.2 The zero-modified subclass

A zero-modified distribution is created by starting with a truncated distribution and then placing an arbitrary amount of probability at zero. This probability, p_0^M, is a parameter. The remaining probabilities are adjusted accordingly. Values of p_k^M can be determined from the corresponding zero-truncated distribution as $p_k^M = (1-p_0^M)p_k^T$ or from the corresponding $(a,b,0)$ distribution as $p_k^M = (1-p_0^M)p_k/(1-p_0)$. The same recursion used for the zero-truncated subclass applies.

The mean is $1 - p_0^M$ times the mean for the corresponding zero-truncated distribution. The variance is $1 - p_0^M$ times the zero-truncated variance plus $p_0^M(1-p_0^M)$ times the square of the zero-truncated mean. The probability generating function is $P^M(z) = p_0^M + (1-p_0^M)P(z)$, where $P(z)$ is the probability generating function for the corresponding zero-truncated distribution.

The maximum likelihood estimator of p_0^M is always the sample relative frequency at 0.

B.4 THE COMPOUND CLASS

Members of this class are obtained by compounding one distribution with another. That is, let N be a discrete distribution, called the **primary distribution** and let $M_1, M_2, \ldots$ be identically and independently distributed with another discrete distribution, called the **secondary distribution**. The compound distribution is $S = M_1 + \cdots + M_N$. The probabilities for the compound distributions are found from

$$p_k = \frac{1}{1 - af_0} \sum_{y=1}^{k} (a + by/k) f_y p_{k-y}$$

for $n = 1, 2, \ldots$, where a and b are the usual values for the primary distribution [which must be a member of the $(a, b, 0)$ class] and f_y is p_y for the secondary distribution. The only two primary distributions used here are Poisson (for which $p_0 = \exp[-\lambda(1 - f_0)]$) and geometric [for which $p_0 = 1/[1 + \beta - \beta f_0]]$. Because this information completely describes these distributions, only the names and starting values are given below.

The moments can be found from the moments of the individual distributions:

$$\mathrm{E}[S] = \mathrm{E}[N]\mathrm{E}[M] \quad \text{and} \quad \mathrm{Var}[S] = \mathrm{E}[N]\,\mathrm{Var}[M] + \mathrm{Var}[N]\mathrm{E}[M]^2.$$

The probability generating function is $P(z) = P_{\text{primary}}[P_{\text{secondary}}(z)]$.

In the following list the primary distribution is always named first. For the first, second, and fourth distributions, the secondary distribution is the $(a, b, 0)$ class member with that name. For the third and the last three distributions (the Poisson–ETNB and its two special cases) the secondary distribution is the zero-truncated version.

B.4.1 Some compound distributions

B.4.1.1 *Poisson–binomial*—$\lambda, q, m (0 < q < 1,\ m$ an integer)

$$\hat{q} = \frac{\hat{\sigma}^2/\hat{\mu} - 1}{m - 1}, \quad \hat{\lambda} = \frac{\hat{\mu}}{m\hat{q}} \quad \text{or} \quad \tilde{q} = 0.5,\ \tilde{\lambda} = \frac{2\hat{\mu}}{m}.$$