Multiple-solution problems in an actuarial science classroom: an example

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Introduction

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- We define multiple-solution problems (MSPs) as mathematical problems that can be approached using different tools or strategies from the same area or different branches of mathematics.
- Encouraging students to develop multiple solutions for given problems has a positive effect on students' understanding and creativity.

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- By solving this problem in different ways, it helps students to connect their knowledge learned from courses in other related disciplines within the SOA exam curriculum.

100 Calculus and Linear Algebra
110 Probability and Statistics
120 Applied Statistical Methods
130 Operations Research
135 Numerical Methods
151 Risk Theory

Mang Kung dice

- It consists of six special cubic dice, with only one side of each die marked according to the Chinese Mang Kung (blind men) way:

Mang Kung dice

- Students are asked to find the probability mass function (PMF) for the random variable S (total face value); that is,

$$Pr(S = s).$$

- The possible values for s are non-negative integers ranging from 0 to 21 = (⚀ + ⚁ + ⚂ + ⚃ + ⚄ + ⚅).
- For the purpose of deriving the exact distribution, we assume that the dice are fair.

[1] The Sample Space Approach

- Most students who have already passed SOA EXAM P should be able to get the answer by counting the sample space:

Table 1. Possible outcomes and PMF of S.

s	Combinations					$Pr(S = s)$
0	(0)					$\frac{15,625}{46,656}$
1	(1)					$\frac{3125}{46,656}$
2	(2)					$\frac{3125}{46,656}$
3	(3)	(1,2)				$\frac{3750}{46,656}$
4	(4)	(1,3)				$\frac{3750}{46,656}$
5	(5)	(1,4)	(2,3)			$\frac{4375}{46,656}$
6	(6)	(1,5)	(2,4)	(1,2,3)		$\frac{4500}{46,656}$
7	(1,6)	(2,5)	(3,4)	(1,2,4)		$\frac{2000}{46,656}$
8	(2,6)	(3,5)	(1,2,5)	(1,3,4)		$\frac{1500}{46,656}$
9	(3,6)	(4,5)	(1,3,5)	(1,2,6)	(2,3,4)	$\frac{1625}{46,656}$
10	(4,6)	(1,3,6)	(1,4,5)	(2,3,5)	(1,2,3,4)	$\frac{1025}{46,656}$
11	(5,6)	(1,4,6)	(2,3,6)	(2,4,5)	(1,2,3,5)	$\frac{1025}{46,656}$
12	(1,5,6)	(2,4,6)	(3,4,5)	(1,2,3,6)	(1,2,4,5)	$\frac{425}{46,656}$
13	(2,5,6)	(3,4,6)	(1,3,4,5)	(1,2,4,6)		$\frac{300}{46,656}$
14	(3,5,6)	(1,2,5,6)	(1,3,4,6)	(2,3,4,5)		$\frac{200}{46,656}$
15	(4,5,6)	(1,3,5,6)	(2,3,4,6)	(1,2,3,4,5)		$\frac{180}{46,656}$
16	(1,4,5,6)	(2,3,5,6)	(1,2,3,4,6)			$\frac{55}{46,656}$
17	(2,4,5,6)	(1,2,3,5,6)				$\frac{30}{46,656}$
18	(3,4,5,6)	(1,2,4,5,6)				$\frac{30}{46,656}$
19	(1,3,4,5,6)					$\frac{5}{46,656}$
20	(2,3,4,5,6)					$\frac{5}{46,656}$
21	(1,2,3,4,5,6)					$\frac{1}{46,656}$

[2] The Integer Partition Approach

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- If students have taken an elementary course in number theory. It is not difficult for them to find out that the solution is related to the integer partition problem in number theory.

[2] The Integer Partition Approach

- For example, in Table 1, the set of possible outcomes that leads to the event $(S = 6)$ is $\{(6), (1,5), (2,4), (1,2,3)\}$. Hence,

$$
\begin{aligned}
Pr(S = 6) = {} & 1 \times \left(\frac{1}{6}\right)^1 \times \left(\frac{5}{6}\right)^5 + \\
& 2 \times \left(\frac{1}{6}\right)^2 \times \left(\frac{5}{6}\right)^4 + \\
& 1 \times \left(\frac{1}{6}\right)^3 \times \left(\frac{5}{6}\right)^3 \\
= {} & \frac{4{,}500}{46{,}656}.
\end{aligned}
$$

[2] The Integer Partition Approach

- In general,

$$Pr(S = s) = \sum_{r=0}^{6} p_{\mathcal{C}}(r, s) \left(\frac{1}{6}\right)^r \left(\frac{5}{6}\right)^{6-r},$$

for $s = 0, 1, \ldots, 21$; where $p_{\mathcal{C}}(r, s)$ denotes the number of partitions of a non-negative integer $s \leq 21$ that has exactly r parts, and those parts are distinct elements from the set $\mathcal{C} = \{1, 2, \ldots, 6\}$.

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- The values of $p_{\mathcal{C}}(r, s)$ can be obtained via the standard bivariate generating function in number theory of partitions.

[3] The Probability Generating Function Approach

- The probability generating function (PGF) g of a distribution S on the non-negative integers is defined by

$$\begin{aligned} g_S(t) &= E[t^S] \\ &= Pr(S = 0) + t\, Pr(S = 1) + t^2\, Pr(S = 2) + \cdots \end{aligned}$$

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- Note that

$$S = D_1 + D_2 + \cdots + D_6$$

with

$$Pr(D_i = i) = \frac{1}{6} \text{ and } Pr(D_i = 0) = \frac{5}{6}$$

[3] The Probability Generating Function Approach

- Hence,

$$\begin{aligned} g_S(t) = E[t^S] &= E[t^{(D_1+D_2+\cdots+D_6)}] \\ &= \prod_{i=1}^{6}(\frac{5}{6} + \frac{1}{6}t^i). \end{aligned}$$

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$$= \prod_{i=1}^{6}(\frac{5}{6} + \frac{1}{6}t^i).$$

- Students with knowledge in SOA VEE Mathematical Statistics should have learned that

$$Pr(S = s) = g_S^{(k)}(0)/k!$$

where $g_S^{(k)}(t)$ is the *kth* derivative of $g_S(t)$ with respect to t.

[4] The Individual Risk Model Approach

- The aggregate loss model is an important topic of SOA EXAM C:

Construction and Evaluation of Actuarial Models Exam—February 2018

C. Aggregate Models
1. Compute relevant parameters and statistics for collective risk models.
2. Evaluate compound models for aggregate claims.
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- The aggregate loss random variable:

$$S = X_1 + X_2 + \cdots + X_N$$

where $S = 0$ when $N = 0$

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- The collective risk model assumes that
  - The frequency distribution of N does not depend on values of $X_1, X_2, \ldots$
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- The individual risk model assumes that
  - n is fixed as the number of insurance contracts
  - $X_j's$ are independent but not necessary identically distributed
  - $X_j's$ has a probability mass at zero (i.e., no loss)

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- In particular, we assume there are 6 independent insurance policies with an insured amount of $b_i = i$ units and a claim probability of $\theta_i = \frac{1}{6}$ for $i = 1, 2, \ldots, 6$.

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- Interestingly, the Mang Kung Dice problem can be solved using the individual risk model in actuarial science.
- In particular, we assume there are 6 independent insurance policies with an insured amount of $b_i = i$ units and a claim probability of $\theta_i = \frac{1}{6}$ for $i = 1, 2, \ldots, 6$.
- The PMF for S can be computed using the following equation, known as the De Pril recursive formula in actuarial science:

$$Pr(S = s) = \frac{1}{s} \sum_{i=1}^{\min(6,s)} \sum_{k=1}^{[s/i]} h(i,k)\, Pr(S = s - ik),$$

for $s = 1, 2, \ldots, 21$.

Mang Kung dice - a generalization

- We may generalise the ordinary six-faced, six-die set of Mang Kung Dice to the n-faced, n-die situation.

Die/Face	1	2	3	4	$\cdots$	n
1	⚀	☐	☐	☐	$\cdots$	☐
2	☐	⚁	☐	☐	$\cdots$	☐
3	☐	☐	⚂	☐	$\cdots$	☐
4	☐	☐	☐	⚃	$\cdots$	☐
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$	☐
n	☐	☐	☐	☐	$\cdots$	n

Some *n*-polyhedral dice

- 12-faced

Some *n*-polyhedral dice

- 24-faced

Some *n*-polyhedral dice

- 30-faced

Some *n*-polyhedral dice

- 60-faced

Some *n*-polyhedral dice

- 120-faced

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- We define the discrete random variable S_n as the total face value of throwing the set of *n*-polyhedral fair Mang Kung dice.
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- We define the discrete random variable S_n as the total face value of throwing the set of *n*-polyhedral fair Mang Kung dice.
- The possible outcomes of S_n are non-negative integers ranging from 0 to $A_n = (n)(n+1)/2$.
- Students (for upper undergraduate or graduate students) are asked to derive the PMF for S_n in different ways.

Mang Kung dice - a generalization

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Check for updates

Multiple-solution problems in a statistics classroom: an example

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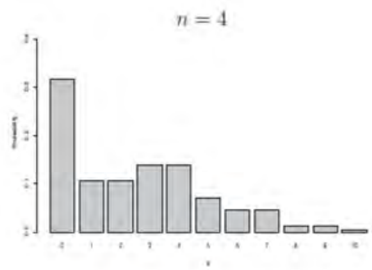
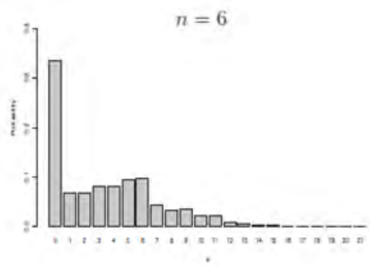
Mang Kung dice - a generalization





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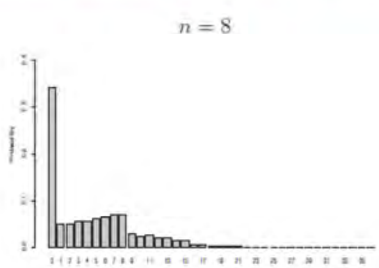

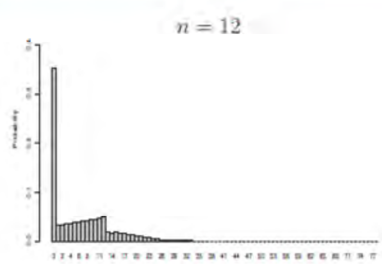

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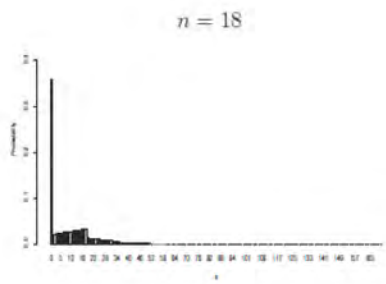

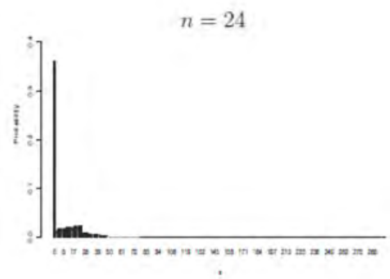

Mang Kung dice - a further generalization

- How about if $n \to \infty$?

Mang Kung dice - a further generalization

- How about if $n \to \infty$?
- One student of mine provided a proof that PMF of S_n is asymtotically equivalent to a compound poisson distribution.

A Final Remark

- The solutions to the MSP example described in this talk are by no means exhaustive – indeed, we welcome audience to share their alternative solutions with us.

THANK YOU!