QFI CORE Model Solutions
Spring 2017

1. **Learning Objectives:**

1. The candidate will understand the fundamentals of stochastic calculus as they apply to option pricing.

Learning Outcomes:

(1c) Understand Ito integral and stochastic differential equations.

(1d) Understand and apply Ito's Lemma.

(1f) Demonstrate understanding of option pricing techniques and theory for equity and interest rate derivatives.

(1g) Demonstrate understanding of the differences and implications of real-world versus risk-neutral probability measures.

(1h) Define and apply the concepts of martingale, market price of risk and measures in single and multiple state variable contexts.

(1i) Understand and apply Girsanov's theorem in changing measures.

Sources:

Problems and Solutions Math Finance, Chin et al p. 131

Commentary on Question:

Overall the candidates did part (a) well. Most did not fully reduce the equation to a single Brownian process and thus lost some marks. Candidates struggled with part (b) in fully proving the martingale property. Most understood the basic concepts of Girsanov's theorem and explained in words, which received partial marks. Due to the fact that candidates weren't able to fully solve part (b), their answers in part (c) and part (d) were impacted. Partial marks were given in these subsequent parts as long as the candidate explained their assumptions and were on the correct solution path.

The question aims to test the candidate's understanding of Ito's lemma, and change of measure (Girsanov's theorem).

1. Continued

Solution:

(a) Express the diffusion process of Z_t as $\frac{dZ_t}{Z_t} = \mu dt + \sigma dW_t$ where W_t is a standard Wiener process under $\mathbb{P}$ by deriving μ, σ, and W_t in terms of $\rho, \mu_1, \mu_2, \sigma_1, \sigma_2, W_{1,t}$, and $W_{2,t}$.

Commentary on Question:
Candidates did this part well. But most left the answer at $\sigma_1 dW_{1,t} + \sigma_2 dW_{2,t}$, without actually solving for the single Brownian process, and thus lost some marks.

Using the product rule (Ito's Lemma)

$$dZ_t = S_{1,t}dS_{2,t} + S_{2,t}dS_{1,t} + dS_{1,t}dS_{2,t}$$

$$dZ_t = S_{1,t}\left(S_{2,t}\mu_2 dt + S_{2,t}\sigma_2 dW_{2,t}\right) + S_{2,t}\left(S_{1,t}\mu_1 dt + S_{1,t}\sigma_1 dW_{1,t}\right) + S_{1,t}S_{2,t}\rho\sigma_1\sigma_2 dt$$

$$\frac{dZ_t}{Z_t} = (\mu_1 + \mu_2 + \rho\sigma_1\sigma_2)dt + \sigma_1 dW_{1,t} + \sigma_2 dW_{2,t}$$

Since $\left(\sigma_1 dW_{1,t} + \sigma_2 dW_{2,t}\right)^2 = \left(\sigma_1^2 + \sigma_2^2 + 2\sigma_1\sigma_2\rho\right)dt$ and $\sigma_1 dW_{1,t} + \sigma_2 dW_{2,t}$ is normally distributed random variable with zero mean, we can construct another $\mathbb{P}$ standard Weiner processes, W_t, such that $\sigma_1 dW_{1,t} + \sigma_2 dW_{2,t} = \sigma dW_t$, where $\sigma = \sqrt{\sigma_1^2 + \sigma_2^2 + 2\sigma_1\sigma_2\rho}$.
Now write $\mu = \mu_1 + \mu_2 + \rho\sigma_1\sigma_2$, hence $\frac{dZ_t}{Z_t} = \mu dt + \sigma dW_t$.

(b) Show, using Girsanov's theorem, that by changing the measure $\mathbb{P}$ to an equivalent risk-neutral measure $\mathbb{Q}$, $e^{-rt}\Pi_t$ is a $\mathbb{Q}$-martingale.

Commentary on Question:
Most candidates did poorly on this question. The candidates did not have a clear idea what the final equation should reduce to and thus some were lost in all the algebra. Most candidates did try to explain in words their thought process and got partial marks for conceptually describing the steps. However, we were looking for the actual solution in a concise mathematical equation.

1. Continued

At time t, we let the value of self-financing portfolio Π_t to be

$$\Pi_t = \phi_t Z_t + \psi_t B_t$$
$$d\Pi_t = \phi_t dZ_t + \psi_t dB_t$$
$$d\Pi_t = \phi_t Z_t(\mu dt + \sigma dW_t) + \psi_t r B_t dt$$
$$d\Pi_t = \phi_t Z_t(\mu dt + \sigma dW_t) + r(\Pi_t - \phi_t Z_t)dt$$
$$d\Pi_t = r\Pi_t dt + \sigma\phi_t Z_t\left(\frac{\mu - r}{\sigma}dt + dW_t\right)$$
$$d(\mathrm{e}^{-rt}\Pi_t) = -r\mathrm{e}^{-rt}\Pi_t dt + \mathrm{e}^{-rt}d\Pi_t$$
$$d(\mathrm{e}^{-rt}\Pi_t) = \mathrm{e}^{-rt}\sigma\phi_t Z_t\left(\frac{\mu - r}{\sigma}dt + dW_t\right)$$

write

$$d\widetilde{W}_t = dW_t + \frac{\mu - r}{\sigma}dt$$

By applying Girsanov's theorem to change the measure $\mathbb{P}$ to an equivalent risk-neutral measure $\mathbb{Q}$, under which $\widetilde{W}_t = W_t + \frac{\mu - r}{\sigma}t$ is a $\mathbb{Q}$ standard Wiener process, then the discounted portfolio $e^{-rt}\Pi_t$ is a $\mathbb{Q}$ martingale.

(c) Find the diffusion process for Z_t under the measure $\mathbb{Q}$.

Commentary on Question:
Part (c) is very straight-forward if the candidate got part (b). However, since most candidate did not completely solve for part (b), not a lot of candidates did well here. There are some candidates that received full marks even though they did not finish part (b). As long as the candidate clearly stated their assumptions, we did not deduct marks for carry-over mistakes from part (b).

Substituting $d\widetilde{W}_t = dW_t + \frac{\mu - r}{\sigma}dt$ in $\frac{dZ_t}{Z_t} = \mu dt + \sigma dW_t$,

$$\frac{dZ_t}{Z_t} = rdt + \sigma d\widetilde{W}_t$$

(d) Calculate the value of the derivative at time t.

Commentary on Question:
Most candidate left this question blank. For the candidates that did attempt this part, most were able to recognize the lognormal distribution and get very close to the answer.

Value of the derivative at time t is $\mathbb{E}^{\mathbb{Q}}\left[e^{-r(T-t)}\Psi(Z_T)|\mathcal{F}_t\right] = e^{-r(T-t)}\mathbb{E}^{\mathbb{Q}}[Z_T^{\tau}|\mathcal{F}_t]$ where $\{\mathcal{F}_t\}_{t\geq 0}$ is the natural filtration.

Since the Z_T is lognormal under $\mathbb{Q}$ −measure the expectation can be evaluated easily.

1. Continued

$$Z_T = Z_t e^{\left(r-\frac{\sigma^2}{2}\right)(T-t)+\sigma(\widetilde{W}_T-\widetilde{W}_t)}$$

$$\mathbb{E}^{\mathbb{Q}}\left[Z_T^{\tau}\middle|\mathcal{F}_t\right] = Z_t^{\tau} e^{\left(r-\frac{\sigma^2}{2}\right)(T-t)\tau}\mathbb{E}^{\mathbb{Q}}\left[\mathrm{e}^{\sigma\tau(\widetilde{W}_T-\widetilde{W}_t)}\middle|\mathcal{F}_t\right]$$

$$\mathbb{E}^{\mathbb{Q}}\left[Z_T^{\tau}\middle|\mathcal{F}_t\right] = Z_t^{\tau} e^{\left(r-\frac{\sigma^2}{2}\right)(T-t)\tau} e^{\frac{\sigma^2\tau^2(T-t)}{2}}$$

2. Learning Objectives:

3. The candidate will understand the quantitative tools and techniques for modeling the term structure of interest rates and pricing interest rate derivatives.

Learning Outcomes:

(3a) Understand and apply the concepts of risk-neutral measure, forward measure, normalization, and the market price of risk, in the pricing of interest rate derivatives.

(3b) Apply the models to price common interest sensitive instruments including: callable bonds, bond options, caps, floors, swaptions, caption, floortions.

Sources:

An Introduction to the Mathematics of Financial Derivatives, Neftci, Hirsa (Chapter 17)

Commentary on Question:

The question tests the concepts of state prices, risk neutral measure, and forward measure using a simple three-period interest rate tree. In order to receive full credits, the candidates need to demonstrate understanding of interest rate tree and how risk neutral and forward measures are used to price interest rate instruments, as well as being able to correctly apply the concepts to the given interest rate tree numerically.

Solution:

(a) Calculate the remaining two risk-neutral probabilities Q_{ud} and Q_{du}.

Commentary on Question:

Candidates did reasonably well on this part, as it is directly adapted from the source text. Many candidates were able to set up the correct equations and solve for the unknowns. However, there were still a number of candidates who did not know how to work with the interest rate tree or apply the concept of risk neutral measure accurately (e.g. trying to match the forward bond price instead of the time 0 bond price with the risk neutral probabilities).

<u>Method 1: Calculate using state prices</u>

The state price ψ_{uu} can be calculated from the risk neutral probability Q_{uu}:

$$Q_{uu} = (1 + 6\%)(1 + 8\%)\psi_{uu}$$
$$\psi_{uu} = 0.218$$

Similarly

$$Q_{dd} = (1 + 6\%)(1 + 4\%)\psi_{dd}$$
$$\psi_{dd} = 0.218$$

2. Continued

Given the traded assets, the state prices ψ_{uu}, ψ_{ud}, ψ_{du}, and ψ_{dd} can be calculate as:

$$\begin{bmatrix} 1 \\ 0.89 \\ 0.84 \end{bmatrix} = \begin{bmatrix} 1.1448 & 1.1448 & 1.1024 & 1.1024 \\ 1 & 1 & 1 & 1 \\ 0.92 & 0.93 & 0.95 & 0.97 \end{bmatrix} \begin{bmatrix} 0.218 \\ \psi_{ud} \\ \psi_{du} \\ 0.218 \end{bmatrix}$$

The answers can be obtained by solving any combination of the 3 equations.
Solving the first 2 equations should give answers close to:

$$\psi_{ud} = 0.229$$
$$\psi_{du} = 0.225$$

Solving the last 2 equations should give answers close to:

$$\psi_{ud} = 0.206$$
$$\psi_{du} = 0.248$$

Solving the first and third equations should give answers close to:

$$\psi_{ud} = 0.222$$
$$\psi_{du} = 0.232$$

Thus the risk neutral probabilities are calculated as:

$$Q_{ud} = 0.229 \times 1.06 \times 1.08 = 0.262 \; or \; 0.236 \; or \; 0.254$$
$$Q_{du} = 0.225 \times 1.06 \times 1.04 = 0.248 \; or \; 0.273 \; or \; 0.256$$

Answers different from the above due to rounding received full credits.

Method 2: Directly calculate the risk neutral probabilities

Equation 1: $Q_{uu} + Q_{ud} + Q_{du} + Q_{dd} = 1$

Plug in: $0.25 + Q_{ud} + Q_{du} + 0.24 = 1$, thus $Q_{ud} + Q_{du} = 0.51$

Equation 2: $B_{0,2} = \frac{Q_{uu}+Q_{ud}}{(1+r_0)(1+r_{1,0})} + \frac{Q_{du}+Q_{dd}}{(1+r_0)(1+r_{1,1})}$

Plug in: $0.89 = \frac{0.25+Q_{ud}}{(1+6\%)(1+8\%)} + \frac{Q_{du}+0.24}{(1+6\%)(1+4\%)}$

Equation 3: $B_{0,3} = \frac{Q_{uu}}{(1+r_0)(1+r_{1,0})(1+r_{2,0})} + \frac{Q_{ud}}{(1+r_0)(1+r_{1,0})(1+r_{2,1})} + \frac{Q_{du}}{(1+r_0)(1+r_{1,1})(1+r_{2,2})} + \frac{Q_{dd}}{(1+r_0)(1+r_{1,1})(1+r_{2,3})}$

Plug in: $0.84 = \frac{0.25}{(1.06)(1.08)(1.09)} + \frac{Q_{ud}}{(1.06)(1.08)(1.07)} + \frac{Q_{du}}{(1.06)(1.04)(1.05)} + \frac{0.24}{(1.06)(1.04)(1.03)}$

2. Continued

The answers can be obtained by solving any combination of the 3 equations.
Solve Equations 1 & 2:

$$0.89 = \frac{0.25 + Q_{ud}}{(1+6\%)(1+8\%)} + \frac{(0.51 - Q_{ud}) + 0.24}{(1+6\%)(1+4\%)}$$
$$-0.00871 = -0.0336Q_{ud}$$
$$Q_{ud} = 0.25933$$
$$Q_{du} = 0.25067$$

Or Solve Equations 1 & 3:

$$0.84 = 0.20035 + \frac{Q_{ud}}{1.06 * 1.08 * 1.07} + \frac{0.51 - Q_{ud}}{1.06 * 1.04 * 1.05} + 0.21137$$
$$-0.01231 = -0.0475Q_{ud}$$
$$Q_{ud} = 0.25891$$
$$Q_{du} = 0.25109$$

Or Solve Equations 2 & 3:

$$0.89 \times 1.06 \times 1.08 \times 1.04 - 0.25 \times 1.04 - 0.24 \times 1.08 = 1.04Q_{ud} + 1.08Q_{du}$$

i.e. $0.50040 = 0.96296Q_{ud} + Q_{du}$

Manipulate Equation 3 similarly, $0.49575 = 0.94496Q_{ud} + Q_{du}$

$$0.00465 = 0.018Q_{ud}$$
$$Q_{ud} = 0.2583$$
$$Q_{du} = 0.49575 - 0.94496 \times 0.2583 = 0.2517$$

Answers different from the above due to rounding received full credits.

<u>Method 3: Calculate the risk neutral probabilities for the first step of the tree</u>

Equation 1: $Q_u + Q_d = 1$

Equation 2: $B_{0,2} = \frac{Q_u}{(1+r_0)(1+r_{1,0})} + \frac{Q_d}{(1+r_0)(1+r_{1,1})}$

Plug in: $0.89 = \frac{Q_u}{(1+6\%)(1+8\%)} + \frac{Q_d}{(1+6\%)(1+4\%)}$

Solve the 2 equations:

$$0.89 = \frac{Q_u}{1.06 \times 1.08} + \frac{1 - Q_u}{1.06 \times 1.04}$$
$$-0.01711 = -0.0336Q_u$$
$$Q_u = 0.5093$$
$$Q_d = 0.4907$$

Then: $Q_{ud} = Q_u - Q_{uu} = 0.2593$, $Q_{du} = Q_d - Q_{dd} = 0.2507$

2. Continued

(b) Determine the time-0 price of a caplet on r_2 with notional of 1000, cap $K = 6\%$ and expiry at time 3 using:

(i) Risk-neutral measure

(ii) Forward measure

Commentary on Question:
Candidates did better with the risk neutral measure than with the forward measure. Most candidates knew the general form of pricing formula under risk neutral measure, but less candidates were able to calculate the forward measure probabilities and correctly apply the pricing formula. For the risk neutral measure, a common mistake was discounting the expected value using only two periods of interest rates. Among those who understood how to price the caplet under the forward measure, a common mistake was normalizing the probabilities and caplet price with the price of bond maturing at time 3.

<u>Risk Neutral Measure</u>

The price can be calculated directly since the risk neutral probabilities have already been calculated in part a).

$$Price = 1000 \times E^Q\left[\frac{Max(r_2 - K, 0)}{(1+r_0)(1+r_1)(1+r_2)}\right]$$
$$= 1000 \times \left[\frac{r_{2,0} - K}{(1+r_0)\left(1+r_{1,0}\right)\left(1+r_{2,0}\right)} Q_{uu} + \frac{r_{2,1} - K}{(1+r_0)\left(1+r_{1,0}\right)\left(1+r_{2,1}\right)} Q_{ud}\right]$$
$$= 1000 \times \left[\frac{9\% - 6\%}{1.06 \times 1.08 \times 1.09} \times 0.25 + \frac{7\% - 6\%}{1.06 \times 1.08 \times 1.07} \times 0.254\right] = 8.084$$

Wrong results from part a) are not penalized again, and rounding differences are not penalized.

<u>Forward Measure</u>

The forward measure probabilities are calculated as

$$\pi_{uu} = \frac{\psi_{uu}}{0.89} = 0.2454 \text{ Or } \pi_{uu} = \frac{Q_{uu}}{0.89 \times 1.06 \times 1.08} = 0.2454$$
$$\pi_{ud} = \frac{\psi_{ud}}{0.89} = 0.2573 \text{ Or } \pi_{ud} = \frac{Q_{ud}}{0.89 \times 1.06 \times 1.08} = 0.2573$$
$$\pi_{du} = \frac{\psi_{du}}{0.89} = 0.2528 \text{ Or } \pi_{du} = \frac{Q_{du}}{0.89 \times 1.06 \times 1.04} = 0.2528$$
$$\pi_{dd} = \frac{\psi_{dd}}{0.89} = 0.2446 \text{ Or } \pi_{dd} = \frac{Q_{dd}}{0.89 \times 1.06 \times 1.04} = 0.2446$$

$$Price = 1000 \times 0.89 \times E^{\pi}\left[Max\left(\frac{r_2 - K}{1 + r_2}, 0\right)\right]$$
$$= 1000 \times 0.89 \times \left[\frac{r_{2,0} - K}{1 + r_{2,0}}\pi_{uu} + \frac{r_{2,1} - K}{1 + r_{2,1}}\pi_{ud}\right]$$
$$= 1000 \times 0.89 \times \left[\frac{9\% - 6\%}{1.09} \times 0.2454 + \frac{7\% - 6\%}{1.07} \times 0.2573\right] = 8.151$$

Wrong results from part a) are not penalized again, and rounding differences are not penalized.

(c) Identify errors of this approach for calculating C_t.

Commentary on Question:
Candidates generally did not do very well on this part. Most candidates criticized the appropriateness of using a geometric Brownian motion for modeling interest rates, instead of recognizing the fundamental error that the forward rate process is not martingale under the risk neutral measure. Some candidates wrote down the formula from Paul Wilmott Introduces Quantitative Finance, Chapter 18, and only commented on the appearance of the formula without showing any understanding, for which no credit was given.

Since the forward rate F_{T-1} will coincide with the LIBOR rate L_{T-1} at time T-1, the forward rate F_t is the underlying of this caplet.

F_t is not a martingale under Q measure, when using Girsanov' theorem to switch from P measure to Q measure, the revised drift is not known, so we cannot use Black Scholes.

Under the forward measure π, the forward rate F_t is a martingale:
$$dF_t = \sigma F_t d\widetilde{W_t}$$
Therefore, the price of a capital with payoff $C_T = \max(L_{T-1} - K, 0)$ at maturity is:
$$C_t = F_t N(d_1) - KN(d_2)$$
where $d_1 = \frac{\ln\left(\frac{F_t}{K}\right) + \frac{1}{2}\sigma^2(T-t)}{\sigma\sqrt{T-t}}$ and $d_2 = d_1 - \sigma\sqrt{T-t}$.

3. Learning Objectives:

1. The candidate will understand the fundamentals of stochastic calculus as they apply to option pricing.

Learning Outcomes:

(1c) Understand Ito integral and stochastic differential equations.

(1d) Understand and apply Ito's Lemma.

(1h) Define and apply the concepts of martingale, market price of risk and measures in single and multiple state variable contexts.

Sources:

Problems and Solutions in Mathematical Finance (Chin/Nel/Olaffson)

Commentary on Question:

To receive full credit for this question, candidates must clearly derive the requested results and provide justification for each step.

Solution:

(a) Prove that $E\left(W_t^4\right) = 3t^2$.

Commentary on Question:

Most candidates received full credit for this part of the problem. Candidates either applied Ito's Lemma or used the fact that W_t is normally distributed and found the fourth moment of the distribution to arrive at the result.

Applying Ito's Lemma to W_t^4:

$$d(W_t^4) = 4W_t^3 dW_t + \frac{1}{2}(12W_t^2)dt$$

$$W_t^4 = \int_0^t 4W_s^3 dW_s + 6\int_0^t W_s^2 ds$$

$$E(W_t^4) = E\left(\int_0^t 4W_s^3 dW_s\right) + 6\int_0^t E(W_s^2)ds$$

$$= 0 + 6\int_0^t s ds = 3t^2.$$

Alternatively, one could note that W_t is normally distributed with mean 0 and variance t, hence the moment generating function is $E(e^{sW_t}) = e^{\frac{t}{2}s^2}$, the 4th moment is thus the 4th derivative at $s=0$, so $E(W_t^4) = 4!\,\frac{1}{2}\left(\frac{t}{2}\right)^2 = 3t^2$.

3. Continued

(b) Derive an expression in terms of t for each of the following:

(i) $Var(X_t)$

(ii) $Var(Y_t)$

(iii) $Cov(X_t, Y_t)$

Commentary on Question:
Candidates struggled with part (b), especially parts (ii) and (iii). Candidates were given some credit for writing down the correct formula for the variance and covariance.The most common mistake was to apply Ito's Isometry wrong, e.g. $E\left(\left(\int_0^t W_s ds\right)^2\right) = E\left(\int_0^t W_s^2 \; ds\right)$.

(i) $Var(X_t) = E\left(\left(\int_0^t W_s ds\right)^2\right) - \left(E\left(\int_0^t W_s ds\right)\right)^2 = E\left(\int_0^t W_s ds \int_0^t W_u du\right) - 0$
$= E\left(\int_0^t \int_0^t W_s W_u dsdu\right)$.

After applying the martingale property and the independent increments property of W_t, we have:

$$Var(X_t) = \int_0^t \int_0^t E(W_s W_u) dsdu$$

$$= \int_0^t \int_0^u E\big(W_s E(W_u | F_s)\big) dsdu + \int_0^t \int_u^t E\big(W_u E(W_s | F_u)\big) dsdu$$

$$= \int_0^t \int_0^u E(W_s^2) dsdu + \int_0^t \int_u^t E(W_u^2) dsdu = \int_0^t \int_0^u s dsdu + \int_0^t \int_u^t u dsdu$$

$$= \int_0^t \frac{u^2}{2} du + \int_0^t u(t-u) du = \frac{t^3}{6} + \left(\frac{t^3}{2} - \frac{t^3}{3}\right) = \frac{t^3}{3}.$$

Alternative solution:
From $d(sW_s) = W_s ds + s dW_s$ we have
$X_t = tW_t - \int_0^t s dW_s = \int_0^t (t-s) dW_s$.
Using Ito Isometry we have
$E(X_t^2) = E\left[\left(\int_0^t (t-s) dW_s\right)^2\right] = \int_0^t (t-s)^2 ds = \frac{t^3}{3}$.
Since $E(X_t) = 0$, $Var(X_t) = E(X_t^2) = \frac{t^3}{3}$.

3. **Continued**

(ii) $Var(Y_t) = E\left(\left(\int_0^t W_s^2 ds\right)^2\right) - \left(E\left(\int_0^t W_s^2 ds\right)\right)^2$

$$= \int_0^t \int_0^t E(W_s^2 W_u^2) ds du - \left(\int_0^t E(W_s^2) ds\right)^2$$

$$= \int_0^t \int_0^u E(E(W_s^2 W_u^2 | F_s)) ds du + \int_0^t \int_u^t E(E(W_s^2 W_u^2 | F_u)) ds du - \left(\int_0^t s ds\right)^2$$

$= \int_0^t \int_0^u E(W_s^2 E(W_u^2 | F_s)) ds du + \int_0^t \int_u^t E(W_u^2 E(W_s^2 | F_u)) ds du - \frac{t^4}{4}.$

When $0 < s < u < t$, after applying the martingale property and the independent increments property of W, we have:

$$E(W_u^2 | F_s) = E\left(\left(W_s + (W_u - W_s)\right)^2 \Big| F_s\right)$$
$$= E(W_s^2 + (W_u - W_s)^2 + 2W_s(W_u - W_s) | F_s)$$
$$= W_s^2 + (u - s).$$

Similarly, when $0 < u < s < t,$ we have

$$E(W_s^2 | F_u) = W_u^2 + (s - u)$$

Thus

$$Var\left(\int_0^t W_s^2 ds\right) = \int_0^t \int_0^u E\left(W_s^2\left(W_s^2 + (u - s)\right)\right) ds du$$
$$+ \int_0^t \int_u^t E\left(W_u^2\left(W_u^2 + (s - u)\right)\right) ds du - \frac{t^4}{4}$$

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Since $E(W_t^4) = 3t^2$ from part (i),

$Var\left(\int_0^t W_s^2 ds\right)$

$$= \int_0^t \int_0^u E\left(W_s^4 + W_s^2(u - s)\right) ds du + \int_0^t \int_u^t E\left(W_u^4 + W_u^2(s - u)\right) ds du - \frac{t^4}{4}$$

$$= \int_0^t \int_0^u \left(3s^2 + s(u - s)\right) ds du + \int_0^t \int_u^t \left(3u^2 + u(s - u)\right) ds du - \frac{t^4}{4}$$

$$= \int_0^t \left(\frac{2}{3}u^3 + \frac{u^3}{2}\right) du + \int_0^t \left(2u^2(t - u) + u\left(\frac{t^2}{2} - \frac{u^2}{2}\right)\right) du - \frac{t^4}{4}$$

$= \int_0^t \frac{7}{6}u^3 du + \left(\frac{2}{3}t^4 - \frac{2}{4}t^4 + \frac{1}{4}t^4 - \frac{1}{8}t^4\right) - \frac{t^4}{4} = \frac{7}{24}t^4 + \frac{7}{24}t^4 - \frac{6t^4}{24} = \frac{1}{3}t^4.$

Alternative solution:

Applying the product rule and Ito's Lemma to $(t - s)W_s^2$ we have:

$d((t - s)W_s^2) = -W_s^2 ds + (t - s) dW_s^2 = -W_s^2 ds + (t - s)(2W_s dW_s + ds).$

Then

$$Y_t = \int_0^t (t - s) 2 W_s dW_s + \int_0^t (t - s) ds$$

3. Continued

Thus, using Ito isometry,

$Var(Y_t) = 4E\left[\left(\int_0^t (t-s)W_s dW_s\right)^2\right] = 4\int_0^t (t-s)^2 E(W_s^2)ds$

$= 4\int_0^t (t-s)^2 s ds = \frac{t^4}{3}$.

(iii)

$$Covar(X_t, Y_t) = E\left(\int_0^t W_s ds \int_0^t W_u^2 du\right) - E\left(\int_0^t W_s ds\right)E\left(\int_0^t W_s^2 ds\right)$$

$$= E\left(\int_0^t W_s ds \int_0^t W_u^2 du\right) = E\left(\int_0^t \int_0^t W_s W_u^2 ds du\right)$$

$$= \int_0^t \int_0^u E\big(E(W_s W_u^2 | F_s)\big) ds du + \int_0^t \int_u^t E\big(E(W_s W_u^2 | F_u)\big) ds du =$$

$$= \int_0^t \int_0^u E(W_s E(W_u^2 | F_s)) ds du + \int_0^t \int_u^t E(W_u^2 E(W_s | F_u)) ds du.$$

From the martingale properties, when $0 < s < u < t$,

$E(W_u^2 | F_s) = E\left(\left(W_s + (W_u - W_s)\right)^2 | F_s\right) = W_s^2 + (u-s)$.

Thus

$$Covar(X_t, Y_t) = \int_0^t \int_0^u E\big(W_s(W_s^2 + u - s)\big) ds du + \int_0^t \int_u^t E(W_u^2 W_u) ds du.$$

Since $E(W_u^3) = E(W_s) = 0$, we have $Covar(X_t, Y_t) = 0$.

Alternative solution:

Recall that $X_t = \int_0^t (t-s) dW_s$ and $Y_t = \int_0^t (t-s) 2W_s dW_s + \int_0^t (t-s) ds$.

Noting that $E(X_t) = 0$ we have

$$Covar(X_t, Y_t) = E(X_t Y_t) - E(X_t)E(Y_t)$$

$$= E\left(\int_0^t (t-s) dW_s \left(\int_0^t (t-s) 2W_s dW_s + \int_0^t (t-s) ds\right)\right)$$

$$= E\left(\int_0^t (t-s) dW_s \int_0^t (t-s) 2W_s dW_s\right)$$

Applying Ito Isometry we find

$$Covar(X_t, Y_t) = \int_0^t E((t-s)^2 2W_s) ds = 0$$

Therefore

$$\Sigma = \begin{bmatrix} \frac{t^3}{3} & 0 \\ 0 & \frac{t^4}{3} \end{bmatrix}.$$

4. **Learning Objectives:**

1. The candidate will understand the fundamentals of stochastic calculus as they apply to option pricing.

Learning Outcomes:

(1a) Understand and apply concepts of probability and statistics important in mathematical finance.

(1c) Understand Ito integral and stochastic differential equations.

(1d) Understand and apply Ito's Lemma.

Sources:

Problems and Solutions Math Finance pp.134-135

Commentary on Question:

Most candidates understood well for the idea of the question

Solution:

(a) Prove by using Ito's lemma that for $t \le T$.

$$Y_T = e^{-\alpha(T-t)}Y_t + \left(\theta - \frac{\sigma^2}{2\alpha}\right)\left(1 - e^{-\alpha(T-t)}\right) + \int_t^T \sigma e^{-\alpha(T-s)} dW_s.$$

Commentary on Question:

Some candidates didn't have techniques for solving mean reverting log price, which is a useful technique for extending to multi variable stochastic processes

It tests whether candidates can discern separable terms in the equation for the variable.

Letting $Y_t = \log S_t$ and applying Ito's lemma,

$$\begin{aligned} dY_t &= \frac{\partial Y_t}{\partial S_t} dS_t + \frac{1}{2}\frac{\partial^2 Y_t}{\partial S_t^2}(dS_t)^2 \\ &= \frac{1}{S_t} dS_t - \frac{1}{2}\frac{1}{S_t^2}(dS_t)^2 \\ &= \alpha\left(\theta - Y_t - \frac{\sigma^2}{2\alpha}\right)dt + \sigma dW_t \end{aligned}$$

4. **Continued**

Define $Z_t = e^{\alpha t}Y_t$ and applying prododuct rule (Ito's lemma)

$$dZ_t = \mathrm{Y_t} d(e^{\alpha t}) + e^{\alpha t}dY_t + d(e^{\alpha t})dY_t$$
$$= \alpha e^{\alpha t}Y_t dt + e^{\alpha t}dY_t$$
$$= \alpha\left(\theta - \frac{\sigma^2}{2\alpha}\right)e^{\alpha t}dt + \sigma e^{\alpha t}dW_t$$

Integrating both sides give

$$\int_t^T dZ_s = \int_t^T \alpha\left(\theta - \frac{\sigma^2}{2\alpha}\right)e^{\alpha s}ds + \int_t^T \sigma e^{\alpha s}dW_s$$
$$Z_T - Z_t = \left(\theta - \frac{\sigma^2}{2\alpha}\right)(e^{\alpha T} - e^{\alpha t}) + \int_t^T \sigma e^{\alpha s}dW_s$$
$$Y_T = e^{-\alpha(T-t)}Y_t + \left(\theta - \frac{\sigma^2}{2\alpha}\right)\left(1 - e^{-\alpha(T-t)}\right) + \int_t^T \sigma e^{-\alpha(T-s)}dW_s$$

(b) Derive the mean and the variance of Y_T at time $t \le T$.

Commentary on Question:
It tests candidates can apply Ito isometry and Gaussian process

Solutions to the stochastic processes are usually in the form of moments of the underlying distribution at certain time.

Y_T is a normal distribution with mean

$$E(Y_T) = e^{-\alpha(T-t)}Y_t + \left(\theta - \frac{\sigma^2}{2\alpha}\right)\left(1 - e^{-\alpha(T-t)}\right)$$

and variance

$$Var(Y_T) = Var\left(\int_t^T \sigma e^{-\alpha(T-s)}dW_s\right)$$
$$= \int_t^T \sigma^2 e^{-2\alpha(T-s)}ds = \frac{\sigma^2\left(1 - e^{-2\alpha(T-t)}\right)}{2\alpha}$$

(c) Derive $E(S_T)$ at time $t \le T$.

Commentary on Question:
Most candidates who solved part (b) arrived at the solution.

Candidates can understand how to derive forward prices from spot price stochastic process.

4. **Continued**

$$E(S_T) = E(e^{Y_T})$$

Since Y_T is normally distributed random variable, using the moment generating function of a normally distributed random variable

$$E(e^{Y_T}) = \exp\left\{E(Y_T) + \frac{1}{2}Var(Y_T)\right\}$$

$$E[S_T] = \exp\left\{e^{-\alpha(T-t)}Y_t + \left(\theta - \frac{\sigma^2}{2\alpha}\right)\left(1 - e^{-\alpha(T-t)}\right) + \frac{\sigma^2\left(1 - e^{-2\alpha(T-t)}\right)}{4\alpha}\right\}$$

5. Learning Objectives:

2. The candidate will understand how to apply the fundamental theory underlying the standard models for pricing financial derivatives. The candidate will understand the implications for option pricing when markets do not satisfy the common assumptions used in option pricing theory such as market completeness, bounded variation, perfect liquidity, etc. The Candidate will understand how to evaluate situations associated with derivatives and hedging activities.

Learning Outcomes:

(2b) Compare and contrast the various kinds of volatility, (eg actual, realized, implied, forward, etc.).

(2d) Understand the different approaches to hedging.

(2e) Understand how to delta hedge and the interplay between hedging assumptions and hedging outcomes.

Sources:

Paul Wilmott Introduces Quantitative Finance, Wilmott, Paul, 2nd Edition

Commentary on Question:

Most candidates did well for parts (a) and (b), but not for the rest of the questions. In particular, few candidates did well in part (c).

Solution:

(a) Calculate the call option price that is consistent with your volatility view.

Commentary on Question:

This question tests direct application of Black-Scholes's option pricing formula. Many candidates did well.

Use the following input in the Black-Scholes's call option formula to get the call option price

S = X = 100

r = 2%

T = 1

σ = actual vol = 20%

$$d_1 = \frac{\ln\frac{S}{X} + \left(r + \frac{\sigma^2}{2}\right)T}{\sigma\sqrt{T}} = \frac{\ln\frac{100}{100} + \left(2\% + \frac{20\%^2}{2}\right) * 1}{\sigma\sqrt{T}} = 0.2$$

5. Continued

$d_2 = d1 - \sigma\sqrt{T} = 0.2 - 20\%\sqrt{1} = 0$

$N(d_1) = 0.5793$

$N(d_2) = 0.5$

$C = SN(d_1) - Xe^{-rT}N(d_2) = 100 * 0.5793 - 100e^{-2\%} * 0.5 = 8.92$

(b) Propose a trading strategy on 9/30/2016 involving the call option and/or its underlying stock that guarantees a profit if your volatility view proves correct.

Commentary on Question:
It tests candidates' understanding of delta hedging. Most candidates are able to propose the correct strategy of "buying one call and shorting the delta share of the underlying stock". Some candidates' proposed using the "implied vol" for delta hedging, however, the implied vol was not directly available in the question. Partial credits were given in this case.

Since $8.92 exceeds actual market price of $6.92, the following trading strategy will guarantee a profit for each call option

(i) buy one call option at market price of $6.92
(ii) delta hedge the call option until option maturity
(iii) use actual vol = 20% to determine delta

The above strategy can be scaled to more call options as long as the strategy does not impact the market price of the options and the underlying stock.

(c) Provide a mathematical derivation to support your proposed strategy in part (b).

Commentary on Question:
Most candidates did not perform well on this question. This question is directly from the text book. Partial credits were given for each step that the candidates answered correctly.

Proof:
For each call option bought and delta hedged, let

Δ^i = call option delta calculated with implied vol
Δ^a = call option delta calculated with actual vol
V^i = call option value calculated with implied vol
V^a = call option value calculated with actual vol

5. Continued

Set up delta hedge portfolio

Position	Value today	Value tomorrow
Long Option:	V^i	$V^i + dV^i$
Short Stock:	$-\Delta^a S$	$-\Delta^a S - \Delta^a dS$
Cash:	$-V^i + \Delta^a S$	$-V^i + \Delta^a S + r(-V^i + \Delta^a S)dt$

Portfolio value change

Change = $dV^i - \Delta^a dS + r\left(-V^i + \Delta^a S\right)dt$ (1)

Because the option is correctly valued at actual vol, we have

$dV^a - \Delta^a dS + r(-V^a + \Delta^a S)dt = 0$ (2)

Combining equations (1) and (2) gives us the change from t to $t + dt$

Change = $dV^i - dV^a - r\left(V^i - V^a\right)dt$

= $e^{rt} d\left(e^{-rt}\left((V^i - V^a\right)\right)$

The present value of this change from t_0 (hedge inception date of 9/30/2016) to t is

$$e^{-r(t-t_0)} e^{rt} d\left(e^{-rt}\left((V^i - V^a\right)\right) = e^{rt_0} d\left(e^{-rt}\left((V^i - V^a\right)\right)$$

Total present value of change from inception to option expiration is

$$e^{rt_0} \int_{t_0}^{T} d\left(e^{-rt}\left((V^i - V^a\right)\right) = V^a - V^i > 0$$

(d) Determine the present value of expected profit of your strategy on 9/30/2016.

Commentary on Question:
Some candidates did well on this question, while others incorrectly determined the present value of profit by further discounting the difference of the two option prices

For each call option bought and delta hedged, total expected present value of profit = $V^a - V^i$= \$8.92 – \$6.92 = \$2.

(e) Determine the net cash position of your strategy on 9/30/2016.

Commentary on Question:
Some candidates did well on this question, while others incorrectly determined the sign of the cash position.

5. **Continued**

For each call option bought and delta hedged:
$\Delta^a = N(d1) = 0.5793$
Cash = $-V^i + \Delta^a S$
$= -6.92 + 0.5793 * 100$
$= 51.01$

(f) Outline pros and cons of your strategy.

Commentary on Question:
Many candidates are able to identify one or two key items on this question. Some candidates are able to identify all key items on this question.

Pros:

- Know exactly what profit will be realized at option expiration.
- Usually the optimal hedge strategy if not constrainted by mark-to-market.

Cons:

- Volatile mark-to-market before option expiration.
- No one can be totally confident on volatility forecast that is used to calculate delta.

6. Learning Objectives:

1. The candidate will understand the fundamentals of stochastic calculus as they apply to option pricing.

2. The candidate will understand how to apply the fundamental theory underlying the standard models for pricing financial derivatives. The candidate will understand the implications for option pricing when markets do not satisfy the common assumptions used in option pricing theory such as market completeness, bounded variation, perfect liquidity, etc. The Candidate will understand how to evaluate situations associated with derivatives and hedging activities.

Learning Outcomes:

(1f) Demonstrate understanding of option pricing techniques and theory for equity and interest rate derivatives.

(1h) Define and apply the concepts of martingale, market price of risk and measures in single and multiple state variable contexts.

(1i) Understand and apply Girsanov's theorem in changing measures.

(2d) Understand the different approaches to hedging.

Sources:

Neftci Ch. 6, 11, 13

Wilmott Ch. 8

Commentary on Question:

This question is testing a candidate's understanding of the Black-Scholes formula and option pricing techniques.

Most candidates did well on parts (a) and (b) but struggled on parts (c) and (d). Most notably on part (d), most candidates did not understand how to explicitly set up the put option and derive a correct Black-Scholes equation.

Solution:

(a) Show that

$$S_t = S_0 e^{\left(r-\frac{\sigma^2}{2}\right)t+\sigma W_t}$$

Commentary on Question:

To receive full marks, candidates need to start with the SDE laid out in the question and derive the final equation using Ito's Lemma.

6. Continued

Apply Ito's Lemma to ln S_t

$$dln(S_t) = \frac{\partial ln(S_t)}{\partial t}(dt) + \frac{\partial ln(S_t)}{\partial S_t}(dS_t) + \frac{1}{2}\frac{\partial^2 ln(S_t)}{\partial S_t^2}(dS_t)^2$$

$$= (0)(dt) + \frac{1}{S_t}(dS_t) - \frac{1}{2S_t^2}(dS_t)^2$$

$$= \frac{1}{S_t}(dS_t) - \frac{1}{2S_t^2}(dS_t)^2$$

$$= (rdt + \sigma dW_t) - \frac{1}{2}\sigma^2 dt$$

Integrate both sides with limits t and 0

$$lnS_t - ln\,S_0 = \; rt \; + \; W_t\sigma - \left(\frac{1}{2}\right)\sigma^2 t$$

$$S_t = S_0 exp(\left(r - \left(\frac{1}{2}\right)\sigma^2\right)t + \sigma W_t)$$

(b) Show that the value of the European chooser option

$$H\left(S_u,u,K,u,T\right) = \max\left(C\left(S_u,u,K,T\right), C\left(S_u,u,K,T\right) + e^{-r(T-u)}K - S_u\right)$$

Commentary on Question:
Most candidates did well and recognized the relationship of Put-Call Parity.

We are given $H(S_u,u,K,u,T) = max\big(C(S_u,u,K,T), P(S_u,u,K,T)\big)$

Since under risk-neutral measure, the discounted stock is a martingale, we have

$$\begin{aligned}C(S_t,t,K,T) - P(S_t,t,K,T) \\ &= e^{-r(T-t)}E[(\max(S_T-K,0) - \max(K-S_T,0)\,|F_t)] \\ &= e^{-r(T-t)}E[(\max(S_T-K,0) + \min(S_T-K,0)\,|F_t)] \\ &= e^{-r(T-t)}E\big[S_T - K|F_t]\big] \\ &= S_t - e^{-r(T-t)}K\end{aligned}$$

This is actually put-call parity.

6. Continued

Therefore

$$H(S_u, u, K, u, T) = \max\left(C(S_u, u, K, T), C(S_u, u, K, T) + e^{-r(T-u)}K - S_u\right)$$

(c) Show that

$$H\left(S_0, 0, K, u, T\right) = C\left(S_0, 0, K, T\right) + e^{-rT}E\left[\max\left(0, K - S_0 e^{rT} e^{\sigma W_u - \frac{u\sigma^2}{2}}\right)\right]$$

where $E[\quad]$ is the expectation under the risk-neutral measure $\mathbb{Q}$.

Commentary on Question:

Most candidates did well, but missed the steps explicitly separating out time 0 and time u.

From part (b)

$$H(S_u, u, K, u, T) = max\left(C(S_u, u, K, T), C(S_u, u, K, T) + e^{-r(T-u)}K - S_u\right)$$

That is

$$H(S_u, u, K, u, T) = C(S_u, u, K, T) + max\left(0, e^{-r(T-u)}K - S_u\right)$$

Noting

$$H(S_0, 0, K, u, T) = e^{-ru}E[H(S_u, u, K, u, T)]$$
$$C(S_0, 0, K, T) = e^{-ru}E[C(S_u, u, K, T)]$$

we calculate

$$\begin{aligned} H(S_0, 0, K, u, T) &= e^{-ru}E[C(S_u, u, K, T) + max\left(0, e^{-r(T-u)}K - S_u\right)] \\ &= e^{-ru}E[C(S_u, u, K, T)] + e^{-ru}E[max\left(0, e^{-r(T-u)}K - S_u\right)] \\ &= C(S_0, 0, K, T) + e^{-rT}E[max\left(0, K - e^{r(T-u)}S_u\right)] \end{aligned}$$

Recall that from part (a)

$$S_t = S_0 e^{\left(r - \frac{\sigma^2}{2}\right)t + \sigma W_t}$$

Substituting this into the above equation we have

$$H(S_0, 0) = C(S_0, 0) + e^{-rT}E[max\left(0, K - S_0 e^{rT} e^{\sigma W_u - \frac{u\sigma^2}{2}}\right)]$$

6. Continued

(d) Show that

$$H\left(S_0, 0, K, u, T\right) = S_0\left(N\left(d_1\right) - N\left(-d_1^*\right)\right) + Ke^{-rT}\left(N\left(-d_2^*\right) - N\left(d_2\right)\right)$$

$$where\ d_1 = \frac{\ln\left(\frac{S_0}{K}\right) + \left(r + \frac{\sigma^2}{2}\right)T}{\sigma\sqrt{T}}$$

$$d_2 = d_1 - \sigma\sqrt{T}$$

$$d_1^* = \frac{\ln\left(\frac{S_0}{K}\right) + \left(rT + \frac{\sigma^2}{2}u\right)}{\sigma\sqrt{u}}$$

$$d_2^* = d_1^* - \sigma\sqrt{u}$$

Commentary on Question:
Most candidates didn't use the correct put option's strike price to derive d_1^*.

Since $e^{-rT}E[\,max((0, K - e^{r(T-u)}S_u) = e^{-ru}E\left[max\left(0, e^{-r(T-u)}K - S_u\right)\right]$
The chooser is equivalent to a portfolio of a long call expiring at time T with strike price K plus a long put with a strike $e^{-r(T-u)}K$ *expiring at u*

$$H(S_0, 0, K, u, T) = C(S_0, 0\,, K, T) + P\left(S_0, 0, e^{-r(T-u)}K\,, u\right)$$
$$= S_0N(d_1) - K\,e^{-rT}N(d_2) + Ke^{-rT}N(-d_2^*) - S_0N(-d_1^*)$$
$$H(S_0, 0, K, u, T) = S_0\left(N(d_1) - N(-d_1^*)\right) + Ke^{-rT}(N(-d_2^*) - N(d_2))$$
$$where\ d_1 = \frac{\ln\left(\frac{S_0}{K}\right) + \left(r + \frac{\sigma^2}{2}\right)T}{\sigma\sqrt{T}}$$
$$d_2 = d_1 - \sigma\sqrt{T}$$

$$d_1^* = (\ln\left(\frac{S_0}{Kexp\left(-r(T-u)\right)}\right) + \left(r + \frac{\sigma^2}{2}\right)u)/\sigma\sqrt{u}$$

$$d_1^* = (\ln\left(\frac{S_0}{K}\right) + \ln(\frac{1}{\exp\left(-r(T-u)\right)} + \left(r + \frac{\sigma^2}{2}\right)u)/\sigma\sqrt{u}$$

6. Continued

$$d_1^* = (\ln\left(\frac{S_0}{K}\right) + \ln\left(\exp\left(r(T-u)\right) + \left(r + \frac{\sigma^2}{2}\right)u\right) / \sigma\sqrt{u}$$

$$d_1^* = (\ln\left(\frac{S_0}{K}\right) + r(T-u) + \left(r + \frac{\sigma^2}{2}\right)u) / \sigma\sqrt{u}$$

$$d_1^* = \frac{\ln\left(\frac{S_0}{K}\right) + \left(rT + \frac{\sigma^2}{2}u\right)}{\sigma\sqrt{u}}$$

$$d_2^* = d_1^* - \sigma\sqrt{u}$$

(e) Derive the delta of the chooser option.

Commentary on Question:

Most candidates got this correctly.

$$H(S_0, 0, K, u, T) = C(S_0, 0, K, T) + P\left(S_0, 0, e^{-r(T-u)}K, u\right)$$

From the formula sheet
Delta of $= \frac{\partial H(S_t, t)}{\partial S_t} = N(d_1) + N(d_1^*) - 1$

(f) Calculate the limits of deltas of European chooser options:

(i) When the underlying stock price S_0 approaches to zero.

(ii) When the underlying stock price S_0 approaches to infinity.

Commentary on Question:
Some candidates didn't derive delta correctly from part (e) but the analysis for limits was following the right logic. Partial marks were given under this situation.

Delta: as stock price tends to 0, it tends to -1
$S \to 0$ implies $ln\left(\frac{S}{K}\right) \to -\infty$ and $d_1, d_1^* \to -\infty, N(d_1), N(d_1^*) \to 0$
Then $delta \to 0 + 0 - 1 = -1$

As stock price tends to infinity, $ln\left(\frac{S}{K}\right) \to \infty$
And $d_1, d_1^* \to \infty, N(d_1), N(d_1^*) \to 1$
Then $delta \to 1 + 1 - 1 = 1$

7. **Learning Objectives:**

2. The candidate will understand how to apply the fundamental theory underlying the standard models for pricing financial derivatives. The candidate will understand the implications for option pricing when markets do not satisfy the common assumptions used in option pricing theory such as market completeness, bounded variation, perfect liquidity, etc. The Candidate will understand how to evaluate situations associated with derivatives and hedging activities.

Learning Outcomes:

(2d) Understand the different approaches to hedging.

Sources:

Paul Wilmott Introduces Quantitative Finance Ch.8

Commentary on Question:

This question is trying to test candidate's understanding of the relationship among option greeks and their importance in hedging.

Solution:

(a)

(i) Derive Gamma in terms of the notations given above.

(ii) Derive Speed in terms of Gamma and the notations given above.

Commentary on Question:

Part (i) Candidates did generally well on this part. Some candidates did not score perfectly as they failed to express $N'(d_1)$ using the given notations in the question.

Part (ii) Candidates did modest on this part but very few received full points. To receive full points, one needs to express Speed in terms of Gamma. Candidates who copied directly from the formula sheet received zero points.

(i) For a European call option on a non-dividend paying stock, $\Delta = N(d_1)$.

And that $d_1 = \frac{\ln\left(\frac{S}{K}\right)+\left(r+\left(\frac{\sigma^2}{2}\right)\right)T}{\sigma\sqrt{T}}$

$\Gamma = \frac{\partial^2 C_t}{\partial S^2} = \frac{\partial(N(d_1))}{\partial S} = \frac{\partial N(d_1)}{\partial d_1} * \frac{\partial d_1}{\partial S}$ by chain rule

$= N'(d_1) * \frac{\frac{1}{S}}{\sigma\sqrt{T}} = \frac{1}{S\sigma\sqrt{T}} N'(d_1)$

where $N'(d_1) = \frac{\partial\left(\frac{1}{\sqrt{(2\pi)}}\int_{\infty}^{d_1} e^{-\frac{y^2}{2}} dy\right)}{\partial d_1} = \frac{e^{-\left(\frac{d_1^2}{2}\right)}}{\sqrt{2\pi}}$

7. Continued

$$speed = \frac{\partial \Gamma}{\partial S} = \frac{\partial\left(\frac{1}{S\;\sigma\sqrt{T}}\frac{e^{-\left(\frac{d_1^2}{2}\right)}}{\sqrt{2\pi}}\right)}{\partial S}$$

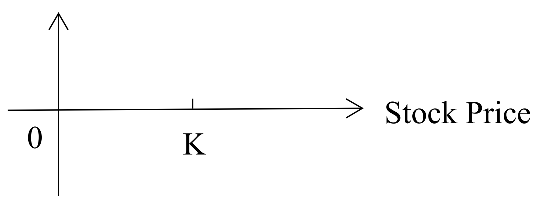
$$= -\frac{1}{S^2\,\sigma\sqrt{T}}\frac{e^{-\left(\frac{d_1^2}{2}\right)}}{\sqrt{2\pi}} + \frac{1}{S\;\sigma\sqrt{T}}\frac{e^{-\left(\frac{d_1^2}{2}\right)}}{\sqrt{2\pi}}(-d_1)\left(\frac{\frac{1}{S}}{\sigma\sqrt{T}}\right)$$

$$= -\frac{1}{S^2\,\sigma\sqrt{T}}\frac{e^{-\left(\frac{d_1^2}{2}\right)}}{\sqrt{2\pi}} + \frac{1}{S^2\,\sigma^2 T}\frac{e^{-\left(\frac{d_1^2}{2}\right)}}{\sqrt{2\pi}}(-d_1)$$

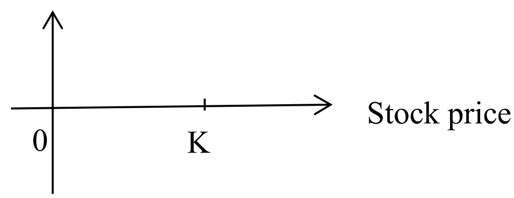
$$= -\frac{\Gamma}{S}\left(1 + \frac{d_1}{\sigma\sqrt{T}}\right)$$

(b) Sketch the following two graphs that illustrate how Gamma and Speed change with the underlying stock price, respectively, based on your derivations in parts (a)(i) and (a)(ii) above. Explain the shape of each of your graphs.

Gamma



Speed

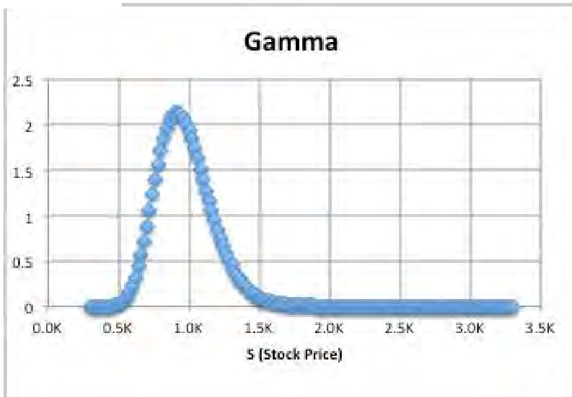


Commentary on Question:
Candidates did poorly on this part. Most candidates tried to approach this question directly from the formula in part (a), which is difficult, and not many were able to do it accurately. Using understanding of what Gamma and Speed truly represent would be the easier approach. Partial points were given to candidates who demonstrated the correct understanding of connecting Speed to the slope of Gamma.

7. Continued

(i)

Gamma

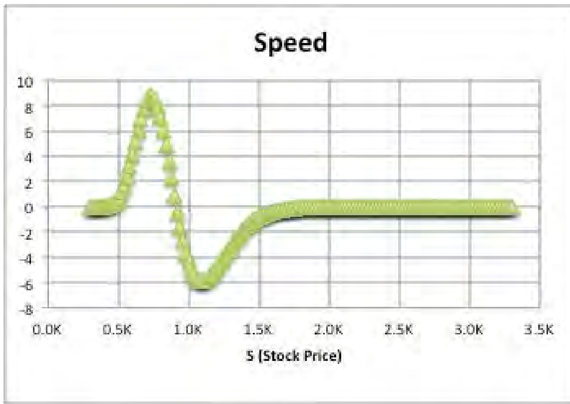


(1) $\lim_{S \to 0^+} \Gamma = 0$
(2) $\lim_{S \to +\infty} \Gamma = 0$
(3) Therefore, the curve of $\Gamma = \Gamma(S)$ increases from 0 at $S = 0$ to the peak at $S = Ke^{-\left(r + \frac{3\sigma^2}{2}\right)t}$ and then decrease to 0 at infinity.
(4) Peak of Gamma is calculated by solving Speed = 0.

7. Continued

(ii)

Speed



For Speed, it is the change of Gamma with respect to price, or the slope of Gamma. So just note the following and trace the shape of Gamma to sketch the graph.

(1) $Speed > 0$ when $S < Ke^{-\left(r+\frac{3\sigma^2}{2}\right)t}$

(2) $Speed < 0$ when $S > Ke^{-\left(r+\frac{3\sigma^2}{2}\right)t}$

(3) $Speed = 0$ when $S = Ke^{-\left(r+\frac{3\sigma^2}{2}\right)t}$

(4) $\lim_{S\to 0^+} \text{Speed} = 0$

(5) $\lim_{S\to +\infty} \text{Speed} = 0$

(c) An intern states that we should reduce the absolute level of Speed, which would result in a more stable Gamma hedge.

Critique the intern's statement.

Commentary on Question:
Candidates did poorly on this part. This question is trying to test the candidates' understanding of gamma hedge. Most candidates understood that lower absolute level of speed would imply less sensitive gamma, but failed to identify that high level of gamma would result in highly sensitive delta.

7. Continued

A high speed would indicate that the gamma is very sensitive to underlying price movement, therefore a lower speed implies less sensitive gamma to underlying price.

However, reducing the absolute level of speed does not guarantee that gamma sensitivity is reduced at all price levels. Furthermore,

1. Stability of gamma at high levels might not be a desirable feature since it would imply option delta is sensitive to the underlying. Given that lower order greeks are usually more important than higher order residuals, if stability of gamma was achieved at the price of highly sensitive delta, it would not be a good feature in a portfolio.
2. Hedging higher-order greeks such as speed is expensive and complicated and usually would result in some other greek imbalances.

8. Learning Objectives:

1. The candidate will understand the fundamentals of stochastic calculus as they apply to option pricing.

3. The candidate will understand the quantitative tools and techniques for modeling the term structure of interest rates and pricing interest rate derivatives.

Learning Outcomes:

(1d) Understand and apply Ito's Lemma.

(3c) Understand and apply popular one-factor interest rate models including Vasicek, Cox-Ross-Ingersoll, Hull-White, Ho-Lee, Black-Derman-Toy, Black-Karasinski.

(3d) Understand the concept of calibration and describe the issues related to calibration, including yield curve fitting.

Sources:

Introduces Quantitative Finance, Wilmott, Paul, 2nd Edition, Chapter 17, 19

An Introduction to the Mathematics of Financial Derivatives, Hirsa and Neftci, 3rd Edition, Chapter 10

Commentary on Question:

This question tests candidates' understanding of HJM and BGM models and their applications in financial markets. It also tests the application of Ito's Lemma to derive stochastic process. In general, some candidates demonstrated weakness on the quantitative part of the question and could hardly write a comprehensive solution to receive full marks.

Solution:

(a) Describe the pros and cons of using the HJM model.

Commentary on Question:
This part of question is straightforward, as long as candidates answer one Pro and one Con, they will get full marks.

Pros:

- HJM models the entire forward rate curve, and hence automatically fits the initial term structure
- Non-Markov nature of HJM model

Cons:

- The continuous compound rates used in HJM are not observable so there is no guarantee that interest rates will stay positive

8. Continued

(b) Show that $m(t,T) = s(t,T)\int_t^T s(t,u)\,du$

(Hint: Express $F(t,T)$ in terms of $P(t,T)$.)

Commentary on Question:
This part of question expects candidates to apply Ito's Lemma by using the relationship of F(t, T) and P(t, T). Most candidates can get full credit for this part.

The arbitrage free relation defines P (t, T) = exp (- $\int_t^T F(t,u)\,du$

Therefore the instantaneous forward rate F (t, T) = $-\frac{\partial}{\partial T}\log P(t,T)$

Given $dP(t,T) = r(t)\,P(t,T)\,dt + v(t,T)\,P(t,T)\,dW$, by Ito's Lemma:

$$d\log P(t,T) = (\frac{\partial}{\partial P}\log P)dP + (\frac{1}{2}\frac{\partial^2}{\partial P^2}\log P)(dP)^2$$
$$= [r(t) - ½\, v(t,T)^2\,]\,dt + v(t,T)\,dW$$

Therefore, dF(t, T) = $\frac{\partial}{\partial T}\left[\frac{1}{2}v(t,T)^2 - r(t)\right]dt - \frac{\partial}{\partial T}v(t,T)dW$

$= \frac{1}{2}\frac{\partial}{\partial T}v(t,T)^2\,dt - \frac{\partial}{\partial T}v(t,T)dW$

$= \frac{1}{2}\,2\,v(\text{t},\text{T})\frac{\partial}{\partial T}v(t,T)dt - \frac{\partial}{\partial T}v(t,T)dW$

$= v(\text{t},\text{T})\frac{\partial}{\partial T}v(t,T)dt - \frac{\partial}{\partial T}v(t,T)dW$

s(t, T) = $-\frac{\partial}{\partial T}v(t,T)$

m(t, T) = $v(\text{t},\text{T})\frac{\partial}{\partial T}v(t,T)$ = s(t, T)$\int_t^T s(t,u)\,du$

(c) Show that in this situation the diffusion coefficient of the bond price SDE is in the same form as the diffusion coefficient of the bond price SDE under the Hull-White model.

Commentary on Question:
This part of question involves comparing the diffusion term between Hull-White and HJM models. Most candidates were able to derive m(t, T) or v(t, T) based on the given information, but they didn't know how to get the bond price SDE or how to get its diffusion coefficient. Some candidates also were unable to identify the correct formulaic form of the Hull-White model short rate process. Few candidates earned full credit on this part.

8. **Continued**

Under the HJM model's special case

Based on part (b), we have s(t, T) = $\sigma e^{-a(T-t)}$
Then v(t, T) = - $\int_t^T S(t,u)\,du = \frac{\sigma}{a}[e^{-a(T-t)} - 1]$

The diffusion coefficient of the bond price SDE

$$P(t,T)v(t,T) = -P(t,T)\frac{\sigma}{a}[1 - \exp(-a(T-t))]$$

Under Hull-white model bond price is given by

$$Z(r,t:T) = \exp(A(t;T) - rB(t;T))$$

with $B(t;T) = \frac{1}{\gamma}[1 - \exp(-\gamma(T-t))]$ and

$$dr = (\eta(t) - r)dt + cdX$$

Using the Ito's Lemma

$$dZ = \frac{\partial Z}{\partial r}dr + \frac{1}{2}\frac{\partial^2 Z}{\partial r^2}(dr)^2$$

Since $(dr)^2 = cdt$ the diffusion coefficient of dz is $c\frac{\partial Z}{\partial r} = -ZcB(t;T)$ which is in the same form as the diffusion coefficient of dP.

(d) Outline the main features of the two models under the risk-neutral measure.

Commentary on Question:
This part is rather straightforward. So long as candidates answered one feature for each model, they receive full credit for this part.

HJM:

- No arbitrage model, volatility term may depend on past values
- Calibrated from instantaneous forward rates

BGM:

- Overcome calibration problems with HJM
- Implies lognormal process for forward LIBOR rates under equivalent martingale measure (discrete version of HJM)
- Volatility function is time-dependent but deterministic

(e) Describe the advantages and disadvantages of using these models to price and hedge exotic interest rate derivatives.

8. Continued

Commentary on Question:
Candidates are expected to specify advantages and disadvantages of these models to value exotic interest rate derivatives. Many candidates made a list of general features of the models but made no reference in pricing or hedging those derivatives. As long as candidates mentioned at least one correct advantage and one correct disadvantage for each model, they will earn full marks.

HJM:
Advantage:
- no need to derive models of spot rates for each individual terms
- the dynamics of forward rates can be used to determine the dynamics of zero-coupon bonds, which can be used to price interest rate swaption

Disadvantage:
- Instantaneous forward rates are unobservable
- It produces a non-Markovian process which will generate non-recombining trees to model exotic interest rate swaptions
- Pricing can be time consuming which requires heavy Monte-Carlo simulation

BGM:
Advantage:
- The discrete measure of LIBOR market model overcomes the problem of HJM that requires unobservable instantaneous forward rates, it just relies on a set of forward rates which are directly observable and related to traded contracts in the market
- It can price any contract whose cash flows can be decomposed into functions of observable forward rates
- It is consistent with Martingale pricing

Disadvantage:
- The lognormal dynamics of forward rate is questionable
- The deterministic volatility is restrictive

9. **Learning Objectives:**

1. The candidate will understand the fundamentals of stochastic calculus as they apply to option pricing.

Learning Outcomes:

(1d) Understand and apply Ito's Lemma.

(1h) Define and apply the concepts of martingale, market price of risk and measures in single and multiple state variable contexts.

Sources:

Chin, Nel, Olafsson, Chapter 2 (definition 2.5, Section 2.2.3 examples 1 and 2)

Commentary on Question:

This question tests the candidates' fundamental understanding of the conditions for a process to be a martingale and the application of Ito's lemma to identify a process to be martingale. In general, candidates answered this question quite well.

Solution:

(a) State the three conditions for a stochastic process to be a martingale.

Commentary on Question:

This part of the question is quite straightforward, just definition 2.5 from the textbook.

A stochastic process X_t is said to be a continuous-time martingale if it satisfies the following conditions:

(i) $E(X_t \mid F_s) = X_s$, *for all* $0 \leq s \leq t$.

(ii) $E(|X_t|) < \infty$

(iii) X_t *is* F_t*- adapted.*

(b) Demonstrate that the process

$$X_t = B_t^2 - t$$

is a martingale by showing that each of the three conditions in part (a) holds.

9. **Continued**

Commentary on Question:
This part of question involves the application of martingale conditions in part (a) by giving a particular form of stochastic process. While many candidates can demonstrate how each of the three conditions are met, some did have trouble showing how conditions (ii) and (iii). For example, some candidates tried to verify meeting the finite condition for $E(X_t)$ instead of the absolute value version $E(|X_t|)$. Some candidates simply stated that the condition F_t-adapted is obvious.

Given B_t is a standard Brownian motion, it follows that:

(i) For all $0 \leq s \leq t$, first we write the expansion of

$$\begin{aligned} B_t^2 - t &= ((B_t - B_s) + B_s)^2 - t \\ &= (B_t - B_s)^2 + 2B_s(B_t - B_s) + B_s{}^2 - t \end{aligned}$$

and using the independent increments between $B_t - B_s$ and B_s, it follows that

$$\begin{aligned} E(X_t \mid F_s) &= E((B_t - B_s)^2 \mid F_s) + E\left(B_s{}^2 \mid F_s\right) - t \\ &= (t - s) + B_s{}^2 - t \\ &= B_s{}^2 - s = X_s \end{aligned}$$

(ii) Because we know that $|B_t^2 - t| \leq |B_t^2| + |t| = B_t^2 + t$, we then have

$$E(|B_t^2 - t|) \leq E(B_t^2) + t = t + t = 2t < \infty$$

(iii) The 3rd condition follows immediately because X_t is a function of B_t.

(c) Derive the stochastic differential equation for the process

$$Y_t = B_t^4 - 6tB_t^2 + ct^2$$

where c is a given constant by using Ito's lemma.

Commentary on Question:
This part of questions tests the application of Ito's Lemma. Most candidates were able to receive full credit, but some candidates derived the wrong version for $\frac{\partial g}{\partial t}$ or forgot the multiple 0.5 for the second derivative term.

9. Continued

By writing $X_t = g(B_t, t) = B_t^4 - 6tB_t^2 + ct^2$, according to Ito's lemma, we have

$$dX_t = \frac{\partial g}{\partial t}dt + \frac{\partial g}{\partial B_t}dB_t + \frac{1}{2}\frac{\partial^2 g}{\partial B_t^2}(dB_t)^2$$

Since

$$\begin{aligned}
\frac{\partial g}{\partial t} &= -6B_t^2 + 2ct \\
\frac{\partial g}{\partial B_t} &= 4B_t^3 - 12tB_t \\
\frac{\partial^2 g}{\partial B_t^2} &= 12B_t^2 - 12t
\end{aligned}$$

the process is equivalent to

$$\begin{aligned}
dX_t &= (-6B_t^2 + 2ct)dt + (4B_t^3 - 12tB_t)dB_t + \frac{1}{2}(12B_t^2 - 12t)dt \\
&= (2ct - 6t)dt + (4B_t^3 - 12tB_t)dB_t
\end{aligned}$$

(d) Determine the value(s) of the constant c, if any, for the process Y_t in part (c) to be a martingale.

Commentary on Question:
As long as candidates know how to make it a martingale by stating that the drift part should be zero and calculate c accordingly, regardless of whether they got the correct dt of dX_t or not, candidates receive full credit.

For this to be a martingale, the drift part must be zero. Thus, we must have $c = 3$.

10. Learning Objectives:

5. The candidate will understand and identify the variety of fixed instruments available for portfolio management. This section deals with fixed income securities. As the name implies the cash flow is often predictable, however there are various risks that affect cash flows of these instruments. In general, the candidates should be able to identify the cash flow pattern and the factors affecting cash flow for commonly available fixed income securities. Candidates should also be comfortable using various interest rate risk quantification measures in the valuation and managing of investment portfolios. Candidates should also understand various strategies of managing the portfolio against given benchmark.

Learning Outcomes:

(5i) Construct and manage portfolios of fixed income securities using the following broad categories.

(i) Managing funds against a target return

(ii) Managing funds against liabilities.

Sources:

Managing Investment Portfolios: A Dynamic Process, Maginn & Tuttle, 3rd Edition, 2007, Ch. 6, Fixed Income Portfolio Management

Commentary on Question:

This question tests on knowledge and application of immunization against liabilities.

Solution:

(a) Critique each of your colleague's statements.

Commentary on Question:

Most candidates did well in this part. The question asks candidates to critique the statements. Therefore, candidates who merely state "correct" or "incorrect" did not receive full credit.

(i) This statement is only true if the yield curve is upward sloping. If the yield curve is downward sloping, the immunization target rate of return would exceed the yield to maturity due to higher reinvestment return.

(ii) This statement is incorrect. The portfolio needs to be rebalanced whenever interest rates change and as time elapses since the previous rebalancing.

(iii) This statement is correct. Since immunized portfolios need to be rebalanced, the liquidity of securities used is a relevant consideration. Illiquid securities involve high transaction costs and make portfolio rebalancing costly.

10. Continued

(iv) This statement is incorrect. This portfolio would also be exposed to the risk of a change in the shape of the yield curve.

(v) This statement is incorrect. The manager must understand the duration and convexity of both the assets and liabilities. Focusing only on the duration of a company's assets will not give a true indication of the total interest rate risk for a company.

(vi) This statement is in correct. The distribution of durations of individual portfolio assets must have a wider range than the distribution of the liabilities.

(b) Calculate the amount of cash required to rebalance the portfolio in order to maintain the dollar duration at the initial level, assuming the proportion by par value is unchanged.

Commentary on Question:
Many candidates were able to calculate the portfolio dollar durations. Some failed to calculate and apply the rebalancing ratio, while some failed to maintain the proportion of par value.

Initial portfolio dollar duration

	Weight of par value	MV	Duration	Dollar Duration
Bond 1	0.33	1,010,697	4.660	47,098.48
Bond 2	0.33	998,619	1.879	18,764.05
Bond 3	0.33	1,097,032	8.598	94,322.81
			Dollar Duration	160,185.34

Portfolio dollar duration after one year (before rebalancing):

	Weight of par value	MV	Duration	Dollar Duration
Bond 1	0.33	1,008,983	3.753	37,867.13
Bond 2	0.33	1,000,054	0.901	9,010.49
Bond 3	0.33	1,088,490	7.791	84,804.26
			Dollar Duration	131,681.87

Rebalancing ratio = 160,185.34 / 131,681.87 = 1.216

Cash required = 0.216 x (1,008,893 + 1,000,054 + 1,088,490)
= 670,482 or $670K

10. Continued

Alternative solution:

Average portfolio duration after one year

$$= \frac{1{,}008{,}983(3.753) + 1{,}000{,}054(0.901) + 1{,}088{,}490(7.791)}{1{,}008{,}983 + 1{,}000{,}054 + 1{,}088{,}490}$$

= 4.2512

Dollar duration needed for rebalancing
= 160,185.34 - 131,681.87 = 28,503.47

Cash required = $\frac{28{,}503.47}{4.2512}$ x 100

= 670,482 or $670K

(c) Calculate the new weights by par value for all three bonds.

Commentary on Question:
Many candidates struggled on this part. Some candidates confused par value with market values and received partial credit. Some failed to increase the par value of Bond 1 and Bond 3 equally.

Since the average portfolio dollar duration fell from the initial value, we need to lengthen the duration. Therefore, one should sell a portion of Bond 2 as it has the shortest duration, and invest into Bond 1 and 3.

Current par weight of Bond 1, 2, 3 = 33.33% each

Let
w_1 = New par weight of Bond 1
w_2 = New par weight of Bond 2
w_3 = New par weight of Bond 3
MV_1 = New market value of Bond 1 = 1,008,983 x $\frac{w_1}{0.3333}$
MV_2 = New market value of Bond 2 = 1,000,054 x $\frac{w_2}{0.3333}$
MV_3 = New market value of Bond 3 = 1,088,490 x $\frac{w_3}{0.3333}$

Set up equations:
(1) $w_1 + w_2 + w_3 = 1$

To maintain the dollar duration,
(2) Initial dollar duration
= 160,185 = (D_1 x MV_1) + (D_2 x MV_2) + (D_3 x MV_3)

10. **Continued**

Since we want to simultaneously increase the par values of Bond 1 and 3, let $w_1 = w_3$

$2w_1 + w_2 = 1$

$160,185 = 0.01 \text{ x } [(3.753 \text{ x } 1,008,983 \text{ x } \frac{w_1}{0.3333}) + (0.901 \text{ x } 1,000,054 \text{ x } \frac{w_2}{0.3333}) + (7.791 \text{ x } 1,088,490 \text{ x } \frac{w_1}{0.3333})]$
$53,395 = 0.01 \text{ x } [(3.753 \text{ x } 1,008,983 \text{ x } w_1) + (0.901 \text{ x } 1,000,054 \text{ x } w_2) + (7.791 \text{ x } 1,088,490 \text{ x } w_1)]$

Solving the equations, new weights for all three bonds:
Bond 1: w_1 = 42.4% or par value of $1,272,212
Bond 2: w_2 = 15.2% or par value of $455,576
Bond 3: w_3 = 42.4% or par value $1,272,212

<u>Alternative solution:</u>

Current dollar duration of Bond 1 = 1,008.983 x 3.753 x 0.01 = 37,867
Current dollar duration of Bond 2 = 1,000,054 x 0.901 x 0.01 = 9,010
Current dollar duration of Bond 3 = 1,088,490 x 7.791 x 0.01 = 84,804

$160,185 = 37,867 (1 + y) + 9,010 (1 - 2y) + 84,804 (1 + y)$
$y = 0.2724$

New weights for all three bonds:
Bond 1: 1,000,000 (1 + 0.2724) = $1.272 million
Bond 2: 1,000,000 (1 – 2(0.2724)) = $0.456 million
Bond 3: 1,000,000 (1 + 0.2724) = $1.272 million

11. Learning Objectives:

5. The candidate will understand and identify the variety of fixed instruments available for portfolio management. This section deals with fixed income securities. As the name implies the cash flow is often predictable, however there are various risks that affect cash flows of these instruments. In general, the candidates should be able to identify the cash flow pattern and the factors affecting cash flow for commonly available fixed income securities. Candidates should also be comfortable using various interest rate risk quantification measures in the valuation and managing of investment portfolios. Candidates should also understand various strategies of managing the portfolio against given benchmark.

Learning Outcomes:

(5a) Explain the cash flow characteristics and pricing of Treasury securities.

(5c) Demonstrate understanding of the different characteristics of securities issued by government agencies or government-sponsored enterprise.

Sources:

Wilmott Chapter 14: 14.7; 14.16
Maginn & Tuttle Chapter 6: Total Return Analysis
Maginn & Tuttle Chapter 6: Credit Options

Commentary on Question:

Majority [parts (a) to (c)] of this question focus on comprehension and analysis, while part (d) and (e) focus on retrieval. Most candidates did well for the retrieval parts, but showed a lack of understanding for the calculation of spot and forward rates.

Solution:

(a) Calculate the following to the nearest 1 basis point:

(i) $f(1,2)$

(ii) $f(2,1)$

(iii) The YTM of a 2-year, annual-coupon, Treasury bond purchased at par a year from now assuming that spot rates in the future are those implied by today's forward rates.

Commentary on Question:

This part was not well done. Many candidates did not understand the definition of forward rates, spot rates, and yield-to-maturity. The model solution below of finding the spot rates first was the general approach, but alternative approaches of calculating the forward rates or YTM were also accepted.

11. Continued

First, calculate spot rates:
100 = 4.5/1.03 + 104.5/(1+s2)^2
s2 = 4.5343%
100 = 5.5/1.03 + 5.5/(1.045343^2) + 105.5/(1+s3)^3
s3 = 5.5856%

f(1,2) = (1.055856^3/1.03)^0.5 – 1 = 6.90%

f(2,1) = (1.055856^3/1.045343^2) – 1 = 7.72%

Let the YTM of a 2-year Treasury bond a year from now = y.
f(1,1) = (1.045343^2/1.03) – 1 = 6.091%

$$100 = \frac{100y}{1.06091} + \frac{100y + 100}{(1.06091)(1.0772)}$$

y = 6.88%

A 3-year A-rated corporate bond with face value of $100 and annual coupon of 6% is purchased today at par. The purchaser has an investment horizon of one year.

Over the next year, the corporate bond is subject to the following potential rating migrations, with, after the payment of (or default on) the coupon, the corresponding end-of-year YTM spread over treasuries.

End-of-year rating	**A**	**BBB**	**Default**
End-of-year YTM spread over Treasuries (bps)	80	200	-
Probability	90%	9%	1%

When there's a default, the recovery value is 40% of face value. The recovery value applies only if the bond defaults.

Assume that the YTMs on 1-year, 2-year, and 3-year Treasuries are the same a year from now.

(b)

(i) Complete the following table:

End-of-year rating	**A**	**BBB**	**Default**
Market value of remaining payments			
Total return over the year			

(ii) Calculate the expected total return for the bond over the one year.

11. Continued

Commentary on Question:
Some candidates misinterpreted the assumption that "the YTMs on 1-year, 2-year, 3-year Treasuries are the same a year from now" and did not apply the spread properly. When calculating the total return over the year, some candidates forgot to add the coupon payment for rating A and BBB.

If the end-of-year rating is A:

MV of remaining payments $= \frac{6}{1.053} + \frac{106}{1.053^2} = 101.30$

Total Return over the year = (6 + 101.30)/100 -1 = 7.30%

If the end-of-year is BBB:

MV of remaining payments $= \frac{6}{1.065} + \frac{106}{1.065^2} = 99.09$

Total Return over the year = (6 + 99.09)/100 -1 = 5.09%

If there's a default:

MV of remaining payments = 100 x 40% = 40

Total Return over the year = 40/100 -1 = -60.0%

End-of-year rating	A	BBB	Default
Market value of remaining payments	101.30	99.09	40
Total return over the year	7.30%	5.09%	-60%

Expected Total Return over the year

= (0.9 X 7.30%) + (0.09 X 5.09%) + [0.01 X (-60.00%)]

= 6.43%

(c) Calculate the investor's revised expected overall total return over the year, given a strike price of $99.5.

Commentary on Question:
Some candidates did not realize that BBB is investment grade. Many candidates did not understand for the overall total return, the premium of the put option should be an addition to the denominator, rather than a subtraction from the numerator.

11. Continued

With the put option, in the Default state, payoff is \$99.5 - \$40 = \$59.5
Premium of the option = \$1
Revised "Default" total return = (40 + 59.5)/101 -1 = -1.49%
Revised "A" total return = (6 + 101.30)/101 -1 = 6.24%
Revised "BBB" total return = (6 + 99.09)/101 -1 = 4.05%

Realizing that the option only pays off in the "default" state.
Able to calculate the payoff/total return in the "default" state, including adjustment for the premium.

With the put option, Expected overall Total Return over the year
= (0.9 X 6.24%) + (0.09 X 4.05%) + [0.01 X (-1.49%)]
= 5.97%

(d) Describe three types of credit risk.

Commentary on Question:
This part was done well.

Default risk – risk that the inssuer may fail to meet its obligations.
Credit spread risk – Risk that the spread between the rate for a risky bond and the rate for a default risk-free bond may vary after the purchase.
Downgrade risk – Risk that one of the major rating agencies will lower its rating for an issuer.

(e) List 2 other instruments that the investor can use to limit the credit risk of the bond.

Commentary on Question:
This part was done well in general.

Credit spread option
Credit swap
Bond put option
(Other correct answers are also acceptable)

12. **Learning Objectives:**

4. The candidate will understand the concept of volatility and some basic models of it.

Learning Outcomes:

(4a) Compare and contrast the various kinds of volatility, (e.g., actual, realized, implied, forward, etc.).

(4b) Understand and apply various techniques for analyzing conditional heteroscedastic models including ARCH and GARCH.

Sources:

Tsay Chapter 3.3, 3.4, 3.5

QFIC-109-15: Chapter 9 of Risk Management and Financial Institutions, Hull, 2nd Edition

Commentary on Question:

Candidates did very poorly in this question. More than 10% of the candidates skipped/ got zeros in this question. A few got decent scores in the question. But still, none of the candidates got a full score.

Solution:

(a) Determine if the GARCH model is appropriate for the data series by analyzing the patterns of the graphs A through D. Support your reasoning.

Commentary on Question:

This question asked the candidates to analysis the charts to determine if the GARCH model is appropriate. When modelling data series, it is very important to know/understand the charts after fitting different models. Around half of the candidates got zeros/very low score in part (a). It is strongly recommended the candidates can understand the charts before writing the exam. Only less than 10% of the candidates got a good score in part (a).

<u>Graph A and B</u>

Graph A: The r_t series shows some serial correlations at lags 1 and 3.

Graph B: The PACF of r_t^2 series shows strong linear dependence.

This suggests a GARCH model.

<u>Graph C and D</u>

Graph C: We can see the $\tilde{a}_t$ series looks like a white noise process

These ACFs fail to suggest any significant serial correlations or conditional heteroscedasticity in the standardized residual series.

Graph D: The PACF of $\tilde{a}_t^2$ series doesn't show any linear dependence.

This confirms that a GARCH model is adequate.

12. Continued

(b) Determine the correct value of α and ABC's closing price on 9/15/2015.

Commentary on Question:
The candidate did poorly in part (b) as well. More than a quarter of candidates got zeros. Around 20% got a full score. The others wrongly calculated the α and led to a wrong ABC's closing price.

Using the GARCH(1,1) model, we have:

For 9/16/2015: $0.0109^2 = 0.00002 + \alpha u_{9/15}^2 + 0.8 * 0.0107^2$

For 9/17/2015: $u_{9/17} = \frac{10.5}{10.0} - 1 = 0.05$

For 9/18/2015: $0.0167^2 = 0.00002 + \alpha * 0.05^2 + 0.8\ 0.0117^2$

From the above, one solves for

$\alpha = 0.05975$

$u_{9/15} = 0.01099$

ABC's closing price on 9/15/2015 = 10.1*(1+0.01099)= 10.21

(c) Provide your best-estimate forecast of annual volatility of ABC's stock price on 9/23/2016 based on the available information in Table 1, assuming that there are 256 trading days in a year.

Commentary on Question:
The candidate did the worst in part (c). More than 30% of candidate skipped the questions/ got zeros. A number of them didn't read the question carefully and put 252 trading days when calculating the annual volatility, while the question stated "assuming that there are 256 trading days in a year". Some of them didn't calculate the long term volatility V_L. Only a handful of them got perfect in part (c).

The target variance is

$$V_L = \frac{\omega}{1-\alpha-\beta}$$

$$V_L = \frac{0.00002}{1-0.06-0.8} = 0.0001429$$

The best-estimate variance is based on the latest data as of 9/20/2016.

$$E(\sigma_{n+t}^2) = V_L + (\alpha+\beta)^t(\sigma_n^2 - V_L)$$

$$E\left(\sigma_{9/23}^2\right) = 0.0001429 + (0.06+0.8)^3(0.017^2 - 0.0001429)$$

$$E\left(\sigma_{9/23}^2\right) = 0.000236$$

Best-estimate annual volatility on 9/23 = $\sqrt{256 * 0.000236} = 24.4\%$

13. Learning Objectives:

5. The candidate will understand and identify the variety of fixed instruments available for portfolio management. This section deals with fixed income securities. As the name implies the cash flow is often predictable, however there are various risks that affect cash flows of these instruments. In general the candidates should be able to identify the cash flow pattern and the factors affecting cash flow for commonly available fixed income securities. Candidates should also be comfortable using various interest rate risk quantification measures in the valuation and managing of investment portfolios. Candidates should also understand various strategies of managing the portfolio against given benchmark.

Learning Outcomes:

(5i) Construct and manage portfolios of fixed income securities using the following broad categories.
- (i) Managing funds against a target return
- (ii) Managing funds against liabilities.

Sources:

Fabozzi et al, The Handbook of Fixed Income Securities
Ch.21, Fixed Income Exchange Traded Funds

Commentary on Question:

This question focuses on retrieval, analysis, comprehension, and calculation. In general, the candidates did well.

Solution:

(a) Compare and contrast TIPS ETF vs. TIPS under the following metrics:

- (i) Intraday liquidity
- (ii) Tax efficiency
- (iii) Risk control

(i) **<u>Intraday Liquiduity:</u>** The ability to trade fixed income ETF (TIPS ETF) throughout the trading day provides investors with great visibility into portfolio valuation, even during periods of volatility and illiquidity

(ii) **<u>Tax Efficiency:</u>** The ETF creation/redemption mechanisum makes ETFs more tax efficient than traditional fixed income (such as TIPS). When ETF shares are redeemed, the ETF fund delivers securities in-kind (meaning that the underlying securities, TIPS, rather than cash, are distributed). This in-kind distribution is not considered a taxable event for capital gains purpose. For a regular fixed income such as TIPS, realized capital gains (or losses) has a tax implication.

13. **Continued**

(iii) **Risk control**: Because TIPS ETF are exchange traded, and because each fund's underlying securities are held in a separate custodial account, fixed income ETFs have minimal counterparty risk

(b) Explain how arbitrage plays a role in an ETF's creation/redemption mechanism.

Commentary on Question:
Many candidates did not address the expense involved in an ETF's creation and redemption.

Description of either (i) or (ii) below was sufficient for 2 grading points.

(i) During periods of strong demand for an ETF, the price of the shares is bid up in the market. If the ETF price is higher than the value of the underlying securities held within the ETF, an arbitrage opportunity may exist. Authorized participants (e.g., broker/dealers) could purchase the underlying fixed-income securities, create new ETF shares and then sell the newly created ETF shares in the market for a profit.

(ii) Conversely, this same set of mechanics operates in markets of strong selling pressure, to help keep the ETF from trading at a persistent discount. Authorized participants could purchase the ETF shares (at a discount), redeem them and then sell the fixed-income securities received from the redemption at a net profit. Arbitrage helps keep the ETF price in line with the value of the underlying securities.

Description of (iii) below is worth another 2 grading points.

(iii) However, A premium or discount can exist and even persist for an ETF as long as it is not large enough to trigger an arbitrage opportunity. This means that the size of the premium/discount will be bounded by the transaction costs participants would incur in executing the underlying arbitrage transaction. As long as the premium or discount is less than these transaction costs, there is no economic incentive to execute the arbitrage opportunity.

(c) Assess whether the TIPS ETF trades at a premium, or a discount, relative to NAV (net asset value), in the ETF creation. Justify your answer by quantifying the ETF's premium/discount.

Commentary on Question:
Some candidates did not realize that well balanced implies a flow factor of 0.5 which is covered in the syllabus material but requires comprehension.
Some candidates successfully calculated the ETF premium of 150bps, but incorrectly said it was trading at a discount by comparing the 150bps to the creation cost of 200bps.

13. **Continued**

The market price-to-NAV relationship of a fixed income ETF can be estimated as follows:

Fixed income ETF premium/discount = (creation cost × flow factor) + execution risk adjustment

Where

Creation cost equals the bid/offer spread in the underlying market,

Flow factor is a scalar between 0 and 1, representing the balance of ETF flows in the market (0 = all sell orders; 1 = all buy orders), and

Execution risk adjustment equals the cost of basket execution and intraday hedging to facilitate creation/redemption.

Note (from the text): Because the execution risk adjustment is a measure of execution risk, its magnitude is driven by the level of volatility and overall liquidity conditions in the market, while its direction is driven by whether the broker-dealer is creating or redeeming ETF shares (generally positive for creation, generally negative for redemption)

Creation cost = 200 bps
Flow factor = 0.5
Execution risk adjustment for ETF creation = 50 bps

ETF premium/discount to NAV = 200 * 0.5 + 50 = 150 bps

The TIPS ETF currently trades at a 150 bp premium

14. **Learning Objectives:**

6. The candidate will understand the variety of equity investments and strategies available for portfolio management.

Learning Outcomes:

(6d) Demonstrate an understanding of equity indices and their construction, including distinguishing among the weighting schemes and their biases.

(6f) compare techniques for characterizing investment style of an investor.

Sources:

Maginn Tuttle, Ch. 7.

Commentary on Question:

In general, the candidates did well in parts (a) to (d). However, for parts (e) and (f), most candidates had difficulties in identifying the price inefficiencies based on the numerical relationship for the source of alphas for different long-short strategies. They also did not do well in explaining the reasons why those inefficiencies exist.
Some candidates did not use appropriately the figures given in the table to justify their analysis.

Solution:

(a) Identify whether the RSI index style is value or growth and explain why.

It is Value Investment Style as it focuses on buying a stock that is deemed relatively cheap in terms of the purchase price of earnings or assets than about a company's future growth prospects. RSI is a Value Index as it focuses on companies with relatively lower price compared to earnings.
A Growth Investment Style focuses on buying stocks that have higher future potential for growth in earnings.

(b) Describe four choices to be made when constructing a stock index and explain how each choice is applied to RSI index.

Four choices when constructing a stock index:

1. The boundaries of the index's universe: the RSI index is an index with narrow universe of a specific group of stocks (e.g. low P/E ratio stocks within Ruritania);
2. Criteria for inclusion: criteria to select stocks to the RSI index is the lowest quartile of companies by P/E ratio within Ruritania;
3. Weighting of the stocks: the equity in the index is weighted equally;
4. Computational method: the return calculation could be price only, or total return series. The RSI index return calculation includes dividend and reinvestment income and thus it uses the total return series method.

14. Continued

(c) Explain how the portfolio fund's construction method and the criteria for exclusion impact RSI index trading costs.

Equally weighted Index: Frequent rebalancing is needed to maintain the equal weights, more weight is given to smaller and potentially less liquid companies. => increase trading cost.

Monthly inclusion / exclusion: This tends to lead to higher volatility as frequent review on which companies are included / excluded from the index will increase rebalancing/trading costs.

From Portfolio fund construction: Full replication requires investment in all components of the index. This increases costs, particularly with an equally weighted approach which may require significant holdings in small and less liquid company stocks.

(d) Recommend changes to the construction of RSI index that would make it easier to track.

Index with the following characteristics would be easier to track:

1. Value-weighted rather than equal weighted: this reduces frequency of rebalancing;

2. Less frequency of index composition review: this reduces requirement to rebalance;

3. Index with buffering: instead of using categorization, index may use quantities with overlay to place partial weighting on stocks close to quartile boundary instead of fully including or fully excluding.

(e) Identify the price inefficiencies based on the data provided.

The short-side alphas were significantly greater than the long-side alphas for the long-short strategy (2.2 > 0.3, 2.8> 0.5, 2.5 > 0.4). This performance difference confirms that price inefficiencies can be found on the short side of the market than on the long side.

The long only strategy's long alphas < the long-short alpha long alphas.(0.2 < 0.3, 0.25< 0.5, 0.2 < 0.4).

14. Continued

(f) Explain why the price inefficiencies identified in part (e) exist.

Short only investment is considered price inefficient because:
1. Relatively few searches for overvalued stocks compared to undervalued stocks due to impediments to short selling;

2. Opportunities to short a short a stock may arise because of management fraud; there are fewer similar opportunities arisen for the long side;

3. Sell-side analysts issue more reports with a buy recommendation than a sell recommendation, as there are much more potential buyers than those already own a stock;

4. Sell-side analysts are reluctant to issue negative opinions on companies' stocks due to vested interest relationship with the companies.

The long-short strategy has the ability to short stocks, which allows fund managers to further exploit positive information on the long side.

In a long-short portfolio, the position weights can be outside the 0% to 100% range because shorting releases money/capital which can be used to take on larger long positions than otherwise possible. In a long-only portfolio, the weight of each stock in the portfolio is limited to the 0% to 100% range.

15. Learning Objectives:

6. The candidate will understand the variety of equity investments and strategies available for portfolio management.

Learning Outcomes:

(6h) Describe the core-satellite approach to portfolio construction with a completeness fund to control overall risk exposures.

Sources:

Maginn and Tuttle Chapter 7.

Commentary on Question:

Candidates did well in the comprehension of the SRI investment and the application of the formulae for the active/misfit returns, the information ratio, and the information coefficient (IC).
Candidates missed part (e) completely. Candidates tried to identify the relationships between active/passive management while the question was about the active management itself (how to generate alphas) and its limitations. Both the question and the answer are clearly identified in the reading material, but the candidates may have not paid attention to this particular issue.

Solution:

(a) Define socially responsible investing (SRI).

Socially responsible investing (SRI) may be defined in many ways, such one of the following for example:

- Socially responsible investing (SRI) integrate ethical values and societal concerns with investment decisions;
- Any investment strategy that seeks to consider both financial return and social good to bring about a social change;
- Investment strategy devoted to conscious creation of social impact through investment;
- Investment practice that seeks to avoid harm by screening companies included in an investment portfolio.

(b)

(i) Describe the two commonly used implementation approaches to achieve SRI.

(ii) List two potential risks associated with SRI from the portfolio management point view.

15. **Continued**

(i) SRI commonly involves negative screening and positive screening strategies to achieve the SRI goal:

(1) Negative screening: apply a set of SRI criteria to reduce an investment universe to a smaller set of securities satisfying SRI criteria, including –

- Industry classification, reflecting concerns for sources judged to be ethically questionable (such as tobacco, gaming, alcohol, and armaments, etc);
- Corporate practices (e.g., practices relating to environmental pollution, human rights, labor standards, animal welfare and integrity in corporate governance.

(2) Positive screening: includes criteria used to identify companies that have ethically desirable characteristics (positive social and/or environmental impact).

(ii) Two concerns related to SRI from the portfolio investment perspective:

(1) Might increase concentration risk: Portfolio manager should track any style bias induced by SRI selection process. For example, applying a negative screening, the portfolio might exclude companies from certain industries or sectors, leading to concentration risk.

(2) Might reduce diversification benefits: by limiting the investment universe to certain areas (with a positive screening process) or by excluding stocks/funds from the total universe, the efficient frontier might get squeezed to a certain degree, unable to achieve the full diversification benefits possible.

SRI might inadvertently increase the portfolio risk by concentrating investment in certain sub-universe (by using positive screening).
Some SRI stocks/funds might have limited liquidity and therefore higher transaction costs when rebalancing the portfolio.

(c) Calculate the following metrics for Managers A and B:

(i) True active return

(ii) Misfit active return

15. Continued

(i) Manager's true active return = Manager's return – Manager normal benchmark
A: = 4.8% - 3.2% = 1.6%
B: = 5.0% - 4.0% = 1.0%

(ii) Manager's misfit return = Manager's normal benchmark – Investor's benchmark
A: = 0
B: = Russell 2000 index return – S&P 500 index return = 4.0% - 3.2% = 0.8%

(d)

(i) Recommend the measure for Jerry to use.

(ii) Assess which manager, A or B, has better investment skills.

Information ratio (IR) computed as (true active return)/(true active risk) is the most accurate.
This IR reflects the efficiency with which a portfolio's tracking risk delivers active returns and is based on the manager's normal benchmark.

$$IR = \frac{E[R_p - R_b]}{\sigma}$$

For Manager A, Information ratio = $\frac{4.8\% - 3.2\%}{1.8\%}$ = 0.89

For Manager B, Information ratio = $\frac{5\% - 4.0\%}{3.2\%}$ = 0.31

Using Information ratio as the measure, Manager A outperforms B.

(e)

(i) Describe how semi-active equity managers generate alpha using stock selection.

(ii) State the Fundamental Law of Active Management and estimate Manager A's Information Coefficient (IC).

(iii) Describe limitations of the semi-active stock-selection approach.

15. Continued

(i) Look at broad themes relating to company's valuation or growth;
Build models to process vast quantity of data;
Apply high degree of risk control and high-breadth strategy as explained by the Fundamental Law of Active Management.

(ii) Fundamental Law of Active Management:

$$IR \approx IC\sqrt{\text{Breadth}}$$

Manager B actively tracks 500 stocks, thus his investment discipline's breath = 500

Given the information ratio calculated previously at 0.89, we can back out his IC is is $\frac{0.89}{\sqrt{500}}$ = 0.04

(iii) Any technique that generates positive alpha may become obsolete.
Quantitative models derived from historical data may become invalid;
Markets may undergo shocks.

16. **Learning Objectives:**

7. The candidate will understand how to develop an investment policy including governance for institutional investors and financial intermediaries.

Learning Outcomes:

(7a) Explain how investment policies and strategies can manage risk and create value.

(7c) Determine how a client's objectives, needs and constraints affect investment strategy and portfolio construction. Include capital, funding objectives, risk appetite and risk-return trade-off, tax, accounting considerations and constraints such as regulators, rating agencies, and liquidity.

(7d) Incorporate financial and non-financial risks into an investment policy, including currency, credit, spread, liquidity, interest rate, equity, insurance product, operational, legal and political risks.

Sources:

Managing Investment Portfolios: A Dynamic Process, Maginn & Tuttle, 3rd Edition, 2007, Ch. 3

Commentary on Question:

This question tests the relationship between an entity's risk management process and its investment objectives & strategies.

Solution:

(a) Determine whether DFK's ability to take risk has increased or decreased based mainly on the change in business mix.

Commentary on Question:

This part was answered reasonably well by the majority of candidates.

Time horizon: Since the duration of liabilities is decreasing, the time horizon should be reduced accordingly and hence decrease the ability to take risk.

Reinvestment risk: As the fixed rate annuity product grows as a percentage of DFK's business, reinvestment risk increases since contract rates are guaranteed using estimates of the rate at which interest payments will be reinvested. The fixed rate nature of the annuities reduces DFK's ability to take risk.

Surplus: DFK's surplus has dropped from $100 million to $45 million, thus reducing its ability to take risk.

16. **Continued**

Liquidity needs: In a rising rate environment, fixed rate annuities are more likely to be subject to disintermediation. Without sufficient liquidity, DFK may be forced to sell securities at a loss to meet surrenders of policies and annuity contract disbursements. The need for greater liquidity reduces DFK's ability to take risk.

(b) Describe two constraints in the investment policy statement that are affected solely by the change in business mix.

Commentary on Question:
Part (b) asks to describe two (2) constraints. If a candidate listed more than 2 constraints, the first 2 were used for grading. Some candidates listed a number of constraints, some of which were not even relevant to DFK's situation.

Examples of two constraints are listed below:

Liquidity - The fixed rate annuities will require a higher level of liquidity in order to meet the current periodic payouts to annuity holders.

Time horizon - time horizon is shorter due to the changing product mix. Since DFK's life insurance contracts have significantly higher duration than the fixed rate annuities, the change in business mix toward annuities has decreased the duration of DFK's liabilities from 14.8 to 12.1

(c) Assess the likely effect of each of the following risks on DFK's surplus if XYZ's forecast is correct:

1. Valuation risk
2. Cash flow volatility risk
3. Credit risk
4. Reinvestment risk

Commentary on Question:
This part asks for the impact on DFK's surplus in a rising interest rate environment along with widening credit spreads. Only partial points were awarded if a candidate did not specifically mention how the surplus was impacted; some candidates only talked about the impact on the particular risk.

In a rising interest rate environment along with widening credit spreads, asset/liability/surplus considerations figure prominently:

16. Continued

Valuation risk:
In a rising interest rate environment and widening credit spreads, a mismatch between the duration of assets and that of its liabilities would affect the surplus. In 2015, after change in the business mix, holding assets with an average duration of 13.2, that exceeds the average duration of liabilities of 12.1, would lead to erosion of surplus over time.

Cash flow volatility risk:
The large holding in mortgage securities adds uncertainty to cash flows. When interest rates rise, slower prepayment rates reduce cash inflows and associated interest on interest yield.

Credit risk:
Credit risk in investment grade and high-yield fixed income holdings will affect surplus. The risk is that credit spreads can widen, leading to lower asset valuations and potential defaults. Either of these situations could reduce surplus.

Reinvestment risk:
The reinvestment risk/return of coupon and principal payments from corporate bonds and mortgage securities is influenced by interest rate volatility. Higher interest rates will lead to higher reinvestment rates, which would have a positive impact on surplus.

17. Learning Objectives:

8. The candidate will understand the theory and techniques of portfolio asset allocation.

Learning Outcomes:

(8b) Propose and critique asset allocation strategies.

(8d) Incorporate risk management principles in investment policy and strategy, including asset allocation.

(8e) Understand and apply the concept of risk factors in the context of asset allocation.

Sources:

Managing Investment Portfolios: A Dynamic Process, Maginn & Tuttle, 3rd Edition Ch. 5

Commentary on Question:

It is a straight forward question and most of them are simple calculations. Most of the candidates did good job on this.

Solution:

(a) Determine the required return for the foundation.

Additive

$R_L = \frac{20}{500} + .02 + .002 = 6.2\%$

OR

Multiplicative

R_L =(1+20/500)* 1.02* 1.002-1=6.3%

(b) Determine which Corner Portfolio, 2 or 3, is preferred based on the following measures, respectively:

(i) Sharpe ratio criterion;

Sharpe ratio is defined as the average return minus the risk-free return divided by the standard deviation of return on an investment

$$S = \frac{E[R - R_f]}{\sqrt{\mathrm{var}[R]}}$$

17. Continued

For portfolio 1, Sharpe ratio = 0.361, so R_f = 9% - 0.361*18% = 2.5%
For portfolio 2, Sharpe ratio = $\frac{8\%-2.5\%}{15\%}$ = 0.367

For portfolio 3, Sharpe ratio = $\frac{7.7\%-2.5\%}{13\%}$ = 0.4

Portfolio 3 has the higher Sharp ratio hence is the better choice

(i) The foundation's expected utility;

$$U_m \;=\; \mathrm{E}(\mathrm{R}_m) - \alpha * \sigma_m^2$$

U_1 = 1.87, so $\alpha = \frac{9\%-1.87\%}{(18\%)^2} = 2.2$
U_3 = $7.7\% - 2.2(13\%)^2 = 3.98\%$
Portfolio 3 has the higher U_m, and is the better choice

(ii) Roy's safety-first criterion.

$$\text{SFRatio} \;=\; \frac{E\,(R_P) - \;R_L}{\sigma_p}$$

Additive
$\text{SFRatio}_2 = \frac{8\%-6.2\%}{15\%} = 0.12$
$\text{SFRatio}_3 = \frac{7.7\%-6.2\%}{13\%} = 0.115$
Portfolio 2 has the higher SFRatio, and would be the better choice.
Or Multiplicative
$\text{SFRatio}_2 = \frac{8\%-6.3\%}{15\%} = 0.114$ (or 0.113)
$\text{SFRatio}_3 = \frac{7.7\%-6.3\%}{13\%} = 0.108$
Portfolio 2 has the higher SFRatio, and would be the better choice.

17. **Continued**

(c) Explain the following concepts of the mean-variance approach:

(i) Efficient frontier, minimum-variance frontier (MVF), and sign-constrained MVF;

- Efficient portfolios offer the maximum expected return for their level of variance of return. Efficient portfolios plot graphically on the efficient frontier.
- Efficient Frontier is part of minimum variance frontier (MVF)
- Each portfolio on MVF represents the portfolio with the smallest variance of return for it is level of expected return.
- Minimum-Variance Optimization (MVO) is used the identify a portfolio with the desired combination of expected return and variance (determine that portfolio's asset-class weight).
- There is a constraint that asset-class weights be non-negative and sum to 1 (each asset class in an efficient portfolio is either 0 weight or positive weight).

(ii) Adjacent corner portfolios;

A sement of the MVF within where:

- Portfolios hold identical assets
- As the MVF passes through a corner portfolio, an asset weight either changes from 0 to positive or from positive to 0.
- Corner portfolios allow to create other minimum-variance portfolios
- Interest rate x% = W*(interest rate P_t) + (1-W) *(interest rate P_{t+1}), where W and (1-W) will become the weights of assets in portfolios P_t and P_{t+1} that will in combination produce the required return.

(iii) The tangency portfolio.

- the tangency portfolio is the efficient portfolio with the highest Sharpe Ratio.
- When the expected return of the Tangency portfolio exceeds the return objective, it may be optimal for the investor to hold the highest Sharpe-ratio efficient portfolio in combination with the risk-free asset (Capital allocation line analysis).
- If the expected return of the tangency portfolio is lower than the return objective, assuming that margin is not allowed, this portfolio is not optimal.

17. Continued

(d) Determine the most appropriate asset allocation for asset classes A to D given the objective of meeting the minimum required return.

The minimum required return is 6.2% or 6.3% from part (b)

Using 6.2% we find
6.2% = W*7.7% + (1-W) *5.8%
W = 0.21; (1-W) = 0.79
Asset class A .21*65% + .79*30% = 0.37
Asset class B .21*5% + .79*10% = 0.09
Asset class C .21*0% + .79*30% = 0.24
Asset class D .21*30% + .79*30% = 0.30
$\sum W = 1$

Credits were given for solutions using 6.3% as well.

(e) Determine whether the foundation should include the new asset in the portfolio, assuming the use of a linear approximation approach to approximate the standard deviation of the efficient portfolio.

Adding the asset class to the portfolio is optimal if the following condition is met:

$$\frac{E(R_{new}) - R_F}{\sigma_{new}} > \frac{E(R_p) - R_F}{\sigma_p} Corr(R_{new}, R_p)$$

The expression says that in order for the investor to gain by adding the asset class, that asset class's Sharpe ratio must exceed the product of the existing portfolio's Sharpe ratio and the correlation of the asset class's rate of return.

For R_p = 6.2%
σ_p = 0.21*13% +0.79* 8%=9.05%
(a linear approximation of standard deviation of efficient portfolio)

$$\frac{E(R_{new}) - R_F}{\sigma_{new}} = \frac{4\% - 2.5\%}{10\%} = 0.15$$
$$\frac{E(R_p) - R_F}{\sigma_p} Corr(R_{new}, R_p) = \frac{6.2\% - 2.5\%}{9.05\%} 0.6 = 0.245$$

17. Continued

The condition is not met, should not include the asset.
Or for $R_p = 6.3\%$
$\sigma_p = 0.26*13\% + 0.74*8\% = 9.30\%$
(a linear approximation of standard deviation of efficient portfolio)

$$\frac{E(R_{new}) - R_F}{\sigma_{new}} = \frac{4\% - 2.5\%}{10\%} = 0.15$$
$$\frac{E(R_p) - R_F}{\sigma_p} Corr(R_{new}, R_p) = \frac{6.3\% - 2.5\%}{9.30\%} 0.6 = 0.245$$

The condition is not met, should not include the asset.

(f) Explain why the estimation error is an important issue in using the mean-variance optimization approach (MVO).

Estimation error
Asset allocations are highly sensitive to small changes in inputs.
The most important inputs in MVO are expected returns. It is the most difficult input to estimate. Forecasting returns, volatilities, and correlations is so difficult and subject to substantial estimation error.

(g) Describe approaches or techniques that can be used to mitigate the impact of the estimation error.

Mitigate estimation error
- The investor should conduct sensitivity (simulation) analysis with different expected return estimates, standard deviations, and expected correlations.
- Use the concept of the resampled efficient frontier.
- The Black-Litterman Approach or Model.
- Factor-based optimization.