QFI CORE Model Solutions
Spring 2018

1. Learning Objectives:

1. The candidate will understand the fundamentals of stochastic calculus as they apply to option pricing.

Learning Outcomes:

(1c) Understand Ito integral and stochastic differential equations.

(1d) Understand and apply Ito's Lemma.

(1i) Understand and apply Girsanov's theorem in changing measures.

Sources:

Neftci, Chapter 15 exercises (2nd edition, pg 367)

Commentary on Question:

Commentary listed underneath question component.

Solution:

(a) Calculate $Prob(W_1 + W_2 > 1)$.

Commentary on Question:

Candidates under-performed on this part.
Transforming W_1+W_2 to $(W_2 - W_1) + 2W_1$ was missed by half of the candidates.

$$E[W_1 + W_2] = E[W_1] + E[W_2] = 0 + 0 = 0$$

$$Var(W_1 + W_2) = Var(W_2 - W_1) + Var(2W_1) = (2 - 1) + 2^2 * 1 = 5$$

$$And\ thus\ Z = (W_1 + W_2) \sim N(0,5)$$

So $Prob(W_1 + W_2 > 1) = 1 - \Phi\left(\frac{1}{\sqrt{5}}\right) \approx 1 - 0.6736 = 0.3264$

(b) Solve the SDE for S_t.

Commentary on Question:

Candidates performed well on this part. Most candidates were able to successfully use Ito's Lemma to derive the SDE.

1. Continued

Set $Y_t = \ln(S_t)$

$\frac{\partial Y}{\partial t} = 0$, $\frac{\partial Y}{\partial S_t} = \frac{1}{S_t}$, $\frac{\partial^2 Y}{\partial S_t^2} = \frac{-1}{S_t^2}$

Applying Ito's Lemma on Y_t

$$dY_t = dln(S_t) = \left(r - f - \frac{\sigma^2}{2}\right)dt + \sigma dW_t$$

Taking integral we find

$$\ln(S_t) - \ln(S_0) = \left(r - f - \frac{\sigma^2}{2}\right)t + \sigma W_t$$

Thus

$$S_t = S_0 e^{\left(r-f-\frac{\sigma^2}{2}\right)t+\sigma W_t}$$

(c) Show that

(i) $dZ_t = Z_t\left[\left(f - r + \sigma^2\right)dt - \sigma dW_t\right].$

(ii) $Z_t e^{(r-f)t}$ is not a martingale under the measure $\mathbb{P}$.

Commentary on Question:
Candidates performed well on this part, especially item (i). Item (ii) was most commonly solved by candidates using Ito's Lemma. An alternate solution using expectation was also acceptable.

(i)

Let $Z_t = \frac{1}{S_t} = \frac{1}{S_0} e^{-(r-f-0.5\sigma^2)t-\sigma W_t}$.
Applying Ito's lemma on Z_t as a function of W_t and t
$dZ_t = Z_t[(f - r + \sigma^2)dt - \sigma dW_t\ ]$ as desired.

1. Continued

Alternative solution using Ito's Lemma with Z_t as a function of S_t:

$$dZ_t = -\frac{dS_t}{S_t^2} + \frac{2}{S_t^3}\frac{(\sigma S_t)^2}{2}dt = -\frac{dS_t}{S_t^2} + \frac{2}{S_t^3}\frac{(\sigma S_t)^2}{2}dt$$
$$= -\frac{(r-f)S_t dt + \sigma S_t dW_t}{S_t^2} + \frac{\sigma^2}{S_t}dt$$
$$= Z_t[(f - r + \sigma^2)dt - \sigma dW_t]$$

(ii)

From Ito's lemma

$$d\left(Z_t e^{(r-f)t}\right) = e^{(r-f)t}(dZ_t + Z_t(r-f)dt)$$

Plugging the result of part (c)(i)

$$d\left(Z_t e^{(r-f)t}\right) = Z_t e^{(r-f)t}(\sigma^2 dt - \sigma dW_t)$$

Since the drift term $e^{(r-f)t}Z_t\sigma^2 dt$ does not vanish, $Z_t e^{(r-f)t}$ is not a martingale.

Alternative solution using Expectation:

Now

$$E[Z_{t+s}|I_t] = \frac{1}{S_0}e^{-(r-f-0.5\sigma^2)(t+s)-\sigma W_t}E\left[e^{-\sigma(W_{t+s}-W_t)}\right]$$
$$= \frac{1}{S_0}e^{-(r-f-0.5\sigma^2)(t+s)-\sigma W_t}e^{0.5\sigma^2 s}$$

so

$$E\left[Z_{t+s}e^{(r-f)(t+s)}\middle|I_t\right] = \frac{1}{S_0}e^{-(r-f-0.5\sigma^2)(t+s)-\sigma W_t+0.5\sigma^2 s+(r-f)(t+s)}$$
$$= \frac{1}{S_0}e^{0.5\sigma^2(t+2s)-\sigma W_t} = Z_t e^{s\sigma^2+(r-f)t} \neq Z_t e^{(r-f)t}$$

(d) Construct a measure $\widetilde{\mathbb{P}}$ such that $Z_t e^{(r-f)t}$ is a martingale under $\widetilde{\mathbb{P}}$.

Commentary on Question:
Candidates did well on this part. Most candidates correctly applied Girsanov's Theorem. Alternate solution using expectation is also acceptable.

Let $X_u = \sigma$.
By Girsanov's Theorem, $\widetilde{W}_t = W_t - \int_0^t X_u du = W_t - \sigma t$ is a martingale under $\tilde{P}$, where $\tilde{P}(A) = \int_A \gamma(W_T)\, dP$.

1. Continued

$d\widetilde{W}_t = dW_t - \sigma dt$

From part c(ii):

$d\left(Z_t e^{(r-f)t}\right) = Z_t e^{(r-f)t}(\sigma^2 dt - \sigma dW_t)$

Substituting $d\widetilde{W}_t = dW_t - \sigma dt$ gives

$d\left(Z_t e^{(r-f)t}\right) = Z_t e^{(r-f)t}\left(\sigma d\widetilde{W}_t\right)$, which as no drift, and as a result is a martingale under $\tilde{P}$.

Alternate solution

Under $\tilde{P}$, $W_t = \widetilde{W}_t + \sigma t$
so $\Delta W_t = \Delta\widetilde{W}_t + \sigma s$ where $\Delta W_t = W_{t+s} - W_t$
Hence

$$E^{\tilde{P}}\left[e^{-\sigma\Delta W_t}\right] = E^{\tilde{P}}\left[e^{-\sigma(\Delta\widetilde{W}_t + \sigma s)}\right] = e^{-\sigma^2 s + 0.5\sigma^2 s}$$

so

$$\begin{aligned} E^{\tilde{P}}[Z_{t+s}|I_t] &= \frac{1}{S_0} e^{-\left(r-f-0.5\sigma^2\right)(t+s)-\sigma W_t} E^{\tilde{P}}\left[e^{-\sigma\Delta W_t}\right] \\ &= \frac{1}{S_0} e^{-\left(r-f-0.5\sigma^2\right)(t+s)-\sigma W_t} e^{-\sigma^2 s + 0.5\sigma^2 s}. \end{aligned}$$

Hence

$E^{\tilde{P}}\left[Z_{t+s} e^{(r-f)(t+s)}|I_t\right] = \frac{1}{S_0} e^{0.5\sigma^2 t - \sigma W_t} = Z_t e^{(r-f)t}$.

2. Learning Objectives:

1. The candidate will understand the fundamentals of stochastic calculus as they apply to option pricing.

2. The candidate will understand how to apply the fundamental theory underlying the standard models for pricing financial derivatives. The candidate will understand the implications for option pricing when markets do not satisfy the common assumptions used in option pricing theory such as market completeness, bounded variation, perfect liquidity, etc. The Candidate will understand how to evaluate situations associated with derivatives and hedging activities.

Learning Outcomes:

(1b) Understand the importance of the no-arbitrage condition in asset pricing.

(1d) Understand and apply Ito's Lemma.

(1j) Understand the Black Scholes Merton PDE (partial differential equation).

(2e) Understand how to delta hedge and the interplay between hedging assumptions and hedging outcomes.

Commentary on Question:
Commentary listed underneath question component.

Solution:

(a) Describe the assumptions of the Black-Scholes model.

Commentary on Question:
Most candidate couldn't include all 4 suggested assumptions.

- The underlying follows a lognormal random walk.
- Hedging can be done continuously.
- There are no transaction costs.
- There are no arbitrage opportunities.

(b) Determine c.

Commentary on Question:
Straightforward question, most candidates were able to apply the formula correctly. Some candidates had trouble solving the quadratic equation at the end.

The price of any contingent claim on a nondividend-paying stock must satisfy the Black-Scholes Equation

$$V_t + rSV_S + 0.5\sigma^2 S^2 V_{SS} = rV.$$

(in the forumla sheet page 14 (6.6))

2. Continued

Since $F_t = 0$, $F_S = -\frac{c}{\sigma^2} S^{-\frac{c}{\sigma^2}-1}$, and $F_{SS} = \frac{c}{\sigma^2}\left(\frac{c}{\sigma^2}+1\right) S^{-\frac{c}{\sigma^2}-2}$
we have

$$rS\left(-\frac{c}{\sigma^2} S^{-\frac{c}{\sigma^2}-1}\right) + 0.5\sigma^2 S^2 \frac{c}{\sigma^2}\left(\frac{c}{\sigma^2}+1\right) S^{-\frac{c}{\sigma^2}-2} = rS^{-\frac{c}{\sigma^2}}$$
$$c^2 + (\sigma^2 - 2r)c - 2r\sigma^2 = 0$$
$$(c-2r)(c+\sigma^2) = 0$$
$$c = 2r \text{ or } -\sigma^2,$$

Since c is a positive constant, we have $c = 2r$.

(c) Show that $V(t,F)$ satisfies the following partial differential equation.

$$\frac{\partial V(t,F)}{\partial t} + rF\frac{\partial V(t,F)}{\partial t} + \frac{2r^2F^2}{\sigma^2}\frac{\partial^2 V(t,F)}{\partial F^2} - rV(t,F) = 0$$

Commentary on Question:
Recognizing the drift term of risk free process to be r is the key to the proof.
Partial credit were given to candidates for correct application of $d\Pi = r\Pi dt$

Consider a self-financing portfolio of Δ_1 long derivative position and a short position in some quantity Δ_2 of the power contract.
The change of portfolio value from time t to $t + dt$ is

$$d\Pi = \Delta_1 dV - \Delta_2 dF.$$

Using Ito's lemma, we have

$$dV = \frac{\partial V}{\partial t}dt + \frac{\partial V}{\partial F}dF + \frac{1}{2}\frac{\partial^2 V}{\partial F^2}(dF)^2.$$
$$dF = F_t dt + F_S dS + 0.5F_{SS}(dS)^2$$
$$= -\frac{c}{\sigma^2} S^{-\frac{c}{\sigma^2}-1}\big(\mu S dt + \sigma S dW(t)\big) + 0.5\frac{c}{\sigma^2}\left(\frac{c}{\sigma^2}+1\right) S^{-\frac{c}{\sigma^2}-2}\sigma^2 S^2 dt$$
$$(dF)^2 = \left(-\frac{c}{\sigma^2} S^{-\frac{c}{\sigma^2}-1}\right)^2 \sigma^2 S^2 \big(dW(t)\big)^2$$
$$(dF)^2 = \frac{c^2}{\sigma^2}F^2 dt$$
$$d\Pi = \Delta_1\left(\frac{\partial V}{\partial t}dt + \frac{\partial V}{\partial F}dF + \frac{1}{2}\frac{\partial^2 V}{\partial F^2}\frac{c^2}{\sigma^2}F^2 dt\right) - \Delta_2 dF$$
$$= \Delta_1\left(\frac{\partial V}{\partial t} + \frac{1}{2}\frac{\partial^2 V}{\partial F^2}\frac{c^2}{\sigma^2}F^2\right)dt + \left(\Delta_1\frac{\partial V}{\partial F} - \Delta_2\right)dF$$

To eliminate the risk in this portfolio, we should set $\frac{\Delta_2}{\Delta_1} = \frac{\partial V}{\partial F}$.
Since the portfolio is risk-free, $d\Pi = r\Pi dt$. Therefore,

2. Continued

$$\Delta_1\left(\frac{\partial V}{\partial t}+\frac{1}{2}\frac{\partial^2 V}{\partial F^2}\frac{c^2}{\sigma^2}F^2\right)=r\Pi=r\Delta_1\left(V-\frac{\partial V}{\partial F}F\right)$$

It follows that $\frac{\partial V}{\partial t}+rF\frac{\partial V}{\partial F}+\frac{c^2F^2}{2\sigma^2}\frac{\partial^2 V}{\partial F^2}-rV=0$ Since $c=2r$,

$$\frac{\partial V}{\partial t}+rF\frac{\partial V}{\partial F}+\frac{2r^2F^2}{\sigma^2}\frac{\partial^2 V}{\partial F^2}-rV=0$$

Alternative solution: Use Willmott (6.6) and then use

$\frac{\partial V}{\partial S}=\frac{\partial V}{\partial F}.\frac{\partial F}{\partial S}$ and

$\frac{\partial^2 V}{\partial S^2}=\frac{\partial}{\partial S}\left(\frac{\partial V}{\partial F}.\frac{\partial F}{\partial S}\right)=\frac{\partial^2 V}{\partial F\partial S}.\frac{\partial F}{\partial S}+\frac{\partial V}{\partial F}.\frac{\partial^2 F}{\partial S^2}=\ \frac{\partial^2 V}{\partial S^2}.\left(\frac{\partial F}{\partial S}\right)^2+\frac{\partial V}{\partial F}.\frac{\partial^2 F}{\partial S^2}$ and

Then substitute values of $\frac{\partial F}{\partial S}$ and $\frac{\partial^2 V}{\partial F^2}$ from part (b).

(d) Show, based on part (c), that the time-t value of a K-strike European put written on the power contract is

$$V(t,F)=Ke^{-r(T-t)}N(-d_2)-FN(-d_1),$$

where $d_1=\dfrac{\ln\left(\frac{F}{K}\right)+\left(r+\frac{2r^2}{\sigma^2}\right)(T-t)}{\frac{2r}{\sigma}\sqrt{T-t}}$, and $d_2=d_1-\frac{2r}{\sigma}\sqrt{T-t}$.

(Hint: Use the fact that the time-t value of a put option on the stock is

$$P(t,S)=Ke^{-r(T-t)}N(-d_2^*)-SN(-d_1^*),$$

where $d_1^*=\dfrac{\ln\left(\frac{S}{K}\right)+\left(r+\frac{\sigma^2}{2}\right)(T-t)}{\sigma\sqrt{T-t}}$, and $d_2^*=d_1^*-\sigma\sqrt{T-t}$.)

Commentary on Question:

Based on the hint, most candidates realized the need to prove stock volatility is $\frac{2r}{\sigma}$, *however many candidates failed to offer steps to prove it.*

The put option price satisfies the following PDE

$$\frac{\partial V(t,F)}{\partial t}+rF\frac{\partial V(t,F)}{\partial F}+\frac{1}{2}\left(\frac{c}{\sigma}\right)^2\frac{\partial^2 V}{\partial F^2}-rV=0,$$

with the boundaries conditions

$$V(t,0)=Ke^{-r(T-t)},\text{ for } F=0$$
$$V(t,F)\to 0,\text{ for } F\to\infty$$

2. **Continued**

$$V(T,F) = \max\,(K - F, 0)$$

This PDE is analogous to the PDE for the price of put option on stock

$$\frac{\partial P(t,S)}{\partial t} + rS\frac{\partial P(t,S)}{\partial S} + \frac{\sigma^2 S^2}{2}\frac{\partial^2 P(t,S)}{\partial S^2} - rV(t,S) = 0,$$

$$P(t,0) = Ke^{-r(T-t)}, \text{ for } S = 0$$

$$P(t,S) \to 0, \text{ for } S \to \infty$$

$$P(T,S) = \max\,(K - S, 0)$$

with the stock volatility σ replaced by $\frac{k}{\sigma}$.

Therefore, analogous to the PDE solution for put option on stock, we have the value of put option on power contract is

$$P(t,F) = Ke^{-r(T-t)}N(-d_2) - FN(-d_1),$$

$$\text{where } d_1 = \frac{\ln\left(\frac{F}{K}\right) + \left(r + \frac{k^2}{2\sigma^2}\right)(T-t)}{\frac{2r}{\sigma}\sqrt{T-t}} = \frac{\ln\left(\frac{F}{K}\right) + \left(r + \frac{2r^2}{\sigma^2}\right)(T-t)}{\frac{2r}{\sigma}\sqrt{T-t}},$$

$$\text{and } d_2 = d_1 - \frac{2r}{\sigma}\sqrt{T-t}.$$

3. Learning Objectives:

1. The candidate will understand the fundamentals of stochastic calculus as they apply to option pricing.

Learning Outcomes:

(1b) Understand the importance of the no-arbitrage condition in asset pricing.

(1c) Understand Ito integral and stochastic differential equations.

(1d) Understand and apply Ito's Lemma.

(1f) Demonstrate understanding of option pricing techniques and theory for equity and interest rate derivatives.

Sources:

Wilmott Ch 6; Neftci Ch 10; Chin, Nell, Olafsson, QFIC-113-17

Commentary on Question:

The question tests the candidates' understanding and application of the stochastic calculus to evaluate the financial instruments. Overall most candidates demonstrated their understanding of the Ito's Lemma. But it became more challenging when candidates were asked to utilize the knowledge to translate it into real-world applications.

Solution:

(a) Derive the SDE satisfied by the process $X_t = S_t^2$.

Commentary on Question:

Candidates generally did well in part (a) by applying the Ito's Lemma correctly. A few candidates failed to rearrange the equation properly after applying Ito's Lemma and got partial scores.

From Ito's Lemma:

$$dX_t = \frac{\partial X_t}{\partial t} dt + \frac{\partial X_t}{\partial S_t} dS_t + \frac{1}{2}\frac{\partial^2 X}{\partial S_t^2}\sigma^2 S_t^2 dt$$

$$\frac{\partial X_t}{\partial t} dt = 0$$

$$\frac{\partial X}{\partial S_t} = 2S_t$$

$$\frac{\partial^2 X_t}{\partial S_t^2} = 2$$

$$dX_t = (2S_t)(rS_t\, dt + \sigma\, S_t\, dW_t) + \frac{1}{2}\frac{\partial^2 X_t}{\partial S_t^2}\sigma^2 S_t^2 dt$$

$$dX_t = X_t(2r + \sigma^2)dt + 2\sigma X_t\ dW_t$$

3. Continued

(b) Derive the following:

(i) The expression for the time-t value f_t in terms of S_t,

(ii) The delta of the contract.

Commentary on Question:
Most candidates were able to make use of the mean and the variance of the lognormal distribution. Common mistake was the confusion in applying the 1/2 to the volatility of 2σ *(used in the adjustment of the drift term), and the 1/2 applied to the V (as part of the mean/first moment of a lognormal distribution) in later steps.*
Candidates got full marks for (ii) by taking derivative with respect to S even if the answer in (i) was incorrect.

(i)

Based on part (a), we know that

$$\ln(X_T) - \ln(X_t) \sim \text{Normal}\left(\left((2r + \sigma^2) - \frac{1}{2}(2\sigma)^2\right)(T-t), (2\sigma)^2(T-t)\right)$$

Let's denote M = $((2r + \sigma^2) - \frac{1}{2}(2\sigma)^2)(T-t)$ and V = $(2\sigma)^2(T-t)$,

Therefore,

$$E_t[S_T^2] = E_t[X_T] = X_t e^{M + \frac{1}{2}V} = S_t^2\, e^{(2r + \sigma^2)\,(T-t)}$$

Note: The above identity may be derived using other approaches like PDE or martingale.

The value of the option is then:

$$f_t = E_t\left[exp^{-r(T-t)}\, S_T^2\right] = S_t^2\, e^{(r + \sigma^2)\,(T-t)}$$

(ii) The delta of the option is

$$\frac{\partial f_t}{\partial S_t} = 2S_t e^{(r + \sigma^2)\,(T-t)}$$

3. Continued

(c) Derive the following:

(i) $g(K)$ for all $K \in [0,\infty)$,

(ii) The expression for f_t in terms of $c_t(K,T)$, assuming that no arbitrage exists.

Commentary on Question:
Candidates generally did poorly in part (c). For those who attempted, most candidates were able to transform the max() function into the integral format over the proper domain. One common mistake was the failure to take the derivative with respect to S correctly.

(i)

$$f(S_T) = \int_0^{\infty} g(K)\,\max(S_T - K, 0)dK$$

$$f(S_T) = \int_0^{S_T} g(K) \times (S_T - K)dK = \int_0^{S_T} g(K) \times S_T\,dK - \int_0^{S_T} g(K) \times KdK$$

Thus, taking derivative with respect to S_T from both sides:

$$\frac{d}{dS_T} f_T = \left(\int_0^{S} \frac{d}{dS_T}(g(K) \times S_T)\,dK + g(S_T) \times S_T \times \frac{dS_T}{dS_T}\right) - \left(g(S_T) \times S_T \times \frac{dS_T}{dS_T}\right)$$

$$\frac{d}{dS_T} f_T = \int_0^{S_T} g(K)\,dK$$

Taking derivative again:

$$\frac{d^2}{d{S_T}^2} f_t = g(S_T)$$

$$g(S_T) = \frac{d^2}{d{S_T}^2} f_T = 2$$

That is,

$$g(K) = 2$$

(ii)
Since no arbitrage exists, if two instruments have identical payoffs at the same points in time, they must have the same prices at all times.
Since

$$f(S_T) = \int_0^\infty 2 \max(S_T - K, 0) dK$$

we have

$$f_t = 2 \int_0^\infty c_t(K, T)\, dK$$

(d) Compare and contrast the hedging strategies that can be developed using the results obtained in (b) and (c).

Commentary on Question:
Most candidates were able to identify the delta hedge strategy, with fewer able to identify the static hedge strategy. Yet very few were able to identify the difference in the model-dependency between the two strategies.

Using results in (c), we can develop model-independent strategies, which does not have to specify the dynamics of the underlying asset, not even its volatility, to find the possible hedge; while
Using results in (b) (which relies on GBM), we typically need to at least know the volatility of the underlying asset. If our model is wrong, then the option value and any hedging strategy will also be wrong. Thus, we are exposed to model risk.

(c) provides portfolio replication static hedging; while
(b) relies on continuous (or dynamic or delta) hedging per the Black Scholes PDE

4. Learning Objectives:

1. The candidate will understand the fundamentals of stochastic calculus as they apply to option pricing.

Learning Outcomes:

(1a) Understand and apply concepts of probability and statistics important in mathematical finance.

(1d) Understand and apply Ito's Lemma.

(1h) Define and apply the concepts of martingale, market price of risk and measures in single and multiple state variable contexts.

Sources:

Nefci Ch.6 Page 88-89, 103

Commentary on Question:

In general part (a) and part (c) were easy and most candidates got full mark. For part (b), most candidates were able to use Ito's Lemma to prove one condition for a martingale but some candidates failed to justify the other two conditions. For part (d), most candidates didn't answer it well.

Solution:

(a) State the conditions for a process to be a martingale.

Commentary on Question:

This was the easiest part of the question, and most candidates got full mark.

A process $\{S_t, t \in [0, \infty)\}$ is a martingale with respect to the family of information sets I_t and with respect to the probablity P, if, for all $t > 0$,

1. S_t is known, given I_t
2. Unconditional "forecasts" are finite: $E[|S_t|] < \infty$
3. $E_t[S_T] = S_t$ for all $t < T$

(b) Show that each of the following processes is a martingale by using Ito's formula to verify that it satisfies all the conditions in part (a).

(i) $W_t^3 - 3tW_t$,

(ii) $e^{\lambda W_t - 0.5\lambda^2 t}$.

Commentary on Question:

Most candidates used Ito's Lemma to prove the conditional expectation or alternatively proved the no drift condition. Some of them failed to justify unconditional finite condition, thus didn't get full mark.

4. Continued

Condition 1 for all three is obvious given the definition of Wiener process.

i) $W_t^3 - 3tW_t$

Condition 2:

$E[|W_t^3 - 3tW_t|] \leq E[|W_t^3|] + 3tE[|W_t|] \leq (E[W_t^4])^{\frac{3}{4}} + 3t(E[W_t^2])^{\frac{1}{2}} = (3t^2)^{\frac{3}{4}} + 3t(t)^{\frac{1}{2}} \leq 6t^{\frac{3}{2}} < \infty$ since $W_t \sim N(0,t)$

Condition 3: Using Ito's Lemma

$d(W_t^3 - 3tW_t) = (3W_t^2 - 3t)dW_t - 3W_t dt + \frac{1}{2}6W_t dt = (3W_t^2 - 3t)dW_t$

Thus

$$W_T^3 - 3TW_T = (W_t^3 - 3tW_t) + \int_t^T (3W_t^2 - 3t)dW_t$$

For any $t < T$,

$E_t[W_T^3 - 3TW_T] = (W_t^3 - 3tW_t) + E_t\left[\int_t^T (3W_s^2 - 3s)dW_s\right] = W_t^3 - 3tW_t$

Therefore $W_t^3 - 3tW_t$ satisfies all 3 conditions for martingale.

ii) $e^{\lambda W_t - \frac{1}{2}\lambda^2 t}$

Condition 2:

Since
$\lambda W_t - \frac{1}{2}\lambda^2 t \sim N\left(-\frac{1}{2}\lambda^2 t, \lambda^2 t\right)$, thus $e^{\lambda W_t - \frac{1}{2}\lambda^2 t}$ is lognormal.

$$E\left[e^{\lambda W_t - \frac{1}{2}\lambda^2 t}\right] = e^{-\frac{1}{2}\lambda^2 t + \frac{1}{2}\lambda^2 t} = 1 < \infty$$

Condition 3: Using Ito's Lemma

$$d\left(e^{\lambda W_t - \frac{\lambda^2}{2}t}\right) = \lambda e^{\lambda W_t - \frac{\lambda^2}{2}t}dW_t - \frac{\lambda^2}{2}e^{\lambda W_t - \frac{\lambda^2}{2}t}dt + \frac{\lambda^2}{2}e^{\lambda W_t - \frac{\lambda^2}{2}t}dt$$
$$= \lambda e^{\lambda W_t - \frac{\lambda^2}{2}t}dW_t$$

4. **Continued**

Similar to i), for any $t < T$,

$$E_t\left[e^{\lambda W_T-\frac{\lambda^2}{2}T}\right] = e^{\lambda W_t-\frac{\lambda^2}{2}t} + E_t\left[\int_t^T \lambda e^{\lambda W_s-\frac{\lambda^2}{2}s} dW_s\right] = e^{\lambda W_t-\frac{\lambda^2}{2}t}$$

Therefore $e^{\lambda W_t-\frac{1}{2}\lambda^2 t}$ satisfies all 3 conditions for martingale.

(c) Show that M_t is a martingale.

Commentary on Question:
Most of the candidates knew how to use tower property to show conditional expectation and got full mark. Some candidates didn't explain M_t *satisfies condition 1 and 2 and thus only got partial mark.*

That $M_t = E[X|I_t]$ satisfies condition 1 follows from the definition of expectation, and that M_t satisfies condition 2 follows from the assumption $E[|X_t|] < \infty$.

We verify that Condition 3 is also satisfied as below:

For any $t < T$, given information set I_t

$$E_t[M_T] = E_t\big[E[X|I_T]\big] = E[X|I_t]$$

where the last equality results from the tower property.

Therefore, $E[M_T] = M_t$

(d) Find the process $g(t, W_t)$ for each of the following martingales such that $M_t = M_0 + \int_0^t g(u, W_u)\, dW_u$ for $t \le T$.

(i) $M_t = E\left[W_T^3 | I_t\right]$,

(ii) $M_t = E\left[e^{\lambda W_T} | I_t\right]$.

4. **Continued**

Commentary on Question:
This was the worst performed part. Most candidates didn't know where to start or tried to write down the Ito's Lemma. A few candidates got partial credits for applying Ito's Lemma with incorrect final answers. Only a few candidates got full marks.

Part (i)

$M_t = E^P\left[W_T^3 \middle| I_t\right]$

From part (a), $W_t^3 - 3tW_t$ is a martingale, so $E[W_T^3 - 3TW_T \big| I_t] = W_t^3 - 3tW_t$
Also $E^P[W_T | I_t] = W_t$ is a martingale by definition.

So, $E^P\left[W_T^3 \middle| I_t\right] = E^P[W_T^3 - 3TW_T \big| I_t] + 3TE^P[W_T | I_t] = W_t^3 + 3(T-t)W_t$
That is

$$M_t = W_t^3 + 3(T-t)W_t = h(t, W_t)$$

Hence
$g(t, X) = \frac{dh}{dw}\ (t, W_t) = 3W_t^2 + 3(T-t).$

Part (ii)

$M_t = E^P\left[e^{\lambda W_T} \middle| I_t\right]$

From part (a), $e^{\lambda W_t - \frac{1}{2}\lambda^2 t}$ is a martingale, so that $E\left[e^{\lambda W_T - \frac{1}{2}\lambda^2 T} \middle| I_t\right] = e^{\lambda W_t - \frac{1}{2}\lambda^2 t}$

So, $E^P\left[e^{\lambda W_T} \middle| I_t\right] = e^{\frac{1}{2}\lambda^2 T} E^P\left[e^{\lambda W_T - \frac{1}{2}\lambda^2 T} \middle| I_t\right] = e^{\frac{1}{2}\lambda^2 T} e^{\lambda W_t - \frac{1}{2}\lambda^2 t} = e^{\lambda W_t + \frac{1}{2}\lambda^2 (T-t)}$
That is

$$M_t = e^{\lambda W_t + \frac{1}{2}\lambda^2 (T-t)} = h(t, W_t)$$

Hence, $g(t, X) = \frac{dh}{dw}\ (t, W_t) = \lambda e^{\lambda W_t + \frac{1}{2}\lambda^2 (T-t)}$ will satisfy the equation.

5. **Learning Objectives:**

2. The candidate will understand how to apply the fundamental theory underlying the standard models for pricing financial derivatives. The candidate will understand the implications for option pricing when markets do not satisfy the common assumptions used in option pricing theory such as market completeness, bounded variation, perfect liquidity, etc. The Candidate will understand how to evaluate situations associated with derivatives and hedging activities.

Learning Outcomes:

(2b) Compare and contrast the various kinds of volatility, (eg actual, realized, implied, forward, etc.).

(2c) Compare and contrast various approaches for setting volatility assumptions in hedging.

(2d) Understand the different approaches to hedging.

(2e) Understand how to delta hedge and the interplay between hedging assumptions and hedging outcomes.

Sources:

Paul Wilmott Introduces Quantitative Finance, Wilmott, Paul, 2nd Edition, 2007
Ch. 8, 10

Commentary on Question:

The performance is fair. Nobody did perfectly in this question

Solution:

(a) Identify each of following partial derivatives with the named variables:

Partial derivatives: $\frac{\partial V}{\partial t}, \quad \frac{\partial V}{\partial S}, \quad \frac{\partial^2 V}{\partial S^2}, \quad \frac{\partial V}{\partial r},$

Named variables: Delta, Gamma, Rho, Speed, Theta, Vega

Commentary on Question:

Most of the candidates answered correctly. A few of them mixed up Rho with Vega

$\frac{\partial V}{\partial t} = \theta\ (theta) \qquad \frac{\partial^2 V}{\partial S^2} = \Gamma(gamma)$

$\frac{\partial V}{\partial S} = \Delta(delta) \qquad \frac{\partial V}{\partial r} = \rho\ (rho)$

5. Continued

(b) Determine your portfolio value and Theta at time t.

Commentary on Question:
Most of the candidates calculated the portfolio value correctly. Some of them mixed up with the formulas and get a positive portfolio value, which is wrong in this question. Half of candidates did not get time decay correctly.

$$\theta + \frac{1}{2}\sigma^2 S^2 \Gamma + rS\Delta - rV = 0$$

$$\theta = rV - rS\Delta - \frac{1}{2}\sigma^2 S^2 \Gamma = r(V - S\Delta) - \frac{1}{2}\sigma^2 S^2 \Gamma$$

$$Portfolio\ value = V - S\Delta = 11.48 - 100 * 0.89 = -77.52$$

$$Time\ decay = r(V - S\Delta) - \frac{1}{2}\sigma^2 S^2 \Gamma$$

$$= 0.01 * (-77.52) - \frac{1}{2} 0.1^2 100^2 0.02 = -1.78$$

(c) Outline characteristics of delta hedging and suggest another hedging strategy that could mitigate the weaknesses of delta hedging.

Commentary on Question:
Overall, candidates did well on this part. Most of them knew the characteristics of Delta hedging and successfully suggested Gamma hedging to mitigate the weakness of Delta hedging

Delta hedging
- Exploits the perfect correlation between the changes in the option value and the changes in the stock price
- Hedge must be continually monitored and frequently adjusted by sales or purchase of the underlying assets
- Due to the frequent adjustment, any dynamic hedging strategy is going to result in losses due to transaction costs
- Portfolio delta = 0

Gamma hedging
- Reduce the size of each re-hedge
- Increase the time between re-hedges
- Reduce costs
- More accurate form of hedging that eliminates the second-order effects (convexity)
- Portfolio delta and gamma = 0

5. **Continued**

(d) Calculate the at-the-money option's implied volatility at time $t+1$ based on the assumption that implied volatilities exhibit

(i) Sticky strike behavior,

(ii) Sticky delta behavior.

Commentary on Question:
Candidates performed poorly in this question. Most of them didn't know what is sticky strike and sticky delta. Only a handful of candidates answered this part correctly

i) The *Sticky Strike rule* states that the implied volatility does not change when the stock price increases or decreases. Therefore, the implied vol when K=120 is 0.003%*(120)^2-0.64%*120+49.5% = 15.9%

ii) The *Sticky Delta rule* states that the volatility smile shifts when the stock price increases or decreases. Therefore, the implied vol for strike K=120 (relative to S = 120) is the same as for K=100 (relative to S = 100).

0.003%*(100)^2-0.64%*(100)+49.5% =15.5%

(e) Identify one market in which sticky strike behavior is commonly observed, and one market in which sticky delta behavior is commonly observed

Commentary on Question:
Same as part (d)

Sticky strike is commonly observed in the equity markets
Sticky delta is commonly observed in FX market

(f) Recommend and justify the volatility measure that should be used for the hedging program in order to be consistent with the CFO's vision.

Commentary on Question:
Performance was good overall. Most of them correctly answered implied volatility. However, some of them did not answer the actual volatility part correctly.

The implied volatility should be used for hedging
It has deterministic profit so it best meets the goal of minimal fluctuation in daily P/L
Hedging with the actual volatility would result in potentially large fluctuations in daily P/L

6. **Learning Objectives:**

2. The candidate will understand how to apply the fundamental theory underlying the standard models for pricing financial derivatives. The candidate will understand the implications for option pricing when markets do not satisfy the common assumptions used in option pricing theory such as market completeness, bounded variation, perfect liquidity, etc. The Candidate will understand how to evaluate situations associated with derivatives and hedging activities.

Learning Outcomes:

(2b) Compare and contrast the various kinds of volatility, (eg actual, realized, implied, forward, etc.).

(2d) Understand the different approaches to hedging.

(2e) Understand how to delta hedge and the interplay between hedging assumptions and hedging outcomes.

Sources:

Paul Wilmott: Introduction to Quantitative Finance, Chapter 8 & 10

Commentary on Question:

This question tests candidates' understanding of different approaches to delta hedging.

Solution:

(a) Calculate the market price of the put option.

Commentary on Question:

This is a straightforward application of Black-Scholes' European option pricing formula. Many candidates did well on this part.

Stock price S = 100
Strike price X = 100
Interest rate r = 2%
Time-to-maturity T = 1
Market implied vol σ = 20%

$$d_1 = \frac{\ln\frac{S}{X} + \left(r + \frac{\sigma^2}{2}\right)T}{\sigma\sqrt{T}} = \frac{\ln\frac{100}{100} + \left(2\% + \frac{20\%^2}{2}\right) * 1}{0.2\sqrt{1}} = 0.2$$

$$d_2 = d_1 - \sigma\sqrt{T} = 0.2 - 20\%\sqrt{1} = 0$$

$$N(-d_1) = 1 - N(d_1) = 1 - \text{N}(0.2) = 0.4207$$

$$N(-d_2) = N(0) = 0.5$$

$$Put = Xe^{-rT}N(-d_2) - SN(-d_1) = 100 * e^{-2\%} * 0.5 - 100 * 0.4207 = 6.94$$

6. **Continued**

(b) Propose a trading strategy today involving the put option and/or its underlying stock such that the strategy will generate daily profit for you until the option expires.

Commentary on Question:
This is a straightforward application of a trading strategy since the put option's market price $6.94 exceeds your fair value of $5.94. The key requirement of the trading strategy is that it generates daily profit for you until the option expires. This necessitates using the market "implied volatility" to determine delta in the hedging strategy. About 20% of the candidates incorrectly proposed using the "actual volatility" in the hedging strategy, which will not meet the question's requirement.

Since the market price of $6.94 exceeds the perceived fair value of $5.94, the following trading strategy will guarantee a daily profit for each put option involved:

(i) Sell one put option at the market price of $6.94
(ii) Short 0.4027 unit of the stock
(iii) Delta hedge the put option until option maturity
(iv) Use the market implied volatility to determine the delta

The above strategy can be scaled to more put options as long as it has no impact on the market price of the options and the stock.

(c) Derive the present value of the total profit of your strategy. Define all terms and symbols used in your derivation.

Commentary on Question:
Many candidates did poorly on this part largely because of the three issues: (i) the derivations were based on using the actual volatility in the hedging strategy. This is incorrect as it does not support the strategy required in part (b); (ii) though the correct final result were provided, no derivation steps were shown. (iii) no answers (skipped) to this part entirely.

Derivation:
For each put option sold and delta hedged, let

$\sigma_i =$ implied volatility
$\Delta_i =$ put option delta calculated with the implied vol σ_i
$V_i =$ put option market value.
$\theta_i =$ put option theta calculated with the implied vol σ_i
$\gamma_i =$ put option gamma calculated with the implied vol σ_i
$\sigma =$ actual volatility that produces my belief price of $5.94
$t_0 =$ start date of the delta hedge strategy

6. Continued

$T =$ expiration date of the put option

Set up the delta hedge portfolio today

Position	Value
Option:	$-V_i$
Stock:	$\Delta_i S$
Cash:	$V_i - \Delta_i S$

Daily portfolio value change

Change = $-dV_i + \Delta_i dS + r(V_i - \Delta_i S)dt$

$$= -\theta_i dt - \frac{1}{2}\sigma^2 S^2 \gamma_i dt + r(V_i - \Delta_i S)dt$$

$$= \frac{1}{2}(\sigma_i^2 - \sigma^2)S^2 \gamma_i dt$$

Daily profit is always positive since $\sigma_i > \sigma$ and $\gamma_i > 0$

The present value of total profit is $\frac{1}{2}(\sigma_i^2 - \sigma^2)\int_{t_0}^{T} e^{-r(T-t_0)} S^2 \gamma_i dt$

(d) Determine the total cash amount generated (or required) for risk-free investment (or borrowing) today as a result of executing your strategy in parts (b) and (c).

Commentary on Question:
This is a straightforward extension of part (b). However, many candidates performed poorly on this question because they misinterpreted put option delta. The negative delta means short selling the stock that generates cash up front, but many candidates incorrectly treated as buying the stock that requires borrowing.

For each put option sold and delta hedged:

$\Delta_i = -N(-d1) = -0.4207$ (negative delta means shorting the stock)

Cash amount generated today = $V_i - \Delta_i S = 6.94 - (-0.4207) * 100$ = 49.01

(e) Outline the advantages and disadvantages of your strategy in parts (b) and (c).

Commentary on Question:
Many candidates did poorly on this question either because they answered the question from the perspective of the hedging strategy that uses actual volatility (rather than the implied volatility that this question asked) or their answers didn't capture the key essence of this question.

6. Continued

Advantages (Pros):

1. No local fluctuations in daily P&L. Continue to make profit daily.
2. Only need to be on the right side of the trade to profit.
3. Delta is based on the implied volatility, which is easy to observe.

Disadvantages (Cons):

4. Don't know how much profit will be made, only that it's positive. (i.e. Profit is path-dependent)

7. **Learning Objectives:**

1. The candidate will understand the fundamentals of stochastic calculus as they apply to option pricing.

3. The candidate will understand the quantitative tools and techniques for modeling the term structure of interest rates and pricing interest rate derivatives.

Learning Outcomes:

(1c) Understand Ito integral and stochastic differential equations.

(1d) Understand and apply Ito's Lemma.

(3b) Apply the models to price common interest sensitive instruments including: callable bonds, bond options, caps, floors, swaptions, caption, floortions.

(3c) Understand and apply popular one-factor interest rate models including Vasicek, Cox-Ross_Ingersoll, Hull-White, Ho-Lee, Black-Derman_Toy, Black-Karasinski.

(3d) Understand the concept of calibration and describe the issues related to calibration, including yield curve fitting.

Sources:

An Introduction to the Mathematics of Financial Derivatives, Chapter 10 & 18

Paul Wilmott Introduces Quantitative Finance, Chapter 17

Commentary on Question:

This question tests the candidates' understanding of short rate models and the abilities to manipulate the model's SDE for purposes of bond pricing and yield curve fitting. Overall, candidates did fine on this question. Many candidates know the techniques to work with stochastic equations and understand the concept of yield curve fitting.

Solution:

(a) Determine the distribution of r_t including its expected value and variance.

Commentary on Question:

Candidates did well on this part. Most candidates were able to derive the expression for r_t from the given short rate process. A common mistake is treating μ_t and σ_t as constants instead of being time-dependent, and integrating them out.

7. **Continued**

By Ito's Lemma,

$$d(e^{\theta t}r_t) = e^{\theta t}dr_t + \theta e^{\theta t}r_t dt = e^{\theta t}(\mu_t - \theta r_t)dt + e^{\theta t}\sigma_t dW_t + \theta e^{\theta t}r_t dt$$
$$= e^{\theta t}\mu_t dt + e^{\theta t}\sigma_t dW_t$$

Thus,

$$e^{\theta t}r_t - r_0 = \int_0^t e^{\theta s}\mu_s ds + \int_0^t e^{\theta s}\sigma_s dW_s$$

Alternatively,

$$e^{\theta t}(dr_t + \theta r_t dt) = e^{\theta t}(\mu_t dt + \sigma_t dW_t)$$
$$\int_0^t d(e^{\theta s}r_s) = e^{\theta t}r_t - r_0 = \int_0^t e^{\theta s}\mu_s ds + \int_0^t e^{\theta s}\sigma_s dW_s$$

Therefore,

$$r_t = e^{-\theta t}r_0 + \int_0^t e^{-\theta(t-s)}\mu_s ds + \int_0^t e^{-\theta(t-s)}\sigma_s dW_s$$

r_t is Gaussian with

$$E(r_t) = e^{-\theta t}r_0 + \int_0^t e^{-\theta(t-s)}\mu_s ds$$

$$Var(r_t) = E\left[\left(\int_0^t e^{-\theta(t-s)}\sigma_s dW_s\right)^2\right] = \int_0^t e^{-2\theta(t-s)}\sigma_s^2 ds$$

(b) Derive the time-0 price $P(0,T)$ of a zero-coupon bond maturing at time T.

Commentary on Question:
Most candidates know the basic bond pricing formula and are able to process the equation with the expression of r_t derived in part a). However, most candidates did not solve the double integral by changing the order of integration, and some candidates missed taking the expectation of the exponential, so that were not able to arrive at the final answer.

By definition,

$$P(0,T) = E\left[e^{-\int_0^T r_t dt}\right] = E\left[e^{-\int_0^T\left(e^{-\theta t}r_0 + \int_0^t e^{-\theta(t-s)}\mu_s ds + \int_0^t e^{-\theta(t-s)}\sigma_s dW_s\right)dt}\right]$$

$$\int_0^T \left(e^{-\theta t}r_0 + \int_0^t e^{-\theta(t-s)}\mu_s ds + \int_0^t e^{-\theta(t-s)}\sigma_s dW_s\right)dt$$
$$= \int_0^T e^{-\theta t}r_0 dt + \int_0^T\int_0^t e^{-\theta(t-s)}\mu_s ds\, dt + \int_0^T\int_0^t e^{-\theta(t-s)}\sigma_s dW_s\, dt$$

7. Continued

Change the order of integration for the second term,

$$\int_0^T \int_0^t e^{-\theta(t-s)} \mu_s ds\, dt = \int_0^T \int_s^T e^{-\theta(t-s)} \mu_s dt\, ds = \int_0^T \mu_s \left(\frac{1-e^{-\theta(T-s)}}{\theta}\right) ds$$
$$= \int_0^T \mu_t \left(\frac{1-e^{-\theta(T-t)}}{\theta}\right) dt$$

The third term is the given equality. Note that,

$$\int_0^T \int_0^t e^{-\theta(t-s)} \sigma_s dW_s\, dt = \int_0^T \sigma_t \left(\frac{1-e^{-\theta(T-t)}}{\theta}\right) dW_t$$

$\int_0^T \sigma_t \left(\frac{1-e^{-\theta(T-t)}}{\theta}\right) dW_t$ is normally distributed with mean $= 0$, and variance $= \int_0^T \sigma_t^2 \left(\frac{1-e^{-\theta(T-t)}}{\theta}\right)^2 dt$

Thus,

$$P(0,T) = exp\left\{-\frac{r_0\left(1-e^{-\theta T}\right)}{\theta} - \int_0^T \mu_t \left(\frac{1-e^{-\theta(T-t)}}{\theta}\right) dt\right\}$$
$$\times E\left[e^{-\int_0^T \sigma_t \left(\frac{1-e^{-\theta(T-t)}}{\theta}\right) dW_t}\right]$$
$$= exp\left\{-\frac{r_0\left(1-e^{-\theta T}\right)}{\theta} - \int_0^T \mu_t \left(\frac{1-e^{-\theta(T-t)}}{\theta}\right) dt + \frac{1}{2}\int_0^T \sigma_t^2 \left(\frac{1-e^{-\theta(T-t)}}{\theta}\right)^2 dt\right\}$$

(c) Express α_t in terms of μ_t.

Commentary on Question:
Candidate did relatively well on this part. Most candidates were able to use similar techniques as part a) to derive the expression for α_t. Again, some candidates treated μ_t as a constant instead of being time-dependent.

$$dr_t = d\alpha_t + dx_t = \frac{\partial \alpha_t}{\partial t} dt - \theta x_t dt + \sigma_t dW_t = (\mu_t - \theta r_t) dt + \sigma_t dW_t$$

Therefore,

$$\frac{\partial \alpha_t}{\partial t} - \theta x_t = \mu_t - \theta r_t = \mu_t - \theta(\alpha_t + x_t)$$
$$\frac{\partial \alpha_t}{\partial t} = \mu_t - \theta \alpha_t$$

7. Continued

Note that $\alpha_0 = r_0$. Multiply both sides by $e^{\theta t}$,

$$\frac{\partial \alpha_t}{\partial t} e^{\theta t} + \theta \alpha_t e^{\theta t} = \mu_t e^{\theta t}$$

$$\frac{\partial}{\partial t}\left(\alpha_t e^{\theta t}\right) = \mu_t e^{\theta t}$$

$$\alpha_t e^{\theta t} - \alpha_0 = \int_0^t \mu_s e^{\theta s} ds$$

Thus,

$$\alpha_t = \alpha_0 e^{-\theta t} + \int_0^t \mu_s e^{-\theta(t-s)} ds$$

Alternatively, by Ito's Lemma

$$de^{\theta t} x_t = e^{\theta t} dx_t + x_t \theta e^{\theta t} dt = -\theta e^{\theta t} x_t dt + e^{\theta t} \sigma_t dW_t + x_t \theta e^{\theta t} dt$$
$$= e^{\theta t} \sigma_t dW_t$$

Thus,

$$\int_0^t de^{\theta s} x_s = e^{\theta t} x_t - x_0 = \int_0^t e^{\theta s} \sigma_s dW_s$$

$$x_t = \int_0^t e^{-\theta(t-s)} \sigma_s dW_s$$

$$\alpha_t = r_t - x_t = r_0 e^{-\theta t} + \int_0^t \mu_s e^{-\theta(t-s)} ds + \int_0^t e^{-\theta(t-s)} \sigma_s dW_s$$
$$- \int_0^t e^{-\theta(t-s)} \sigma_s dW_s = \alpha_0 e^{-\theta t} + \int_0^t \mu_s e^{-\theta(t-s)} ds$$

(d) Show that

$$\alpha_t = r_0 e^{-\theta t} + f(0,t) - f(0,0) e^{-\theta t} + \int_0^t \frac{\sigma_v^2}{\theta} e^{-\theta(t-v)} \left(1 - e^{-\theta(t-v)}\right) dv$$

Commentary on Question:
Candidates did not do well on this part, some as a result of treating μ_t as a constant in part c) and did not have a correct expression to work with. Only a few candidates were able to correctly integrate out the second and third terms, and even less for the last term.

Sub in the given equation $\mu_t = \frac{\partial f(0,t)}{\partial t} + \theta f(0,t) + \int_0^t \sigma_s^2 e^{-2\theta(t-s)} ds$

$$\alpha_t = \alpha_0 e^{-\theta t} + \int_0^t \left(\frac{\partial f(0,s)}{\partial s} + \theta f(0,s) + \int_0^s \sigma_v^2 e^{-2\theta(s-v)} dv \right) e^{-\theta(t-s)} ds$$

7. Continued

$$\int_0^t \left(\frac{\partial f(0,s)}{\partial s} + \theta f(0,s)\right) e^{-\theta(t-s)} ds = \int_0^t d\left(f(0,s)e^{-\theta(t-s)}\right) = f(0,t) - f(0,0)e^{-\theta t}$$

$$\int_0^t \left(\int_0^s \sigma_v^2 e^{-2\theta(s-v)} dv\right) e^{-\theta(t-s)} ds = \int_0^t \sigma_v^2 e^{-\theta(t-v)} \left(\int_v^t e^{-\theta(s-v)} ds\right) dv$$
$$= \int_0^t \sigma_v^2 e^{-\theta(t-v)} \left(\int_v^t e^{-\theta(t-s)} ds\right) dv = \int_0^t \sigma_v^2 e^{-\theta(t-v)} \frac{1 - e^{-\theta(t-v)}}{\theta} dv$$

Thus,

$$\alpha_t = \alpha_0 e^{-\theta t} + f(0,t) - f(0,0)e^{-\theta t} + \int_0^t \frac{\sigma_v^2}{\theta} e^{-\theta(t-v)}\left(1 - e^{-\theta(t-v)}\right) dv$$

Using the initial condition that $x_0 = 0, r_0 = \alpha_0 + x_0$, i.e. $r_0 = \alpha_0$

$$\alpha_t = r_0 e^{-\theta t} + f(0,t) - f(0,0)e^{-\theta t} + \int_0^t \frac{\sigma_v^2}{\theta} e^{-\theta(t-v)}\left(1 - e^{-\theta(t-v)}\right) dv$$

Note α_t is deterministic depending on the term structure $f(0,t)$ itself only.

(e) Describe one use/application and one disadvantage of yield curve fitting for a one-factor model.

Commentary on Question:
Most candidates understand the concept of yield curve fitting and know what it's used for. Less people are able to describe the disadvantage of yield curve fitting properly.

Use/Application:
Yield curve fitting allows the model to match the observed market prices of bonds, which is important for hedging purposes, as trades are made at the market prices. By using the fitted model, we should be able make money on the hedged portfolios.

Disadvantage:
Often due to the steep slope at the short end of the yield curve, yield curve fitting is inconsistent, as the model fitted in a week/month would be closer to a parallel shift of the model fitted now, instead of dropping off the one week/month period that has passed. So it may be dangerous to use model fitted to observed yield curve for interest rate speculation purposes.

8. **Learning Objectives:**

3. The candidate will understand the quantitative tools and techniques for modeling the term structure of interest rates and pricing interest rate derivatives.

Learning Outcomes:

(3c) Understand and apply popular one-factor interest rate models including Vasicek, Cox-Ross_Ingersoll, Hull-White, Ho-Lee, Black-Derman_Toy, Black-Karasinski.

(3d) Understand the concept of calibration and describe the issues related to calibration, including yield curve fitting.

(3e) Understand and differentiate between the classical approach to interest rate modelling and the HJM modeling approach, including the basic philosophy, arbitrage conditions, assumptions, and practical implementations.

(3f) Understand and apply the HJM and BGM/Libor Market model.

Sources:

An Introduction to the Mathematics of Financial Derivatives, Neftci, Hirsa, 3rd Edition, Ch. 19

Paul Wilmott Introduces Quantitative Finance, Wilmott, Paul, 2nd Edition, Ch. 16-18

Commentary on Question:

Commentary listed underneath question component.

Solution:

(a) Compare and contrast the HJM approach versus the classical approach with regard to pricing interest-sensitive securities.

Commentary on Question:

Candidates performed well on this part. Most candidates were able to differentiate the HJM approach from the classical approach.

- HJM approach eliminates the need to model the expected rate of change of the spot rate.

- In the HJM framework, the spot rate behaves in a non-Markovian fashion. This point is important because current empirical work indicates that spot rate behavior in reality may fail to be Markovian.

- The HJM approach implies arbitrage-free dynamics. Hence, it reproduces market prices of default-free pure discount bonds.

8. Continued

(b) Derive an expression $\eta^*(t)$ for $\eta(t)$ calibrated to the market data such that $r(t^*) = r^*$ and $Z(r^*; t^*, t) = M(t^*, t)$.

Commentary on Question:
Candidates performed as expected on this part. Most candidates were able to identify that this model was Ho-Lee and derive the correct calibration solution.

We want to solve η*(t) so that:

$$M(t^*, t) = Z(r^*;\ t^*, t) = \ e^{A(t^*,t)-B(t^*,t)r^*}$$

Differentiating with regard to t, we get:

$$\frac{\partial}{\partial t}\log\big(M(t^*, t)\big) = \frac{\partial A(t^*, t)}{\partial t} - \frac{\partial B(t^*, t)}{\partial t} r^*$$

Since :

$$\frac{\partial A(t^*,t)}{\partial t} = -\int_{t^*}^{t} \eta^*(s)ds + \frac{1}{2}\beta(t - t^*)^2$$

$$\frac{\partial B(t^*,t)}{\partial t} = 1$$

Then,

$$\frac{\partial}{\partial t}\log\big(M(t^*, t)\big) = -\int_{t^*}^{t} \eta^*(s)ds + \frac{1}{2}\beta(t - t^*)^2 - r^*$$

Differentiating with regard to t again, we get:

$$\frac{\partial^2}{\partial t^2}\log\big(M(t^*, t)\big) = -\eta^*(t) + \beta(t - t^*)$$

And finally:

$$\eta^*(t) = \beta(t - t^*) - \frac{\partial^2}{\partial t^2}\log\big(M(t^*, t)\big)$$

8. Continued

(c) Recommend an approach that should be used to fit the above spot interest rate model to the initial term structure.

Commentary on Question:
Candidates performed as expected on this part. Many candidates recognized that a closed-form solution was available for this model and should be used for calibration.

- Under the Ho-Lee model, we can analytically compute the price of zero-coupon bonds in closed-form. Hence, the closed-form solution method is the most convenient way to fit the initial term structure.

- We can minimize the distance between the closed-form solution and the observed yield curve by choosing the optimal parameter β.

- Furthermore, under the Ho-Lee model, the specification of $\eta^*(t)$ explicitly assumes that the model is calibrated using closed-form solution.

- Tree models and Monte Carlo are less relevant to calibrate this model since we have a closed-form solution.

(d) Derive the stochastic differential equation governing $F(t,T)$.

Commentary on Question:
Candidates performed below expectations on this part. Most candidates outlined the relationship between F(t,T) and Z(t,T). However, only a few candidates were able to come up with the correct SDE for F(t,T).

We know that:

$$F(t,T) = -\frac{\partial}{\partial T}\log\big(Z(t,T)\big) = -\frac{\partial}{\partial T}(A(t,T) - B(t,T)r_t)$$

Hence,

$$F(t,T) = \int_t^T \eta(s)\, ds\ - \frac{1}{2}\beta\ (T-t)^2 + r_t$$

Finally,

$$dF(t,T) = -\eta(t)dt + \beta(T-t)dt + dr_t$$
$$dF(t,T) = \beta(T-t)dt + \beta^{1/2}\, dW_t$$

(e) Assess the relationship between the forward rate dynamics and the initial calibration of the above spot interest rate model.

8. Continued

Commentary on Question:
Candidates did poorly on this part. Most candidates were not able to identify that the forward rate dynamics did not depend on the initial calibration of the spot rate.

From part (d), we have :

$$dF(t,T) = \beta(T - t)dt + \sqrt{\beta}dW_t$$

Hence, the forward rate dynamic is independent of the initial calibration of the model (i.e. $dF(t, T)$ does not depend on $\eta(t)$).

9. Learning Objectives:

4. The candidate will understand the concept of volatility and some basic models of it.

Learning Outcomes:

(4a) Compare and contrast the various kinds of volatility, (eg actual, realized, implied, forward, etc.).

(4b) Understand and apply various techniques for analyzing conditional heteroscedastic models including ARCH and GARCH.

Sources:

QFIC-109-15: Chapter 9 of *Risk Management and Financial Institutions,* Hull, 2nd Ed.

Tsay, Time Series, Chapter 3

Commentary on Question:

This question tests candidates' understanding of GARCH model in the area of model parameter estimation, long-term target variance, volatility forecast and key features of GARCH model.

Solution:

(a) Calculate the maximum value of the log likelihood function, ignoring the constant multiplicative factor, of your model using the given ABC stock prices.

Commentary on Question:

This question tests candidates' ability to estimate parameters of a GARCH(1,1) model with the maximum likelihood method. Though few were able to get the correct maximum likelihood function, many received partial credits through correct answers to daily price return (u_t) and/or variance estimate for day 4 (σ_4^2).

Day (t)	1	2	3	4	…
ABC Stock Price (P_t)	\$10.1	\$11.2	\$10.0	\$10.5	…
$u_t = \frac{P_t}{P_{t-1}} - 1$		10.89%	-10.71%	5.00%	…

For Day 3:

$$\sigma_3^2 = u_2^2 = 10.89\%^2 = 0.0119$$

For Day 4:

$$\sigma_4^2 = \omega + \alpha u_3^2 + \beta \sigma_3^2$$

$$= 0.00002 + 0.1*(-10.71\%)^2 + 0.8*0.0119 = 0.0107$$

9. **Continued**

Required quantity = $\sum(-\frac{u_t^2}{\sigma_t^2} - ln\,\sigma_t^2)$

= – (–10.71%)^2/0.0119 – ln0.0119 – (5%)^2/0.0107 – ln0.0107

= 7.770

(b) Determine the long-run average annual volatility implied by your GARCH(1,1) model.

Commentary on Question:
This question tests candidates' understanding of target variance / volatility in GARCH model. Many candidates did well with this question.

The daily long run average variance is $V_L = \frac{\omega}{1-\alpha-\beta} = \frac{0.00002}{1-0.1-0.8} = 0.0002$

Annual volatility = $\sqrt{252 * 0.0002} = 22.4\%$

(c) Determine the number N.

Commentary on Question:
This question tests candidates' understanding of volatility forecast in GARCH model. Less than 50% of candidates did well with this question due to incorrect application of the formula and/or calculation errors.

$E(\sigma_N^2) = V_L + (\alpha + \beta)^{N-4}(\sigma_4^2 - V_L)$

$5.9\%^2 = 0.0002 + (0.1 + 0.8)^{N-4}(0.0107 - 0.0002)$

Solving for N from the above equation yields $N = 15$

(d) Describe strengths and weaknesses of GARCH models.

Commentary on Question:
This questions tests candidates' knowledge on the key features of GARCH model. Many candidates did well with this question.

Strengths:

1. Large volatlity tends to be followed by another large volatlity generaring the well-known behaviour of volatility clustering inf financial time series.

2. The tail distribution is heavier than that of normal distribution.

9. Continued

3. It provides a simple parametric function that can be used to describe volatiility evolution.

Weakness:

4. It does not provide asymmetric effects between positive and negative asset returns.

10. **Learning Objectives:**

6. The candidate will understand the variety of equity investments and strategies available for portfolio management.

7. The candidate will understand how to develop an investment policy including governance for institutional investors and financial intermediaries.

Learning Outcomes:

(6i) Explain alpha and beta separation as an approach to active management and demonstrate the use of portable alpha.

(6j) Describe the process of identifying, selecting, and contracting with equity managers.

(7c) Determine how a client's objectives, needs and constraints affect investment strategy and portfolio construction. Include capital, funding objectives, risk appetite and risk-return trade-off, tax, accounting considerations and constraints such as regulators, rating agencies, and liquidity.

Sources:

Managing Investment Portfolios: A Dynamic Process, Maginn & Tuttle, 3rd Edition, 2007, Ch 3

Managing Institutional Investor Portfolios, Ch 7 Equity Portfolio Management.

QFIC-110-15 – Liquidity as an Investment Style.

Commentary on Question:

The results in general were lower than expected, particularly on part b), c) and d). But candidates performed well on the calculation of the Information Ratio in part e) and its use for recommending if the Manager should be hired.

On part b) and c), the explanation of the pure beta exposure related to the alpha and beta separation strategy was done correctly. However, candidates did not so well in describing how a pure alpha exposure can be obtained through the liquidity investment style.
Also, in part d), the question was to describe the liquidity investment style itself while many candidates instead mostly answered why liquidity can be seen as an investment style.

Solution:

(a) Identify investment constraints that *ABC* needs to consider in its investment policy.

10. Continued

- Investment income on surplus is subject to taxation, so taxes need to be considered.
- In the United States most states limit the value of life companies' common stock holdings to 20% of admitted assets.
- Regulatory requirements limit non-US investments to a certain percentage of admitted assets in most states.
- Regulatory requirements require that investments be evaluated according to the prudent investor rule.
- Due to large surplus, liquidity requirements for this segment are low, allowing for more illiquid investments.

(b) Explain this concept and describe how *ABC* can obtain this exposure.

Beta measures the exposure of the portfolio relative to a major index and serves as a measure of portfolio to the systematic risk of the portfolio.

Pure beta exposure can be obtained by investing in an inexpensive passively managed fund (index fund).

Fund should seek to replicate the returns from a US large cap rules-based index such as the Russell Top 200 or S&P 500.

(c) Describe how *ABC* can obtain this exposure.

Pure alpha exposure to the liquidity investment style can be obtained by going long less liquid stocks and shorting more liquid stocks or shorting a benchmark index, such as the S&P 500.

(d) Describe the liquidity investment style in detail and compare it to other investment styles such as size, value, and momentum.

Analyze the liquidity investment style

Liquidity can be measured as stock turnover (i.e. volume divided by shares outstanding), which historically is negatively correlated with long-term returns.

Less liquid stocks have costs – take longer to trade and have higher transaction costs (larger bid/ask spreads), which translates to higher returns with longer holding periods and less trading.

10. Continued

Risks to the liquidity investment style – attempting to liquidate less liquid securities in a crisis, alpha generated may decrease if the liquidity factor becomes more common, data may have been overfitted and results may not persist in the future.

Liquidity factor is more pervasive among smaller stocks but holds for larger cap stocks as well.

Liquidity factor of large cap stocks instead of small cap stocks may produce lower alpha, but also comes with lower transaction costs.

Rebalance infrequently to minimize transaction costs.

Compare to other investment styles

Liquidity, value, momentum, and size embody characteristics that the market seeks to avoid and thus may offer a risk premium. Liquidity has the added benefit of not being as prevalent as the other factors, which may provide for higher risk premiums.
It should be noted that when trying to gain alpha exposure one would hedge the beta exposure.

(e) Evaluate based solely on this information whether the asset manager should be hired.

Mean of difference between portfolio and benchmark =

[(4%-2%) + (9%-10%) + (-2%-(-5%)) + (-3%-(-3%))] / 4 = 1%

Standard deviation of difference between portfolio and benchmark =
SQRT {[(2%-1%)^2 + (-1%-1%)^2 + (3%-1%)^2 + (0%-1%)^2]/ (4-3)} = 1.83%

Note: Credit should be given if the candidate uses the population standard deviation (dividing by 4) rather than the sample standard deviation (dividing by 3)

Information Ratio = 1% / 1.83% = 0.55

Information Ratio is above 0.5, which is frequently viewed as distinguishing top-quartile equity managers. Therefore, the equity manager should be hired based on their historical performance.

Information Ratio measures the amount of excess return per unit of excess risk taken relative to the benchmark.

10. Continued

A high Information Ratio indicates a high degree of consistency in annual performance.

Caution should be exercised in placing funds with the fund manager due to trying to extrapolate future performance from relatively short performance history.

11. Learning Objectives:

5. The candidate will understand and identify the variety of fixed instruments available for portfolio management. This section deals with fixed income securities. As the name implies the cash flow is often predictable, however there are various risks that affect cash flows of these instruments. In general the candidates should be able to identify the cash flow pattern and the factors affecting cash flow for commonly available fixed income securities. Candidates should also be comfortable using various interest rate risk quantification measures in the valuation and managing of investment portfolios. Candidates should also understand various strategies of managing the portfolio against given benchmark.

Learning Outcomes:

(5b) Describe the cash flow of various corporate bonds considering underlying risks such as interest rate, credit and event risk.

(5c) Demonstrate an understanding of the characteristics of leveraged loans.

Sources:

The Handbook of Fixed Income Securities, Fabozzi, Frank, 8th Edition, Ch. 12

Managing Investment Portfolios: A Dynamic Process, Maginn & Tuttle, 3rd Edition, Ch. 6

Commentary on Question:

Commentary listed underneath question component.

Solution:

(a) Compare the reinvestment risk for Bond A and Bond B.

Commentary on Question:

Candidates did fairly well on this question, however most of them failed to state that Bond B can be called as long as the funds used for repurchase do not come from issuing another bond. To earn full points, candidates need to clearly state the relative reinvestment risk between Bond A and B and support their conclusions.

In the First 3 years, Bond B has higher reinvestment risk. Bond A cannot be called for first 3 years while Bond B can be called.

Bond B can be called as long as the funds to call the bond do not come from issuing another bond).

However, Bond B has lower reinvestment risk thank Bond A due to higher call price for the last 4 years.

11. **Continued**

Bond B has lower reinvestment risk due to lower coupon therefore less likely to be called.

(b) Calculate the call price if JKL Holding calls Bond C, with outstanding par value of $300 million, on Sep 30, 2022.

Commentary on Question:
Candidates did well on this question. Some candidates calculated the make-whole redemption price and call price at $300 million scale, in which case, no point was deducted as long as the unit call price was calculated correctly.

Discount Rate / Make-whole Rate = 5-yr UST + 20bps = 3.5% + 0.2% = 3.7%

CF Date	Cash Flow	Discounted CF @ 3.70%
9/30/23	6	5.79
9/30/24	6	5.58
9/30/25	6	5.38
9/30/26	6	5.19
9/30/27	106	88.39
Make-whole redemption price		110.33

Call price = 110.33

(c) Develop a strategy to achieve your goal using leverage as a tool.

Commentary on Question:
Most candidates came up with the appropriate leveraging strategy. However, most of them failed to mention to roll over the repo agreement when term is up. To earn full points, candidates need to calculate the borrowing amount correctly and describe the leveraging strategy clearly.

Let X = borrowed funds
[(5 + X) * 5.25% - X * 4%]/5 = 6%
X = 3 million

Strategy:

- Borrow $3 million at 4% using repo agreement
- Invest $8 million in Bond B (5.25%)
- Roll over the repo agreement when term is up

11. Continued

(d) List two actions that you can take to lower the repo rate.

Commentary on Question:
Candidates did poorly on this question. Most of them mentioned "enhanced collateral quality" as one of the actions.

Any two of the followings:

- Cash payment to the trustee – The trustee then can call the bonds pro rata or by lot for redemption.
- Deliver bonds to the trustee – Bonds are purchase by the issuer in the open market. Preferable to issuer when bonds are selling below par.
- Certify to the trustee that it has used unfunded property credits in lieu of the sinking fund (made property and plant investments that have not been used for issuing bonded debt). May be used by electric utility bond issues.

(e) List two advantages and two disadvantages of a sinking-fund bond from a bondholder's perspective.

Commentary on Question:
Candidates did well on listing the advantages of sinking-fund bond, but poorly on listing the disadvantages.

Advantages:

- Default risk is reduced due to orderly retirement of the issue.
- If bond prices decline, price support/stability may be provided by the issuer because it must enter the market on the buy side to satisfy sinking fund requirement.
- Sinking fund provision enhances the liquidity of the debt.

Disadvantages:

- Bonds may be called at the special sinking-fund call price when interest rates are lower than rates at time of issuance.
- Some indentures permit variable periodic payments, which reduces default risk protection.

12. Learning Objectives:

8. The candidate will understand the theory and techniques of portfolio asset allocation.

Learning Outcomes:

(8b) Propose and critique asset allocation strategies.

Sources:

The Handbook of Fixed Income Securities, Fabozzi, Frank, 8th Edition, Ch. 21

Managing Investment Portfolios: A Dynamic Process, Maginn & Tuttle, 3rd Edition, Ch. 5

Commentary on Question:

Commentary listed underneath question component.

Solution:

(a) Describe three advantages of using fixed-income ETFs in general and how you are using Smart-Bond ETF in this situation.

Commentary on Question:
Most candidates did well on this part. Some failed to address the second part of the question or didn't mention or describe cash equitization.

ETF provides access to a wider range of fixed income sector.
It has lower management cost.
It is transparent since the holdings are published daily.
It provides intraday liquidity.
It is tax efficient.
It provides investment breadth.
It has minimal counterparty risk.
It has lower tracking error when benchmarked to a broad index.

In this situation, I am using ETF for cash equitization by investing excess cash short-term to maintain market exposure.

(b) Describe three other investment strategies involving ETFs that you can use in managing this insurance company investment portfolio.

Commentary on Question:
Most candidates did poorly on this part. Some disregarded the investment policy that prohibits using derivatives.

12. Continued

1. Transition management
   - Maintain market exposure when restructuring portfolios to avoid performance gaps and excessive costs.

2. Tactical allocations
   - Move in and out of the market quickly to take advantage of a market opportunity.

3. Portfolio rebalancing
   - Access liquidity easily to maintain a strategic asset allocation without accessing the less-liquid portion of the portfolio.

(c) Evaluate the dealer's comment that "Smart-Bond ETF is a good choice since it has very good liquidity" based on the information given.

Commentary on Question:
Many candidates did well on this part. To receive full credit, candidates need to assess implications of the statements on the liquidity of the ETF.

ETFs develop their own independent exchange liquidity through the secondary market as trading volume grows and will tend to be more liquid than underlying bonds– the dealer is right about this. However, when the bond market is illiquid, the liquidity of fixed income ETFs will also be reduced. High creation cost also indicates low liquidity in the bond market, which leads to low liquidity of ETF. Furthermore, the difficulty of sourcing bonds to execute ETF trades indicates that the market is less liquid and less transparent.

(d) Explain how an arbitrage opportunity driven by strong supply of ETFs can be capitalized by:

(i) An authorized participant

(ii) An investor

Commentary on Question:
Candidates generally did well. Some candidates misstated that ETF prices will be high when supply is high.

(i) Authorized participant
During periods of strong supply, an AP could purchase ETF shares in the open market at a discount. Then he/she can deliver them to the ETF provider in exchange for the underlying bond holdings (i.e. redeem the shares) and sell the bonds at a net profit.

12. **Continued**

(ii) Investor
An investor can buy the ETF at a discount during strong supply period. Then the investor can short sell a basket of correlated securities or derivatives.

(e) Determine using mean-variance analysis whether you should invest in Smart-Bond ETF.

Commentary on Question:
Most candidates did well on this part.

Portfolio yield = (0.28*2.8% + 0.32*3.3% + 0.2*3.5% + 0.05*5%)
= 2.79%

Sharpe ratio of portfolio
= (3.28% - 1.50%)/1%
= 1.29

Portfolio Sharpe ratio * Correlation
= 1.29 x 0.6 = 0.77

ETF yield = 12*0.3% = 3.6%
Or 1.003^12 – 1 = 3.66%

Sharpe ratio of ETF
= (12*0.3% - 1.5%)/1.3%
= 1.62

Since ETF Sharpe ratio > Portfolio Sharpe ratio * Correlation, should invest in ETF.

(f) Describe two special issues, other than currency risk, that you should consider if you decide to invest in Global-Bond ETF in addition to Smart-Bond ETF.

Commentary on Question:
Most candidates received some credit on this part.

Increased correlations in time of stress
- Correlation across international markets tends to increase in times of market breaks or crashes.

12. Continued

Emerging market concerns
- E.g. political risk, sovereign risk, limited free float of shares, limitations on the amount of nondomestic ownership, quality of company information, non-normality of returns.

(There could be other acceptable answers not listed.)

13. Learning Objectives:

5. The candidate will understand and identify the variety of fixed instruments available for portfolio management. This section deals with fixed income securities. As the name implies the cash flow is often predictable, however there are various risks that affect cash flows of these instruments. In general the candidates should be able to identify the cash flow pattern and the factors affecting cash flow for commonly available fixed income securities. Candidates should also be comfortable using various interest rate risk quantification measures in the valuation and managing of investment portfolios. Candidates should also understand various strategies of managing the portfolio against given benchmark.

Learning Outcomes:

(5a) Demonstrate an understanding of par yield curves, sport curves, and forward curves and their relationship to traded security prices; and understanding of bootstrapping and interpolation.

(5f) Construct and manage portfolios of fixed income securities using the following broad categories.

- (i) Managing funds against a target return
- (ii) Managing funds against liabilities.

Sources:

Wilmott Chapter 14 : 14.7, Pages 327-329 and 14.16, Pages 339-346

Maginn & Tuttle Chapter 6 , Pages 346-351, 360-361, 365

Commentary on Question:

This question is meant to test candidates understanding and application of forward rates as well as ability to construct and critique immunization portfolio. Most candidates did relatively well.

Solution:

(a) Calculate the following to the nearest 1 basis point:

(i) $f(0,5)$

(ii) $f(2,5)$

(iii) Price of a 2-year, annual-coupon, Treasury bond [with par value of $100 and coupon of 5%] purchased a year from now assuming that spot rates in the future are those implied by today's forward rates.

13. **Continued**

Commentary on Question:
One possible solution is shown below. Alternative solutions are given full credit as well provided that the candidates demonstrated understanding of forward rates and calculated the rates correctly. For part (i), some candidates mixed up the concept for spot rate and that for YTM and incorrectly used YTM as the 3-year spot rate. For part (iii), some candidates did not calculate the price at time of purchase (i.e. at time 1).

First, calculate spot rates:
s1=f(0,1) = 2.5%
s2=[(1.025)(1+f(1,1))]^0.5 -1 = (1.025*1.037)^0.5 – 1
s2 = 3.0982%
100 = 3.4/(1+s1) + 3.4/(1+s2)^2 + 103.4/(1+s3)^3
f(0,3) = s3 = 3.4175%

(1+f(0,3))^3*(1+f(3,2))^2=(1+f(0,5))^5
(1+.034175)^3*(1+.045)^2=(1+f(0,5))^5
f(0,5)=3.8491%

(1+s2)^2 * (1+f(2,5))^5 = (1+s3)^3 * (1+f(3,4))^4

f(2,5) = 4.8111%

Let P(1,3) be the price of Treasury at time 1
(1+s1) * (1+f(1,2))^2 = (1+s3)^3
f(1,2) = 3.8793%

$$P(1,3) = \frac{5}{1.037} + \frac{105}{(1.038793\text{^}2)}$$

P(1,3) = 102.1257

(b) Construct an immunizing portfolio involving only Bonds 1 and 2.

Commentary on Question:
Most candidates realized that an immunizing portfolio should match the duration of the liability. Not many candidates realized that the immunizing portfolio should match the market value of the liability as well. Some candidates did not discount the liability to obtain present value.

Realizing that immunization strategy requires selecting securities such that both the portfolio amount (market value) and porfolio duration match those of liabilities.

13. Continued

Realizing that bond 1 and 2 have duration shorter and longer than that of the obligation respectively, making it possible to construct a portfolio with these 2 bonds

Assume amounts invested in bond 1 and 2 are V1 and V2 respectively.
D1 and D2 are durations of bond 1 and 2 respectively.

Present value of liability = 1000000/(1+f(0,5))^5=827,914
Modified duration of liability = 5/(1+ f(0,5)) = 4.815

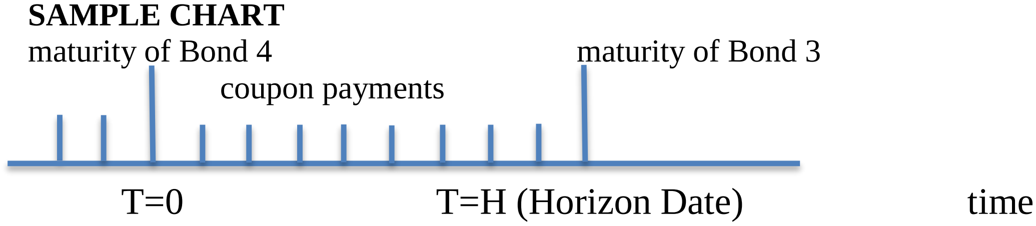
V1 + V2 = 827,914
D1 * V1 + D2 * V2 = 4.815* 827,914

Solve the equations above, get
V1 = 536,467; V2 = 291,447

(c) Illustrate cash flows of Portfolio I and Portfolio II in separate cash-flow charts. The charts do not need to be perfectly scaled.

Commentary on Question:
Most candidates did well with this part of the question.

SAMPLE CHART



mark out coupon payments of Portfolio 1
mark out coupon payments of Portfolio 2
mark out maturities of Portfolio 1
mark out maturities of Portfolio 2

(d) Identify the strategy of each of the two portfolios based on the cash flow charts and explain which portfolio provides better immunization when interest rates move in a parallel way.

Commentary on Question:
Many candidates did not realize that both barbell and bullet portfolios are immune to parallel interest rate shifts.

13. Continued

Portfolio II is called a barbell (high-risk) portfolio, while Portfolio I is called a bullet portfolio (low-risk).
If both portfolios have the duration equal to the horizon length, both portfolios are immune to parallel rate changes.

(e) Explain which portfolio provides better immunization under such scenario.

Commentary on Question:
Most candidates recognized that bullet portfolio provides better immunization since barbell portfolio had higher reinvestment risk. However, some candidates did not address that barbell portfolio had more outstanding assets at the end of the investment horizon.

Bullet portfolio provides better immunization under such scenario
As Company A expects short rate will decline and long rate will go up, the decline would be substantially higher for the barbell portfolio. i.e. Barbell portfolio has a higher reinvestment risk.
Barbell Portfolio experiences the lower reinvestment rates longer than the bullet portfolio does.
More of Portfolio is still outstanding at the end of the investment horizon, which means that the same rate increase causes much more of a capital loss.

14. Learning Objectives:

6. The candidate will understand the variety of equity investments and strategies available for portfolio management.

Learning Outcomes:

(6a) Explain the nature and role of equity investments within portfolios that may include other asset classes.

(6c) explain the basic active equity selection strategies including value, growth and combination approaches.

Sources:

Chapter 7 Equity Portfolio Management in *Managing Investment Portfolios*

Commentary on Question:

The candidates did well in general, especially in part b) where they had to identify the investment style of the funds. However, in part c), the results are mixed and comments will be made in that section.

Solution:

(a) Outline the benefits of including equities in their portfolio.

- Investing in equities offers a diversification benefit to the other investment options;
- Equity is considered as an inflation hedge because its returns are sufficient to preserve purchasing power;
- Equities have high historical long term real rates of return play a growth role in the portfolio;
- Potential dividend payments;
- Easy to invest, and track.

(b) Identify the investment styles for Portfolios A and B.

Portfolio A is a value portfolio.

<u>Justification:</u>

- low P/E ratio (ratio below 10);
- low P/B Ratio;
- high dividend yield;
- large exposure to utilities, relatively low exposure to IT and health care.

Portfolio B is a growth portfolio.

<u>Justification:</u>

- high P/E ratio (ratio above 20);
- high P/B ratio;
- low dividend yield;
- high exposure to health care, relatively low exposure to finance and utilities.

14. Continued

(c) Recommend appropriate portfolios for your two clients, respectively, from Portfolios A, B, and C.

Commentary on Question:
The scores are not as high as expected. Many candidates did recommend a combination of the funds or identified an inappropriate choice.

The choices of the appropriate fund had to be mostly based on the investment style of the fund and the conditions of the market for Bob (value or growth, bull market following a period of contraction), including the particular characteristics of the investors and their objectives for both Bob and Sue.

Bob – Portfolio A – a value portfolio. All else being equal, value stocks are relatively cheap coming out of a recession and typically tend to outperform growth stocks post-recession. As a result, in order for the individual investor to maximize their return, a value investment strategy would be most optimal.

Sue – Portfolio C - a fixed income portfolio – Active equity selection would be outside her risk tolerance and as such would not be consistent with her IPS. It would be more beneficial to invest in a fixed income fund to have guaranteed return to meet her debt payments rather than the additional risk for investment.

15. Learning Objectives:

7. The candidate will understand how to develop an investment policy including governance for institutional investors and financial intermediaries.

Learning Outcomes:

(7a) Explain how investment policies and strategies can manage risk and create value.

(7c) Determine how a client's objectives, needs and constraints affect investment strategy and portfolio construction. Include capital, funding objectives, risk appetite and risk-return trade-off, tax, accounting considerations and constraints such as regulators, rating agencies, and liquidity.

Sources:

Managing Investment Portfolios: A Dynamic Process, Maginn & Tuttle, 3rd Edition, 2007 Chapter 1 and Chapter 3

Commentary on Question:

This question is to test the candidate's understanding of Investment Policy Statement. The candidate should be able to analyze the individual component within IPS given certain situations.

Solution:

(a) Assess the following by describing the factors to be considered.

(i) Risk and return objectives.

(ii) Liquidity needs and time horizon.

Commentary on Question:

Most of the candidates listed the factors to be considered for risk and return objectives, liquidity needs and time horizon. However, some candidates did not assess them based on pension plan described, for example, whether the risk tolerance & liquidity needs are high/low, or whether the time horizon is short/long after considering the factors.

(i)

Risk tolerance is high because

a) The average age of the worker is 20 years from retirement.
b) The fund is 100% funded.

Return objective

c) To adequately fund pension liability on an inflation adjusted basis.
d) To provide steady pension income for the company.
e) Maximize return given the stated risk objective.

15. Continued

(ii)
Liquidity requirement is low because
a) Company and member are actively contributing
b) Younger workforce
c) The fund is 100% funded.

Time horizon is long given the younger workforce and the high funding level.

(b) Assess the appropriateness of the following investments against your IPS:

(i) Invest in US Treasury bonds that mature in 1 to 3 years.

(ii) Invest in an ETF that actively invests in oil companies.

(iii) Invest in an illiquid long duration fixed income fund that focuses on renewable energy.

Commentary on Question:
Full credits were given if the candidate clearly stated the risk, return, liquidity and time horizon associated with each investment. Overall, candidate did well on this question.

i) Short term US Treasury bond is not appropriate because:
a) Risk is low but this is not a concern for the plan.
b) Expected return is too low relative to other options.
c) Liquidity is high but this is not a big concern for the plan.
d) Maturity of the bonds is much shorter than the investment horizon of the fund.

ii) An ETF that actively invests in oil companies is not appropriate because:
a) Risk is too high due to the correlation between the energy sector and the company's business operation.
b) Return maybe high, but expected return is correlated with company's ability to contribute to the fund.
c) Liquidity is high but this is not a big concern for the plan.
d) Time horizon of the plan is long, so equity investment is appropriate.

iii) A long duration fixed income fund that invests in green energy is appropriate because:
a) Risk is high but within risk tolerance. In addition there maybe diversification effect due to low or negative correlation with the company's business operation.
b) Expected return is high.
c) Liquidity is moderate, but the liquidity need of the DB plan is also low.
d) Duration of the fund is long, which is good because duration of the liability is also long.

16. Learning Objectives:

8. The candidate will understand the theory and techniques of portfolio asset allocation.

Learning Outcomes:

(8a) Explain the impact of asset allocation, relative to various investor goals and constraints.

(8b) Propose and critique asset allocation strategies.

Sources:

Managing Investment Portfolios (Chapter 5)

QFIC-112-16 Risk-Factors as Building Blocks for Portfolio Diversification: The Chemistry of Asset Allocation

QFIC-111-16 Stop Playing with Your Optimizer

Commentary on Question:

This question is to test candidates' understanding of portfolio optimization and asset allocation strategies. To receive full credits, candidates were required to correctly find the optimal portfolio mix, as well as analyze the portfolio based on different scenarios.

Solution:

(a) Calculate X, Y, and Z for Corner Portfolio C within the table.

Commentary on Question:

Most candidates are able to find the X,Y,Z values; some candidates have omitted the correlation when finding Y.

X: ER(P) = 65% * 11% + 5% * 9% + 30% * 7% = 9.7%

Y: Var(P) =

65%^2*20%^2+5%^2*18%^2+30%^2*14%^2+2*(65%*5%*20%*18%*0.8+65%*30%*20%*14%*0.35+5%*30%*18%*14%*0.35) =2.47%,

So Std Dev (P3) = 2.47%^0.5 = 15.7%

Z: SR (P) = (9.7% - 2%)/15.7% = 49%

(b) Determine, using the linear interpolation between the corner portfolios, the strategic asset allocation (weights for each of the 5 asset classes) that is most appropriate for Tom based on the traditional MVO analysis.

Commentary on Question:

Most candidates calculated correctly and found the optimal portfolio mix. A small portion of candidates mistakenly used the mix from Corner portfolio of B and D instead of using C and D.

16. Continued

The required return of the asset portfolio is 8%, which lies in corner portfolio C and D, set the weight for Portfolio C as W:

8% = W * 9.7% + (1 – W) * 7.9%, W = 5.56%

Then

Weight for Asset Class 1 = 65% * 5.56% + 35% * 94.44% = 36.67%
Weight for Asset Class 2 = 5%
Weight for Asset Class 3 = 0%
Weight for Asset Class 4 = 65% * 5.56% + 35% * 94.44% = 28.33%
Weight for Asset Class 5 = 30%

(c) Justify the asset allocation chosen in part (b), based on each of the following perspectives:

(i) Investment Objectives

(ii) Investment Constraints

(iii) Other information Tom provided

Commentary on Question:
Most candidates did well in answering (i). Few candidates mentioned the Investment constraint of one-period evaluation in (ii). Only a small portion of the candidates answered question (iii) correctly.

(i). Investment Objectives
- Return: expected return is 8% and lies on the efficient frontier, which meets the return requirement
- Risk: the return deviation would be smaller than linear approximation of corner portfolio 3 and 4 (e.g. 5.56% * 15.7% + 94.44% * 11.6% = 11.85% < 14%), so it meets the risk requirement.

(ii). Investment Constraints
- Time horizon: the MVO analysis is a one-period evaluation; Once Tom retires, the asset allocation will need to consider the retirement payment outflow;
- Tax consideration: the MVO analysis did not address impact from tax.

(iii). Other information
- Borrowing: No borrowing is involved in the optimization (no negative weighting of any asset class), which meets the requirement;
- Risk adjusted performance: the sharp ratio increases from corner portfolio 1 to portfolio 5, while the return is decreasing moving from portfolio 1 to 5. Since the optimal asset allocation is between portfolio 3 and 4, it generates the highest sharp ratio at the required return level of 8%.

16. Continued

(d) Critique the traditional MVO asset allocation method and propose another method under the following circumstances:

(i) Tom is planning for early retirement next year and will start withdrawing retirement payments from the fund.

(ii) Tom is no longer confident about his market expectations, especially the expected returns.

Commentary on Question:
Only a small portion of candidates received full credits for this question, which requires a comprehensive analysis based on different scenarios.

(i) The current MVO method is focus on asset only and did not consider liability, so it does not reflect the future retirement payment liability. In addition, the MVO is for 1-perid only, which does not reflect rebalancing.

Recommend using the Asset Liability Management (ALM) method, as it reflect the liability through funding ratio and net worth, and focus on the surplus efficient frontier. (Can also combine ALM with Monte Carlo simulation to reflect long term horizon).

(ii) The current MVO method is very sensitive to accuracy of inputs, especially expected return; asset allocation optimization would be very different if the expected return from Tom is changed. Furthermore, under MVO method, extreme asset allocation happens and there is risk to allocate a large weight on certain asset class (optimization recommends allocating large weights on asset classes 1, 4 and 5 but little weights on classes 2 and 3).

Recommend using the Black-Litterman method, as it back solves the expected return based on market capitalization weights (instead of using Tom's expectation). Under BL method, extreme views from investors hold could also be eliminated (avoid extreme weighting in certain asset class (more diversified).

(e) Recommend a change in asset allocation after considering Tom's current human capital.

Commentary on Question:
Most candidates were only able to answer that the human capital level is low, but could not see that human capital is bond like characteristic and thus provided wrong recommendations.

16. Continued

Tom is retiring in 5 years, and he has accumulated a significant amount of financial asset ready for retirement. Therefore, Tom's human capital is small while financial capital is large.
Since human capital is bond like characteristic, including human capital would shift the optimal asset allocation to more equity weighting, for diversification purpose. However, Tom only has 5 years left to retirement, his human capital is small so the shift of weighting to equity should be small.
Therefore, the asset allocation would put slightly more weight on asset class 1 and 2, and less weight on asset class 4 and 5, compared to the optimal weight in b) above.