SOCIETY OF ACTUARIES®

Quantitative Finance and Investment Core

Exam QFICORE

MORNING SESSION

Date: Wednesday, October 31, 2018
Time: 8:30 a.m. – 11:45 a.m.

INSTRUCTIONS TO CANDIDATES

General Instructions

1. This examination has a total of 100 points. It consists of a morning session (worth 60 points) and an afternoon session (worth 40 points).

    a) The morning session consists of 9 questions numbered 1 through 9.

    b) The afternoon session consists of 7 questions numbered 10 through 16.

    The points for each question are indicated at the beginning of the question.

2. Failure to stop writing after time is called will result in the disqualification of your answers or further disciplinary action.

3. While every attempt is made to avoid defective questions, sometimes they do occur. If you believe a question is defective, the supervisor or proctor cannot give you any guidance beyond the instructions on the exam booklet.

Written-Answer Instructions

1. Write your candidate number at the top of each sheet. Your name must not appear.

2. Write on only one side of a sheet. Start each question on a fresh sheet. On each sheet, write the number of the question that you are answering. Do not answer more than one question on a single sheet.

3. The answer should be confined to the question as set.

4. When you are asked to calculate, show all your work including any applicable formulas. When you are asked to recommend, provide proper justification supporting your recommendation.

5. When you finish, insert all your written-answer sheets into the Essay Answer Envelope. Be sure to hand in all your answer sheets because they cannot be accepted later. Seal the envelope and write your candidate number in the space provided on the outside of the envelope. Check the appropriate box to indicate morning or afternoon session for Exam QFICORE.

6. Be sure your written-answer envelope is signed because if it is not, your examination will not be graded.

Recognized by the Canadian Institute of Actuaries.

Tournez le cahier d'examen pour la version française.

1. (*6 points*) Let W_t be a standard Brownian motion.

(a) (*1.5 points*) Show that $E\left(W_s W_t W_u\right) = 0$ for $s \le t \le u$.

Let $M_t = \int_0^t v \, dW_v$ for $t \ge 0$.

(b) (*1.5 points*) Show that $Cov\left(M_s, M_t\right) = \dfrac{s^3}{3}$ for $s \le t$.

(c) (*2 points*) Show that $E\left(e^{\theta M_t}\right) = e^{\frac{\theta^2}{2}\left(\frac{t^3}{3}\right)}$ for any given constant θ.

Therefore M_t has the same moment generating function as $N\left(0, \dfrac{t^3}{3}\right)$ and thus

$$M_t \sim N\left(0, \frac{t^3}{3}\right).$$

Define
$$X_t = \begin{cases} 0 & \text{if } t = 0 \\ \dfrac{\sqrt{3}}{t} M_t & \text{if } t > 0. \end{cases}$$

Then
$$X_t \sim N(0, t).$$

(d) (*1 point*) Show that X_t is actually not a Brownian motion.
Hint: Derive $Var\left(X_{t+u} - X_t\right)$ using the result of part (b).

2. (*5 points*) Let W_t be a standard Wiener process and t denote the time.

    (a)   (*0.5 points*) State the conditions for a process to be a martingale.

    (b)   (*1 point*) Show that W_t^2 is not a martingale by identifying a condition in part (a) that is not satisfied.

    (c)   (*1 point*) Determine a deterministic function $f(t)$ such that $W_t^2 - f(t)$ is a martingale.

Let $X_t = t^2 W_t - 2\int_0^t u W_u \, du.$

    (d)   (*2.5 points*) Show that X_t is a martingale by verifying that all the conditions in part (a) hold.

3. (*8 points*) Assume that the stock price S_t follows the following stochastic differential equation:

$$dS_t = \mu S_t dt + \sigma S_t dW_t$$

where μ, σ are positive constants, and W_t is a standard Wiener process under a real-world measure $\mathbb{P}$.

Given $t_0 > 0$, your colleague states that for $t > t_0$

$$\ln\left(S_t\right) = \ln\left(S_{t_0}\right) + \left(r - \frac{\sigma^2}{2}\right)(t - t_0) + \sigma\left(\widetilde{W}_t - \widetilde{W}_{t_0}\right)$$

where

- r is the constant risk-free interest rate.
- $\widetilde{W}_t$ is a standard Wiener process under the risk-neutral measure $\mathbb{Q}$ equivalent to $\mathbb{P}$.

(a) (*2 points*) Show that your colleague's statement is correct.

Hint: Express $\widetilde{W}_t = W_t - X_t$ and determine X_t.

For $t > t_0$, denote by $E_{t_0}^{\mathbb{Q}}\left[\int_{t_0}^t (\widetilde{W}_s - \widetilde{W}_{t_0})ds\right]$ and $Var_{t_0}^{\mathbb{Q}}\left[\int_{t_0}^t (\widetilde{W}_s - \widetilde{W}_{t_0})ds\right]$ the expectation and the variance of $\int_{t_0}^t (\widetilde{W}_s - \widetilde{W}_{t_0})ds$ at time t_0, respectively.

(b) (*0.5 points*) Calculate $E_{t_0}^{\mathbb{Q}}\left[\int_{t_0}^t (\widetilde{W}_s - \widetilde{W}_{t_0})ds\right]$.

(c) (*2.5 points*) Show that

$$Var_{t_0}^{\mathbb{Q}}\left[\int_{t_0}^t (\widetilde{W}_s - \widetilde{W}_{t_0})ds\right] = \frac{(t - t_0)^3}{3}$$

3. **Continued**

Let

$$G_t = e^{\frac{1}{t}\int_0^t \ln S_u \, du} \quad \text{for } t > 0 \text{ and } G_0 = S_0$$

(d)   (*0.5 points*) Show that for $t > t_0$,

$$\ln G_t = \frac{t_0}{t} \ln G_{t_0} + \frac{1}{t}\int_{t_0}^t \ln S_u \, du$$

(e)   (*1 point*) Show that the distribution $\ln G_t$ for $t > t_0$ given information up through time t_0 under the measure $\mathbb{Q}$ is:

$$\ln G_t \sim N\left(\frac{t_0}{t}\ln G_{t_0} + \frac{t-t_0}{t}\ln S_{t_0} + \left(r - \frac{\sigma^2}{2}\right)\frac{(t-t_0)^2}{2t}, \frac{(t-t_0)^3 \sigma^2}{3t^2}\right)$$

Now consider a forward contract on G_t struck at some level K. This contract has a payoff at maturity $T > 0$:

$$F_T = G_T - K$$

(f)   (*1.5 points*) Derive K such that the forward contract has zero value at time 0.

4. (*9 points*) In the Ho-Lee model the risk neutral process for the short rate r_t is as follows:

$$dr_t = \theta_t dt + \sigma dW_t$$

where θ_t is deterministic and time-dependent, σ is constant, and W_t is a standard Wiener process under the risk-neutral measure $\mathbb{Q}$.

(a) (*1.5 points*) Show that for $0 \le t \le T$

$$\int_t^T r_s ds = (T-t)r_t + \int_t^T (T-s)\theta_s ds + \sigma \int_t^T (W_s - W_t) ds$$

Let $E_t^{\mathbb{Q}}[\bullet]$ be the expectation at time t under the measure $\mathbb{Q}$.

Consider the risk-free zero-coupon bond maturing at time T. Let $Z(t,T) = E_t^{\mathbb{Q}}\left[e^{-\int_t^T r_s ds} \right]$

be its price at time $t \le T$.

(b) (*1 point*) Derive the following formula using the result of part (a):

$$Z(t,T) = e^{A(t,T) - r_t(T-t)}$$

where

$$A(t,T) = -\int_t^T \theta_s (T-s) ds + \frac{1}{6}\sigma^2 (T-t)^3$$

Hint: The variance of $\int_t^T (W_s - W_t) ds$ at time t equals $\frac{1}{3}(T-t)^3$.

(c) (*1.5 points*) Show that

$$E_0^{\mathbb{Q}}\left[Z(t,T) \right] = e^{-\left[r_0(T-t) + \int_0^T \theta_s(T-s)ds - \int_0^t \theta_s(t-s)ds \right] + \frac{1}{6}(T-t)^2(T+2t)}$$

4. **Continued**

Let $f(t,T_1,T_2)$ be the forward rate from time T_1 to T_2 observed at time $t \geq 0$.

(d)　　(*2 points*)　Show that $df(t,T_1,T_2) = \frac{1}{2}(T_1 + T_2 - 2t)\sigma^2 dt + \sigma dW_t$.

Let $F = f(0,T_1,T_2)$, and $\bar{F} = E_0^{\mathbb{Q}}\left[f(T_1,T_1,T_2) \right]$ the expected spot rate from T_1 to T_2 as of time 0.

(e)　　(*1.5 points*)　Derive the convexity adjustment $\bar{F} - F$ using the result of part (d).

Assume the short rate now follows a model other than Ho-Lee.

(f)　　(*1.5 points*)　Determine and justify whether $E_0^{\mathbb{Q}}\left[Z(T_1,T_2) \right]$ is larger or smaller than $e^{-\bar{F}(T_2 - T_1)}$.

5. (*6 points*) You manage a portfolio of variable annuities invested in the Russell 2000 Index. The variable annuities have a minimum guarantee that protects the initial deposit amount. The Chief Economist at your company expects upcoming turbulence in the market and expects a drop of up to 30%. You are asked to hedge the minimum guarantee using options based on the classical Black-Scholes model.

Assume the following:

- Risk-free interest rate 5% per annum
- There is no dividend yield on underlying portfolio
- The current Russell 2000 Index level is 100
- Russell 2000 Index implied volatility σ is constant at 24% per annum
- The multiplier for an option contract on the Russell 2000 Index is 1000
- The time to maturity of options related to the Russell 2000 Index is 1 year
  There are 250 days in 1 year.

The delta of the variable annuity portfolio is $2.4 million.

(a)   (*1 point*) Construct a static hedge for the portfolio using a bear put spread and justify your strike levels.

(b)   (*2 points*) Calculate the number of options on the Russell 2000 Index needed to implement the bear put spread static hedge in part (a) that would match the delta of the liability using the Black-Scholes model.

(c)   (*0.5 points*) Critique the delta-neutral approach used in part (b) and suggest an alternative.

Due to the slow decision-making process, one day has elapsed. You are given the following for the bear put spread:

- Yesterday's market price was $6.72.
- Yesterday's implied volatility (standard deviation) was 24% per annum.
- Yesterday's Vega was 30.11.
- Today's market price is $6.93.

(d)   (*1 point*) Estimate today's implied volatility (standard deviation) of the Russell 2000 Index underlying the bear put spread based on the information given above.

5. **Continued**

Based on historical data, the trader estimates the actual volatility (standard deviation) of the Russell 2000 Index to be 28%.

You are delta hedging the bear put spread with today's implied volatility (standard deviation) estimated in part (d) above and assume the Gamma of the spread is 0.0125.

    (e)   (*1.5 points*) Determine the one-day mark-to-market profit or loss on your delta-hedged portfolio of one contract of bear put spread based on historical data,

6. (*6 points*) BNR sells a two-year equity-linked investment guarantee product. Its guarantee can be considered as a short two-year European put option on an underlying equity with the guaranteed level as the strike price K. The liability may be evaluated using the Black-Scholes option pricing formula.

BNR uses a replicating portfolio to hedge the equity risk of its liability. The portfolio consists of the underlying equity index and a risk-free asset. The replicating portfolio is rebalanced every month.

The parameters for the Black-Scholes option value calculation are given below:

- Risk-free rate $\quad r = 6\%$ (annual continuous rate)
- Dividend yield $\quad \delta = 0\%$

At time 0:

- $S_0 \qquad = 100$
- $K \qquad = 100$
- $N(d_1) \qquad = 0.7142$
- $N(d_2) \qquad = 0.6114$

(a) (*1 point*) Calculate the implied volatility for the put option.

(b) (*2 points*) Calculate the amounts invested in the risk-free asset and the equity index in the replicating portfolio at time 0.

After one month, the equity index value is 99.57.

(c) (*1 point*) Calculate the value of the hedging portfolio right before rebalancing.

The hedge error can be defined as the difference between the value of liability and the value of hedging portfolio before rebalancing.

(d) (*1.5 points*) Calculate the hedge error assuming that volatility of the stock and the risk-free interest rates have not changed.

(e) (*0.5 points*) Explain two possible sources of the hedge error.

THIS PAGE INTENTIONALLY LEFT BLANK

7. (*8 points*) Assume that the spot rate $r = r(t)$ follows a stochastic differential equation (SDE):

$$dr = \mu dt + \sigma dW_t,$$

where $\mu = \mu(r,t)$ and $\sigma = \sigma(r,t)$ are functions of r and t such that $\sigma > 0$, and W_t is a standard Wiener process under the measure $\mathbb{P}$.

Let $V = V(r,t;T)$ be the price at time t of a zero-coupon bond of \$1 maturing at time $T (T \geq t)$.

Consider $\zeta(r,t,T) = \dfrac{\dfrac{\partial V}{\partial t} + \dfrac{1}{2}\sigma^2 \dfrac{\partial^2 V}{\partial r^2} - rV}{\dfrac{\partial V}{\partial r}}$.

(a) (*2.5 points*) Show that $\zeta = \zeta(r,t,T)$ is independent of the maturity T by constructing a self-financing and instantaneously risk-free portfolio with Δ_1 units of a bond maturing at time T_1 and Δ_2 units of a bond maturing at time T_2.

Thus we have

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 \frac{\partial^2 V}{\partial r^2} + (\mu - \lambda\sigma)\frac{\partial V}{\partial r} - rV = 0,$$

where $\lambda = \lambda(r,t)$ is a function of r and t defined by $\lambda = \dfrac{\mu + \zeta}{\sigma}$.

Let $Z = XV$ where $X = \exp(-\int_0^t r(s)ds)$.

(b) (*1.5 points*) Show by using Ito's Lemma that $dZ = \sigma X \dfrac{\partial V}{\partial r}(\lambda dt + dW_t)$.

7. **Continued**

Denote by $E_t^{\mathbb{Q}}[\bullet]$ the expectation at time t under a measure $\mathbb{Q}$.

(c) (*1.5 points*) Show by using Girsanov's theorem that for an appropriate measure $\mathbb{Q}$ equivalent to $\mathbb{P}$:

$$V(r,t;T) = E_t^{\mathbb{Q}}[\exp(-\int_t^T r(s)\,ds)].$$

(d) (*1 point*) Show that $dr = -\zeta dt + \sigma dW_t^{\mathbb{Q}}$ where $W_t^{\mathbb{Q}}$ is a standard Wiener process under the measure $\mathbb{Q}$.

For a specific choice of parameterization of $r(t)$ you are given

$$V(r,t;T) = \exp\big(A(t,T) - B(t,T)*r\big),$$

where

$$A(t,T) = -\frac{\alpha(T-t)^2}{2} + \frac{\beta^2}{6}(T-t)^3$$

for some constants α and β, and

$$B(t,T) = T - t.$$

(e) (*1.5 points*) Determine the drift and the diffusion of the SDE for $r(t)$ under the measure $\mathbb{Q}$ in terms of α and β.

8. *(7 points)*

(a) *(1.5 points)* Describe the properties of the following two models for interest rates:

    (i)   The one-factor Vasicek model

    (ii)   The Cox-Ingersoll-Ross model

The Vasicek term structure model is described by the following stochastic differential equation (SDE):

$$dr(t) = \alpha(\theta - r(t))dt + \sigma dW_t$$

where $\alpha, \theta, \sigma > 0$ are constant and W_t is a Brownian motion.

(b) *(1 point)* Show that the solution to the SDE is

$$r(t) = \theta + (r(0) - \theta)e^{-\alpha t} + \sigma e^{-\alpha t} \int_0^t e^{\alpha s} dW_s$$

Next consider the corresponding Cox-Ingersoll-Ross model below:

$$dr_t = \alpha(\theta - r_t)dt + \sigma\sqrt{r_t}\, dW_t.$$

It can be shown as in part (b) that the solution to the SDE satisfies the following:

$$r_t = \theta + (r_0 - \theta)e^{-\alpha t} + \sigma e^{-\alpha t} \int_0^t e^{\alpha s} \sqrt{r_s}\, dW_s$$

Now you are to compare the solutions $r(t)$ and r_t in terms of their expectations and variances.

(c) *(1 point)* Determine the expectations of $r(t)$ and r_t, respectively.

(d) *(2.5 points)* Determine the variances of $r(t)$ and r_t, respectively.

(e) *(1 point)* Show that the expectations and the variances in parts (c) and (d) have the following limits as $t \to \infty$

    (i) $\lim_{t\to\infty} E[r(t)] = \lim_{t\to\infty} E[r_t] = \theta$

(ii) $\displaystyle\lim_{t\to\infty}Var\big[r(t)\big]=\frac{\sigma^2}{2\alpha}$ and $\displaystyle\lim_{t\to\infty}Var\big[r_t\big]=\frac{\sigma^2}{2\alpha}\theta$

9. (*5 points*) The following shows the ABC company's daily stock price data:

Day (t)	1	2	3	4	...
ABC Stock Price (P_t)	$10.0	$10.4	$9.8	$10.3	...

Using the maximum likelihood method and some manual steps in compiling the data, your assistant estimated a GARCH(1,1) model for the volatility of ABC stock price as below:

$$u_t = \frac{P_t}{P_{t-1}} - 1, \qquad t = 2,3,\ldots$$

$$\sigma_3^2 = u_2^2,$$

$$\sigma_t^2 = w + 0.25u_{t-1}^2 + 0.82\sigma_{t-1}^2, \qquad t = 4,5,\ldots$$

(a) (*0.5 points*) Identify a potential issue with the above model.

You confirmed that the above GARCH(1,1) model was estimated correctly.

(b) (*1 point*) Propose a simpler alternative model that addresses the issues you identified in part (a) and outline the pros and cons of your proposed model vs. the original model.

(c) (*2 points*) Determine the parameters of your proposed model in part (b) by using the maximum likelihood method and the available stock price data.

Assume that there are 252 trading days per year.

(d) (*0.5 points*) Calculate the forecasted Day 8 annualized volatility as of Day 4 using the model you established in part (c).

(e) (*1 point*) Compare and contrast your model to IGARCH(1,1) model.

END OF EXAMINATION
Morning Session

USE THIS PAGE FOR YOUR SCRATCH WORK