QFI CORE Model Solutions
Fall 2017

1.

Learning Objectives:

1. The candidate will understand the fundamentals of stochastic calculus as they apply to option pricing.

Learning Outcomes:

(1c) Understand Ito integral and stochastic differential equations.

(1h) Define and apply the concepts of martingale, market price of risk and measures in single and multiple state variable contexts.

Sources:

Problems and Solutions Math Finance pp.57, 65, 72

Commentary on Question:

Commentary listed underneath question component.

Solution:

(a) Find the correlation coefficient of W_{t_1} and W_{t_2} for $0 < t_1 \le t_2$.

Commentary on Question:
Most candidates received full credit for this part.

We have that
$Var(W_t) = t,\ E(W_t) = 0$, so for $t_2 > t_1$, $E\left(W_{t_2}W_{t_1}\right) = E\left(E\left(W_{t_2}W_{t_1}\right)|\mathcal{F}_{t_1}\right) =$
$E\left(E\left(W_{t_2}|\mathcal{F}_{t_1}\right)W_{t_1}\right) = E\left(W_{t_1}W_{t_1}\right) = t_1$.
Hence,

$$\rho = \frac{Cov\left(W_{t_2}, W_{t_1}\right)}{\sqrt{Var\left(W_{t_2}\right)}\sqrt{Var\left(W_{t_1}\right)}} = \frac{t_1}{\sqrt{t_2 \cdot t_1}} = \sqrt{\frac{t_1}{t_2}}.$$

(b) Find the variance of $\int_0^t W_s ds$.

Commentary on Question:
This part was the most difficult for candidates, although most were able to correctly derive the variance.

1. Continued

$$Var\left(\int_0^t W_s ds\right) = E\left\{\left(\int_0^t W_s ds\right) - E\left(\int_0^t W_s ds\right)\right\}^2$$
$$= E\left(\int_0^t W_s ds\right)^2.$$
$$E\left(\int_{s=0}^{s=t} W_s ds\right)^2 = E\left[\left(\int_{s=0}^{s=t} W_s ds\right)\left(\int_{u=0}^{u=t} W_u du\right)\right]$$
$$= E\left(\int_{s=0}^{s=t}\int_{u=0}^{u=t} W_s W_u du ds\right) = \int_{s=0}^{s=t}\int_{u=0}^{u=t} E(W_s W_u)\, du ds.$$

As $E(W_s W_u) = \min(s,u)$,

$$\int_{s=0}^{s=t}\int_{u=0}^{u=t} \min(s,u)\, du ds = \int_{s=0}^{s=t}\int_{u=0}^{u=s} u\, du ds + \int_{s=0}^{s=t}\int_{u=s}^{u=t} s\, du ds$$
$$= \int_{s=0}^{s=t} \frac{1}{2} s^2 ds + \int_{s=0}^{s=t} s(t-s) ds = \frac{t^3}{3}.$$

Alternative solution:
Since

$$d\big((t-s)W_s\big) = (t-s)dW_s - W_s ds$$

taking integral we have
$\int_0^t d\big((t-s)W_s\big) = \int_0^t (t-s)dW_s - \int_0^t W_s ds$.Since $\int_0^t d\big((t-s)W_s\big) = (t-t)W_t - (t-0)W_0 = 0$, we find

$$\int_0^t W_s ds = \int_0^t (t-s)dW_s.$$

Using Ito's isometry we find

$$var\left(\int_0^t W_s ds\right) = var\left(\int_0^t (t-s)dW_s\right) = \int_0^t (t-s)^2 ds = \frac{t^3}{3}$$

(c) Find the deterministic function m_t with $m_0 = 1$ such that $E\left[m_t e^{\lambda W_t} \middle| F_s\right] = m_s e^{\lambda W_s}$ for all $s \le t$ and a given constant λ.

Commentary on Question:
Most candidates successfully derived the function.

The drift term of

$$d\left(m_t e^{\lambda W_t}\right) = e^{\lambda W_t}\frac{dm_t}{dt}dt + m_t de^{\lambda W_t}$$

is given by

$$\left(e^{\lambda W_t}\frac{dm_t}{dt} + m_t\frac{\lambda^2}{2}e^{\lambda W_t}\right)dt.$$

Since $m_t e^{\lambda W_t}$ is a martingale, we must have

1. Continued

$$e^{\lambda W_t}\frac{dm_t}{dt}+m_t\frac{\lambda^2}{2}e^{\lambda W_t}=0.$$

That is,

$\frac{dm_t}{dt}=-\frac{\lambda^2}{2}m_t.$

Solving the PDE, we find

$m_t=m_0\exp\left(-\frac{\lambda^2 t}{2}\right).$

2. **Learning Objectives:**

1. The candidate will understand the fundamentals of stochastic calculus as they apply to option pricing.

Learning Outcomes:

(1a) Understand and apply concepts of probability and statistics important in mathematical finance.

(1d) Understand and apply Ito's Lemma.

(1e) Understand and apply Jensen's Inequality.

(1j) Understand the Black Scholes Merton PDE (partial differential equation).

Sources:

An Introduction to the Mathematics of Financial Derivatives, Ch. 3,10,12
Frequently Asked Questions in Quantitative Finance, Q23

Commentary on Question:

This question tests candidates' ability to apply Ito's Lemma and Jenson's Inequality, and to derive Black-Scholes-Merton PDE for a given derivative.

Solution:

(a) Show that A_t follows the process for $t > 0$:

$$dA_t = \frac{1}{t}(S_t - A_t)dt$$

Commentary on Question:

This part is straightforward and candidates did generally well. Some candidates did not realize that A_t is not stochastic on S_t and tried to take derivative of dA_t/dS_t.

Take derivatives of both sides, by product rule:

$dA_t = -\frac{\mathrm{dt}}{t^2}\int_0^t S_u du + \frac{1}{t}\mathrm{d}(\int_0^t S_u du\ )$

$= \ -\frac{\mathrm{dt}}{t}A_t + \frac{1}{t}S_t\mathrm{dt}$

Hence, A_t follows the process $dA_t = \frac{1}{t}(S_t - A_t)dt$

(b) Show that V satisfies the following PDE:

$$\frac{\partial V}{\partial t} + \frac{S-A}{t}\frac{\partial V}{\partial A} + \frac{1}{2}\sigma^2 S^2\frac{\partial^2 V}{\partial S^2} + rS\frac{\partial V}{\partial S} - rV = 0$$

2. Continued

Commentary on Question:
Candidates did moderately well on this part. Most candidates were able to approach the question using Ito's Lemma and construction of risk-free portfolio. To receive full mark, one needs to show the complete formula of $\mathrm{dV_t}$ *using Ito's Lemma. Partial points were deducted if the covariance terms were missing. Another common mistake is to assume* $\mathrm{V_t}$ *(rather than the constructed portfolio) to be riskless.*

Step 1 is to apply two-dimensional Ito's lemma to V_t
The two processes are:

$$dS_t = \mu S_t dt + \sigma S_t dW_t$$

$$dA_t = \frac{1}{t}(S_t - A_t)dt$$

Note that A_t is contains no stochastic terms, so ${dA_t}^2 = 0$, $dS_t dA_t = 0$.

Now substitute into Ito's Lemma:

$$dV_t = \frac{\partial V_t}{\partial t}dt + \frac{\partial V_t}{\partial S_t}dS_t + \frac{\partial V_t}{\partial A_t}dA_t + \frac{1}{2}\left[\frac{\partial^2 V_t}{\partial {S_t}^2}{dS_t}^2 + \frac{\partial^2 A_t}{\partial {A_t}^2}{dA_t}^2 + 2\frac{\partial V_t}{\partial S_t}\frac{\partial V_t}{\partial A_t} * dS_t dA_t\right]dt$$

$$dV_t = \frac{\partial V_t}{\partial t}dt + \frac{\partial V_t}{\partial S_t}dS_t + \frac{\partial V_t}{\partial A_t}dA_t + \frac{1}{2}\left[\frac{\partial^2 V_t}{\partial {S_t}^2}{\sigma_s}^2{S_t}^2 + 0\right]dt$$

$$\mathrm{dV_t} = \frac{\partial \mathrm{V_t}}{\partial t}dt + \frac{\partial \mathrm{V_t}}{\partial S_t}dS_t + \frac{\partial \mathrm{V_t}}{\partial A_t}\frac{1}{t}(S_t - A_t)dt + \frac{1}{2}\frac{\partial^2 \mathrm{V_t}}{\partial {S_t}^2}{\sigma_s}^2{S_t}^2 dt$$

$$\mathrm{dV_t} = \left(\frac{\partial \mathrm{V_t}}{\partial \mathrm{t}} + \frac{\partial \mathrm{V_t}}{\partial A_t}\frac{1}{t}(S_t - A_t) + \frac{1}{2}\frac{\partial^2 \mathrm{V_t}}{\partial {S_t}^2}{\sigma_s}^2{\mathrm{S_t}}^2\right)dt + \frac{\partial \mathrm{V_t}}{\partial S_t}dS_t$$

Step 2 is to construct a riskless self-financing portfolio containing Δ_1 option and short Δ_2 units of underlying assets to hedge the option.

$\Pi_t = \Delta_1 \mathrm{V_t} - \Delta_2 S_t$

The change in the value of a self-financing portfolio: $d\Pi_t = \Delta_1 d\mathrm{V_t} - \Delta_2 dS_t$

$$d\Pi_t = \Delta_1\left\{\left(\frac{\partial \mathrm{V_t}}{\partial \mathrm{t}} + \frac{\partial \mathrm{V_t}}{\partial A_t}\frac{1}{t}(S_t - A_t) + \frac{1}{2}\frac{\partial^2 \mathrm{V_t}}{\partial {S_t}^2}{\sigma_s}^2{\mathrm{S_t}}^2\right)dt + \frac{\partial \mathrm{V_t}}{\partial S_t}dS_t\right\} - \Delta_2 dS_t = \Delta_1\left(\frac{\partial \mathrm{V_t}}{\partial \mathrm{t}} + \frac{\partial \mathrm{V_t}}{\partial A_t}\frac{1}{t}(S_t - A_t) + \frac{1}{2}\frac{\partial^2 \mathrm{V_t}}{\partial {S_t}^2}{\sigma_s}^2{\mathrm{S_t}}^2\right)dt + \left(\Delta_1\frac{\partial \mathrm{V_t}}{\partial S_t} - \Delta_2\right)dS_t$$

Choose $\Delta_2/\Delta_1 = \frac{\partial \mathrm{V_t}}{\partial S_t}$ to hedge the risk.

$$d\Pi_t = \Delta_1\left(\frac{\partial \mathrm{V_t}}{\partial \mathrm{t}} + \frac{\partial \mathrm{V_t}}{\partial A_t}\frac{1}{t}(S_t - A_t) + \frac{1}{2}\frac{\partial^2 \mathrm{V_t}}{\partial {S_t}^2}\sigma^2{\mathrm{S_t}}^2\right)dt \; (1)$$

For Π_t to be riskless, $\mathrm{d}\Pi_t = r\Pi_t dt = (r\Delta_1 \mathrm{V_t} - r\Delta_2 S_t)dt$ (2)

2. Continued

Equating (1) & (2) gives:

$$\Delta_1\left(\frac{\partial \mathrm{V_t}}{\partial \mathrm{t}}+\frac{\partial \mathrm{V_t}}{\partial A_t}\frac{1}{t}(S_t-A_t)+\frac{1}{2}\frac{\partial^2 \mathrm{V_t}}{\partial {S_t}^2}\sigma^2 {\mathrm{S_t}}^2\right)dt=(r\Delta_1\mathrm{V_t}-r\Delta_2 S_t)dt$$

$$\frac{\partial \mathrm{V_t}}{\partial \mathrm{t}}+\frac{\partial \mathrm{V_t}}{\partial A_t}\frac{1}{t}(S_t-A_t)+\frac{1}{2}\frac{\partial^2 \mathrm{V_t}}{\partial {S_t}^2}\sigma^2 {\mathrm{S_t}}^2+r\frac{\Delta_2}{\Delta_1}S_t-r\mathrm{V_t}=0$$

Since $\frac{\Delta_2}{\Delta_1}=\frac{\partial \mathrm{V_t}}{\partial S_t}$ we have

$$\frac{\partial V}{\partial \mathrm{t}}+\frac{\partial V}{\partial A}\frac{S-A}{t}+\frac{1}{2}\frac{\partial^2 V}{\partial S^2}\sigma^2\mathrm{S}^2+rS\frac{\partial V}{\partial S}-rV=0.$$

(c) Show that $P \le \frac{1}{n+1}\sum_{i=0}^{n} C_i$ where C_i is the price of a European option with strike price K and maturity date t_i.

Commentary on Question:
Candidates did poorly on this part. Most candidates were trying to approach this part through Jenson's Inequality but were not able to apply it correctly.

The payoff of a European Call Option $(S_t-K)_+$ is a convex function. By applying Jenson's Inequality, $\left(\sum_{i=0}^{n}(S_{t_i}-K)\right)_+ \le \sum_{i=0}^{n}(S_{t_i}-K)_+$

Let P be the value of the European Call option based on Arithmetic Average. Then $P=e^{-rt_n}E\left[\left(\frac{1}{n+1}\sum_{i=0}^{n}S_{t_i}-K\right)_+\right]=\frac{1}{n+1}e^{-rt_n}E\left[\left(\sum_{i=0}^{n}S_{t_i}-K(n+1)\right)_+\right]=\frac{1}{n+1}e^{-rt_n}E\left[\left(\sum_{i=0}^{n}(S_{t_i}-K)\right)_+\right] \le \frac{1}{n+1}e^{-rt_n}\sum_{i=0}^{n}E\left[(S_{t_i}-K)_+\right]=\frac{1}{n+1}\sum_{i=0}^{n}e^{-rt_n}E\left[(S_{t_i}-K)_+\right]$

$e^{-rt_i}E\left[(S_{t_i}-K)_+\right]$ is the value of a normal European Call option with maturity t_i and Strike K. Since $e^{-rt_n}\le e^{-rt_i}$, the price of this Asian European Call option is less than (or equal to) the average value of series of European Call options maturing in any $t_i<t_n$, and with the same Strike K. In other words, $t_n>t_i$ implies that $e^{-rt_n}\le e^{-rt_i}$, which leads to

$$P\le\frac{1}{n+1}\sum_{i=0}^{n}e^{-rt_i}E\left[(S_{t_i}-K)_+\right]=\frac{1}{n+1}\sum_{i=0}^{n}C_i$$

3. **Learning Objectives:**

1. The candidate will understand the fundamentals of stochastic calculus as they apply to option pricing.

Learning Outcomes:

(1d) Understand and apply Ito's Lemma.

(1f) Demonstrate understanding of option pricing techniques and theory for equity and interest rate derivatives.

(1h) Define and apply the concepts of martingale, market price of risk and measures in single and multiple state variable contexts.

(1i) Understand and apply Girsanov's theorem in changing measures.

Sources:

Introduction to Math of Financial Derivatives, Hirsa and Neftci, Chapters 8, 10, 15

Commentary on Question:

This question tests the candidates understanding of several important concepts in financial derivatives pricing, including the Girsanov theorem, martingale, and risk neutral measures. The successful candidates would need to understand how to transform different stochastics processes into appropriate measures and price derivatives under them. The ability to handle calculations under the stochastic calculus environment is also important.

Solution:

(a)

(i) State conditions for a process to be a standard Wiener process.

(ii) Show that Z_t is a standard Wiener process under $\mathbb{P}$ by verifying that Z_t satisfies each of the conditions in part (a)(i).

Commentary on Question:

Candidates did relatively well on this part, as most candidates were able to identify the majority if not all of the properties of a standard Wiener process, and apply the definitions to the specific process given. A common mistake was that many candidates calculated the variance of Z_t instead of the increments $Z_t - Z_s$, in which case a small partial mark was deducted.

3. Continued

(i) For a process W_t to be a standard Wiener Process, it has to satisfy the following conditions:

- W_t is a square integrable martingale with $W_0 = 0$ and $E[(W_t - W_s)^2] = t - s$ where $s \leq t$
- W_t has continuous path over t

(ii) Given that the functions $g(u)$ and $h(u)$ are continuous and that W_{1t} and W_{2t} are both Wiener processes, we have obviously

(1) Z_t is a square integrable martingale with $Z_0 = 0$

(2) The trajectories of Z_t are continuous over t

It remains to prove that $E\left[\left(Z_{t_2} - Z_{t_1}\right)^2\right] = t_2 - t_1,$ if $t_1 \leq t_2$

Since $Z_{t_2} - Z_{t_1} = \int_{t_1}^{t_2} \frac{g(u)}{\sqrt{g_{(u)}^2 + h_{(u)}^2 + 2\rho g(u) h(u)}} dW_{1u} + \int_{t_1}^{t_2} \frac{h(u)}{\sqrt{g_{(u)}^2 + h_{(u)}^2 + 2\rho g(u) h(u)}} dW_{2u}$

$$
\begin{aligned}
E\left[\left(Z_{t_2} - Z_{t_1}\right)^2\right] &= E\left[\left(\int_{t_1}^{t_2} \frac{f(s)}{\sqrt{f_{(s)}^2 + g_{(s)}^2 + 2\rho f(s) g(s)}} dW_{1s} + \int_{t_1}^{t_2} \frac{g(s)}{\sqrt{f_{(s)}^2 + g_{(s)}^2 + 2\rho f(s) g(s)}} dW_{2s}\right)^2\right] \\
&= E\left[\left(\int_{t_1}^{t_2} \frac{f(s)}{\sqrt{f_{(s)}^2 + g_{(s)}^2 + 2\rho f(s) g(s)}} dW_{1s}\right)^2 + \left(\int_{t_1}^{t_2} \frac{g(s)}{\sqrt{f_{(s)}^2 + g_{(s)}^2 + 2\rho f(s) g(s)}} dW_{2s}\right)^2\right. \\
&\quad \left. + 2\left(\int_{t_1}^{t_2} \frac{f(s)}{\sqrt{f_{(s)}^2 + g_{(s)}^2 + 2\rho f(s) g(s)}} dW_{1s}\right)\left(\int_{t_1}^{t_2} \frac{g(s)}{\sqrt{f_{(s)}^2 + g_{(s)}^2 + 2\rho f(s) g(s)}} dW_{2s}\right)\right] \\
&= E\left[\int_{t_1}^{t_2} \frac{f_{(s)}^2 + g_{(s)}^2}{f_{(s)}^2 + g_{(s)}^2 + 2\rho f(s) g(s)} ds + \int_{t_1}^{t_2} \frac{2\rho f(s) g(s)}{f_{(s)}^2 + g_{(s)}^2 + 2\rho f(s) g(s)} ds\right] = E\left[\int_{t_1}^{t_2} 1 ds\right] \\
&= t_2 - t_1
\end{aligned}
$$

(b) Show that $dY_t = \left(\gamma + \mu + \rho\sigma_s\sigma_f\right) Y_t dt + \sigma_s Y_t dW_{1t} + \sigma_f Y_t dW_{2t}$.

Commentary on Question:
Candidates did very well on this part. Most candidates were able to apply Ito's Lemma correctly.

3. Continued

Apply the multivariate Ito's Lemma to Y_t,

$$
\begin{aligned}
dY_t = S_t df_t + f_t dS_t + \frac{1}{2} 2 df_t dS_t \\
&= S_t\left(\gamma f_t dt + \sigma_f f_t dW_{2t}\right) + f_t(\mu S_t dt + \sigma_s S_t dW_{1t}) \\
&+ (\mu S_t dt + \sigma_s S_t dW_{1t})\left(\gamma f_t dt + \sigma_f f_t dW_{2t}\right) \\
&= (\gamma + \mu) S_t f_t dt + (\sigma_s S_t f_t) dW_{1t} + \left(\sigma_f S_t f_t\right) dW_{2t} + \sigma_s \sigma_f S_t f_t dW_{1t} dW_{2t} \\
&= \left(\gamma + \mu + \rho \sigma_s \sigma_f\right) S_t f_t dt + (\sigma_s S_t f_t) dW_{1t} + \left(\sigma_f S_t f_t\right) dW_{2t} \\
&= \left(\gamma + \mu + \rho \sigma_s \sigma_f\right) Y_t dt + \sigma_s Y_t dW_{1t} + \sigma_f Y_t dW_{2t}
\end{aligned}
$$

(c)

(i) Show that $dY_t = r_D Y_t dt + \zeta Y_t d\tilde{W}$ where $\tilde{W}_t$ is a standard Wiener process under an equivalent martingale measure $\mathbb{Q}$ such that

$$d\tilde{W}_t = dW_t + \eta dt.$$

(ii) Solve for η in terms of the given constants.

Commentary on Question:
Candidates didn't do quite well on this part. Many candidates did not realize that this part is testing the Girsanov theorem, and some candidates who did know the Girsanov theorem only mentioned the name but did not show much understanding. Many candidates were able to solve for η, which earned partial marks. However, some candidates assumed a drift of 0 instead of $r_D Y_t$ when solving for η.

According to the Girsanov Theorem, we can find a process X_t and a new Wiener process $\widetilde{W_t}$ such that $d\widetilde{W_t} = dW_t + dX_t$.
The SDE for Y_t has a drift of $r_D Y_t$ under the target equivalent martingale measure $\mathbb{Q}$.
Therefore, we should choose

$$dX_t = \frac{\gamma + \mu + \rho \sigma_s \sigma_f - r_D}{\zeta} dt$$

Thus

$$d\widetilde{W}_t = dW_t + \eta dt$$

where

$$\eta = \frac{\gamma + \mu + \rho \sigma_s \sigma_f - r_D}{\zeta}$$

3. Continued

(d) Derive the SDEs followed by f_t and S_t under the measure $\mathbb{Q}$.

Commentary on Question:
Candidates did poorly on this part. Very few candidates were able to derive the SDE for f_t,, and even less for S_t. Common mistakes include directly setting the drifts of f_t and S_t to either r_D or r_F, without realizing that f_t itself is not a traded asset, and that the question is asking for the SDE under the domestic risk neutral measure $\mathbb{Q}$ while the asset S is foreign.

<u>SDE of f_t:</u>
Due to the no arbitrage condition, measure $\mathbb{Q}$ is also the risk neutral measure for the foreign risk free bond. Therefore, the bond value in domestic currency $de^{-r_F(T-t)}f_t$ has a drift of r_D under measure $\mathbb{Q}$.

Alternatively, $de^{-r_F(T-t)}f_t e^{-r_D t}$ has a drift of 0.

Suppose under measure $\mathbb{Q}$, f_t follows the SDE:

$$df_t = \tilde{\gamma} f_t dt + \sigma_f f_t dW_{3t}$$

where W_{3t} is a standard Wiener process.

Then by Ito's Lemma:

$$de^{-r_F(T-t)}f_t = r_F e^{-r_F(T-t)} f_t dt + e^{-r_F(T-t)} df_t = e^{-r_F(T-t)} f_t (r_F + \tilde{\gamma}) dt + \sigma_f e^{-r_F(T-t)} f_t dW_{3t}$$

Thus,

$$r_F + \tilde{\gamma} = r_D$$

$$\tilde{\gamma} = r_D - r_F$$

$$df_t = (r_D - r_F) f_t dt + \sigma_f f_t dW_{3t}$$

<u>SDE of S_t:</u>
Suppose under measure $\mathbb{Q}$, S_t follows the SDE:

$$dS_t = \tilde{\mu} S_t dt + \sigma_s S_t dW_{4t}$$

where W_{4t} is a standard Wiener process.

Then by Ito's Lemma:

$$dY_t = \left(\tilde{\gamma} + \tilde{\mu} + \rho\sigma_s\sigma_f\right) Y_t dt + \sigma_s Y_t dW_{4t} + \sigma_f Y_t dW_{3t}$$

Thus,

$$\tilde{\gamma} + \tilde{\mu} + \rho\sigma_s\sigma_f = r_D$$

$$\tilde{\mu} = r_D - \tilde{\gamma} - \rho\sigma_s\sigma_f = r_D - (r_D - r_F) - \rho\sigma_s\sigma_f = r_F - \rho\sigma_s\sigma_f$$

$$dS_t = (r_F - \rho\sigma_s\sigma_f) S_t dt + \sigma_s S_t dW_{4t}$$

3. Continued

(e) Calculate the price in domestic currency of a call on asset S_t with strike price $K = 100$ (domestic currency) and maturity $T = 1$ given the following:

$\sigma_s = 0.1, \sigma_f = 0.15, \rho = 0.2, r_D = 0.06, S_0 = 100$ and $f_0 = 1.05$.

Commentary on Question:
Candidates did OK on this part. Candidates generally know the Black-Scholes formula for call options, but still many candidates were not able to plug in the appropriate numbers into the formula. Also, some candidates over thought the question and did not realize that the SDE for Y (S in domestic currency) has already been derived in part b).

With the results of part b), price of the call can be calculated by the standard Black-Scholes formula

$$C = S_0 f_0 N(d_1) - K e^{-r_D T} N(d_2)$$

with

$$\tilde{\sigma} = \sqrt{\sigma_s^2 + \sigma_f^2 + 2\rho\sigma_s\sigma_f} = \sqrt{0.01 + 0.0225 + 0.006} = 0.1962$$

Thus

$$d_1 = \frac{ln(1.05) + (0.06 + \frac{0.0385}{2})}{0.1962} = 0.6526$$

$$d_2 = d_1 - 0.1962 = 0.4563$$

$$C = 105 \times 0.743 - 100 \times e^{-0.06} \times 0.676 = 14.36$$

4. **Learning Objectives:**

1. The candidate will understand the fundamentals of stochastic calculus as they apply to option pricing.

Learning Outcomes:

(1a) Understand and apply concepts of probability and statistics important in mathematical finance.

(1c) Understand Ito integral and stochastic differential equations.

(1d) Understand and apply Ito's Lemma.

Sources:

Chin, et al., Chapter 2

Neftci, Chapters 9-10

Commentary on Question:

Candidates did not do very well on this question, only a few of them managed to get full mark by using Ito's lemma. Some of them didn't even know Ito's lemma or work backward to derive the solution.

Solution:

(a) Verify, by using Ito's lemma, that the solution to the SDE can be expressed in the form

$$X_t = (1-t)\int_0^t \frac{dB_u}{1-u}$$

Commentary on Question:

Some of the candidates use product rule or basic integration, solutions like this did not get full mark. If the candidate uses product rule as shown below, they can only get half of the full mark at most because the question asks to use Ito's Lemma.

Ito's lemma states that

$$dY_t = \left[\frac{\partial g}{\partial t} + \mu(X_t, t)\frac{\partial g}{\partial X_t} + \frac{1}{2}\sigma(X_t, t)^2 \frac{\partial^2 g}{\partial X_t^2}\right] dt + \sigma(X_t, t)\frac{\partial g}{\partial X_t} dB_t$$

By choosing $Y_t = g(X_t, t) = \frac{X_t}{1-t}$ and applying Ito's lemma, it becomes straightforward to show that

4. Continued

$$
\begin{aligned}
dY_t &= \frac{X_t}{(1-t)^2}\,dt + \frac{dX_t}{1-t} \\
&= \frac{1}{1-t}\left(dX_t + \frac{X_t}{1-t}\,dt\right) \\
&= \frac{1}{1-t}\left(-\frac{X_t}{1-t}dt + dB_t + \frac{X_t}{1-t}\,dt\right) = \frac{1}{1-t}dB_t
\end{aligned}
$$

and integrating both sides from 0 to t, we get

$$\frac{X_t}{1-t} = X_0 + \int_0^t \frac{dB_s}{1-s} = \int_0^t \frac{dB_s}{1-s}$$

and the result follows.

Alternatively solution using product rule (receiving only a half of the full mark).

Let $Y_t = (1-t)\int_0^t \frac{dB_s}{1-s}$. By product rule

$$dY_t = -\int_0^t \frac{dB_s}{1-s}\,dt + dB_t = -\frac{Y_t}{1-t}dt + dB_t$$

Thus $d(Y_t - X_t) = -\frac{Y_t - X_t}{1-t}dt$

That is, $Z_t = Y_t - X_t$ is deteministic satisfying the ODE

$$\frac{dZ_t}{dt} = -\frac{Z_t}{1-t}.$$

i.e.

$$Z_t = Z_0(1-t)$$

It follows that $Z_t = 0$ since $Z_0 = 0$, that is $X_t = Y_t = (1-t)\int_0^t \frac{dB_s}{1-s}$

(b) Show that for any $0 \le s \le t < 1$ the covariance between X_s and X_t is given by

$$Cov(X_s, X_t) = s - st$$

Commentary on Question:
Candidates did not get full mark on this question because they did not show the complete solution. Some of them failed to use Ito's isometry property, or failed to separate the integration in the integral.

4. **Continued**

Consider the case where $s \le t$, then we have

$$\begin{aligned} E(X_s X_t) &= (1-s)(1-t)E\left(\int_0^s \frac{dB_u}{1-u}\int_0^t \frac{dB_v}{1-v}\right) \\ &= (1-s)(1-t)E\left[\int_0^s \frac{dB_u}{1-u}\left(\int_0^s \frac{dB_v}{1-v} + \int_s^t \frac{dB_v}{1-v}\right)\right] \\ &= (1-s)(1-t)E\left[\left(\int_0^s \frac{dB_u}{1-u}\right)^2\right] \end{aligned}$$

by independent increment. Using Ito's isometry property, we get

$$\begin{aligned} E(X_s X_t) &= (1-s)(1-t)E\left[\int_0^s \left(\frac{1}{1-u}\right)^2 du\right] \\ &= (1-s)(1-t)\ \frac{s}{1-s} \\ &= s(1-t) \end{aligned}$$

Please note that there is a typo in the Chin et al. text (page 138): On line 6, $\frac{1}{1-t}$ should be $\frac{t}{1-t}$

Since $E(X_s) = 0$, we have $cov(X_s, X_t) = E(X_s X_t) = s - st.$

(c) Show that the process X_t does not have independent increments.

Commentary on Question:
Most of the candidate who got full credits on this part used the alternative solution.

To show that the process does not have independent increments, it is sufficient to show the covariance of the increments is not zero.

Consider $0 < s < t < u < v$ so that we have:

$$cov(X_t - X_s, X_v - X_u) = E[(X_t - X_s)(X_v - X_u)] - E(X_t - X_s)E(X_v - X_u)$$

By simplifying RHS:

$$\begin{aligned} cov(X_t - X_s, X_v - X_u) &= cov(X_t, X_v) - cov(X_t, X_u) \\ &\qquad - cov(X_s, X_v) + cov(X_s, X_u) \\ &= (t - vt) - (t - ut) - (s - sv) + (s - us) \\ &= t(u - v) - s(u - v) \\ &= (t - s)(u - v) \neq 0 \end{aligned}$$

4. **Continued**

Alternative Solution:

$$\begin{aligned} cov(X_t - X_s, X_s) &= cov(X_t, X_s) - cov(X_s, X_s) \\ &= (s - st) - (\mathrm{s} - \mathrm{s}^2) \\ &= (s)(s - t) \neq 0 \end{aligned}$$

5. **Learning Objectives:**

1. The candidate will understand the fundamentals of stochastic calculus as they apply to option pricing.

Learning Outcomes:

(1a) Understand and apply concepts of probability and statistics important in mathematical finance.

(1c) Understand Ito integral and stochastic differential equations.

(1d) Understand and apply Ito's Lemma.

Sources:

Neftci, Chapters 10-15, Chapter 7

Chin et al Page 132 Problem 10

Commentary on Question:

Commentary listed underneath question component.

Solution:

(a) Verify that the solution to the SDE can be expressed in the form

$$X_t = e^{-\beta t}\left[x_0 + \frac{\alpha}{\beta}\left(e^{\beta t} - 1\right) + \sigma\int_0^t e^{\beta s} dB_s\right].$$

Commentary on Question:

Most candidates successfully verified the solution using Ito's Lemma.

By choosing $Y_t = e^{\beta t}X_t$ and applying Ito's calculus, we get

$$\begin{aligned} dY_t &= \beta e^{\beta t}X_t dt + e^{\beta t}dX_t \\ dY_t &= \beta e^{\beta t}X_t dt + e^{\beta t}(\alpha - \beta X_t)dt + + e^{\beta t}\sigma dB_t \\ dY_t &= \alpha e^{\beta t}dt + \sigma e^{\beta t}dB_t. \end{aligned}$$

Integrating from 0 to t, we have

$$\begin{aligned} e^{\beta t}X_t &= x_0 + \int_0^t \alpha e^{\beta s} ds + \int_0^t \sigma e^{\beta s} dB_s \\ e^{\beta t}X_t &= x_0 + \frac{\alpha}{\beta}\left(e^{\beta t} - 1\right) + \sigma\int_0^t e^{\beta s} dB_{s.} \end{aligned}$$

Multiplying both sides by $e^{-\beta t}$ we get the desired result.

5. Continued

(b) Derive a closed-form expression for $Cov(X_s, X_t)$ for $s \le t$.

Commentary on Question:
This was the most challenging part for candidates.

From part (a)
$E(X_s) = e^{-\beta t}\left(x_0 + \frac{\alpha}{\beta}\left(e^{\beta t} - 1\right)\right)$.
Thus
$X_s - E(X_s) = \sigma e^{-\beta s}\int_0^s e^{\beta u} dB_u$.
It follows that

$$Cov(X_s, X_t) = E\left(\left(\sigma e^{-\beta s}\int_0^s e^{\beta u} dB_u\right)\left(\sigma e^{-\beta t}\int_0^t e^{\beta u} dB_u\right)\right)$$

$$= \sigma^2 e^{-\beta(s+t)} E\left(\int_0^s e^{\beta u} dB_u\left(\int_0^s e^{\beta u} dB_u + \int_s^t e^{\beta u} dB_u\right)\right)$$

$= \sigma^2 e^{-\beta(s+t)}\left(E\left(\left(\int_0^s e^{\beta u} dB_u\right)^2\right) + E\left(\int_0^s e^{\beta u} dB_u \int_s^t e^{\beta u} dB_u\right)\right)$.

Since $\int_0^s e^{\beta u} dB_u$ and $\int_s^t e^{\beta u} dB_u$ are independent,

$$E\left(\int_0^s e^{\beta u} dB_u \int_s^t e^{\beta u} dB_u\right) = E\left(\int_0^s e^{\beta u} dB_u\right) E\left(\int_s^t e^{\beta u} dB_u\right) = 0.$$

Moreover, from Ito's isometry
$E\left(\left(\int_0^s e^{\beta u} dB_u\right)^2\right) = E\left(\int_0^s e^{2\beta u} du\right)$. Thus

$Cov(X_s, X_t) = \sigma^2 e^{-\beta(s+t)} E\left(\int_0^s e^{2\beta u} du\right) = \frac{\sigma^2}{2\beta} e^{-\beta(s+t)}\left(e^{2\beta s} - 1\right)$.

(c) Deduce from part (b) that

$$Var(X_t) = \frac{\sigma^2}{2\beta}\left(1 - e^{-2\beta t}\right).$$

Commentary on Question:
Most candidates successfully derived this expression, even if they incorrectly answered (b).

5. Continued

By setting $s = t$ in part (b), we get

$$Var(X_t) = \frac{\sigma^2}{2\beta} e^{-2\beta t}\left(e^{2\beta t} - 1\right) = \frac{\sigma^2}{2\beta}\left(1 - e^{-2\beta t}\right).$$

(d) Show that the limiting distribution of this process X_t as $t \to \infty$ is a normal distribution with mean $\dfrac{\alpha}{\beta}$ and variance $\dfrac{\sigma^2}{2\beta}$.

Commentary on Question:
In order to receive full credit for this part, candidates had to explain that the limiting distribution was normal in addition to finding the mean and variance.

Because B_t is a Gaussian process and the addition of a deterministic function simply results in another Gaussian process, we can conclude that the limiting distribution is also a Normal distribution with mean

$$E(X_t) = e^{-\beta t}\left[x_0 + \frac{\alpha}{\beta}\left(e^{\beta t} - 1\right)\right] \to \frac{\alpha}{\beta} \text{ as } t \to \infty$$

and variance

$$Var(X_t) = \frac{\sigma^2}{2\beta}\left(1 - e^{-2\beta t}\right) \to \frac{\sigma^2}{2\beta} \; as \; t \to \infty.$$

This is also called the stationary distribution of the process.

(e) Explain how the dynamics of the model are affected by the values of the parameters α and β.

Commentary on Question:
Candidates needed to explain how the mean was affected by the parameters to receive full credit.

By rewriting the O-U process in the form

$$dX_t = \beta\left(\frac{\alpha}{\beta} - X_t\right)dt + \sigma dB_t,$$

5. Continued

then this is expressed in the form of Eq. (11.78) as in Neftci. Thus, we see that interest rates will fluctuate around the mean $\frac{\alpha}{\beta}$ so that increasing the value of α, while fixing β, will cause the process to revert back to a larger value, and vice-versa.

Equation 11.78 in Nefci is the special case when alpha is 0

The parameter β controls how long it will take for the process to revert back to this mean value. The larger its value, the faster the process will revert back toward this mean. In addition, β also affects the mean value in the sense that increasing its value will lower the mean value, and vice versa.

6. Learning Objectives:

1. The candidate will understand the fundamentals of stochastic calculus as they apply to option pricing.

Learning Outcomes:

(1b) Understand the importance of the no-arbitrage condition in asset pricing.

(1g) Demonstrate understanding of the differences and implications of real-world versus risk-neutral probability measures.

Sources:

Neftci Chapter 2

Commentary on Question:

Overall, there are 2 extremes on the score. Either they got all 4 parts correct, or just part (a) and (c) as these 2 parts are relatively straightforward.

Solution:

(a) Show that risk-neutral probabilities do not exist.

Commentary on Question:
Most candidates got full credit on this one.

Risk neutral probabilities exist if and only if there are no-arbitrage opportunities

$$\begin{pmatrix} A_0 \\ B_0 \\ C_0 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 \\ 80 & 100 & 120 \\ 0 & 0 & 140 \end{pmatrix} \begin{pmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \end{pmatrix}$$

There are no-arbitrage opportunities if and only if positive constants ψ_1, ψ_2, and ψ_3 can be found that satisfy the above system of equations.

$$\begin{cases} \psi_1 + \psi_2 + \psi_3 = 0.8 \\ 80\psi_1 + 100\psi_2 + 120\psi_3 = 70 \\ 140\psi_3 = 28 \end{cases}$$

Therefore

$$\psi_1 = 0.7$$
$$\psi_2 = -0.1$$
$$\psi_3 = 0.2$$

Since ψ_2 is negative, thus arbitrage opportunities exists and risk neutral probabilities do not exist.

6. Continued

Alternative Solution:
The risk neutral probabilities are artificial probabilities such that the expectations calculated with them, once discounted by the risk-free rate, equals the current asset value.

Asset A has no variance in its payoff at time 1, thus the return implied by this assets is the risk-free rate. And the risk-neutral probabilities should satisfy the following system of equations:

$$\begin{cases} Q_1 + Q_2 + Q_3 = 1 \\ (80Q_1 + 100Q_2 + 120Q_3) \times 0.8 = 70 \\ 140\psi_3 \times 0.8 = 28 \end{cases}$$

Solving the equations, we have

$$Q_1 = 0.875$$
$$Q_2 = -0.125$$
$$Q_3 = 0.25$$

Since Q_2 is negative, thus it's not a probability, and risk neutral probabilities do not exist.

(b) Construct an arbitrage portfolio which takes the position of -1 in asset *C*.

Commentary on Question:
Most of the candidates failed to provide the system of inequality, and attempted to answer this question by trial and error. But, only some of them could get the correct solution with this approach.

Suppose an arbitrage portfolio is composed of θ_A, θ_B, and θ_C positions in the assets A, B and C respectively, then:

1) $S_0\theta \leq 0$ and $D\theta > 0$, or
2) $S_0\theta < 0$ and $D\theta \geq 0$

One arbitrage portfolio can be found by:

$$\begin{cases} 0.8\theta_A + 70\theta_B + 28 \times (-1) = 0 \\ \theta_A + 80\theta_B > 0 \\ \theta_A + 100\theta_B > 0 \\ \theta_A + 120\theta_B + 140 \times (-1) > 0 \end{cases}$$

6. **Continued**

Re-organizing the first equation, and sub into the system of inequalities:

$$\begin{cases} -7.5\theta_B + 35 > 0 \\ 12.5\theta_B + 35 > 0 \\ 32.5\theta_B - 105 > 0 \end{cases}$$

Thus $\frac{42}{13} < \theta_B < \frac{14}{3}$.

If we choose $\theta_B = 4$, then $\theta_A = -315$.

Other answers are possible, as long as the arbitrage condition are met.

(c) Calculate the risk-neutral probabilities of the three states of the world.

Commentary on Question:
Most candidates got full mark on this one.

Calculate the state prices implied by assets A, B and C:

$$\begin{cases} \psi_1 + \psi_2 + \psi_3 = 0.9 \\ 80\psi_1 + 100\psi_2 + 120\psi_3 = 78 \\ 140\psi_3 = 14 \end{cases}$$

Therefore

$$\psi_1 = 0.7$$
$$\psi_2 = 0.1$$
$$\psi_3 = 0.1$$

Since positive constants ψ_1, ψ_2, and ψ_3 can be found, there aren't arbitrage opportunities.

Asset A is the risk-free money market instrument, and its price 0.9 is the risk-free rate of discount. The risk-neutral probabilities should be:

$$Q_1 = \frac{0.7}{0.9} = 0.78$$

$$Q_2 = Q_3 = \frac{0.1}{0.9} = 0.11$$

(d) Construct an arbitrage portfolio using all 4 assets.

Commentary on Question:
Most candidates did poor on this one. A few of them got partial mark by identifying the underpriced asset D. Even so, some of them failed to indicate whether it should be a buy or sell position on asset D and most of them failed to construct the correct portfolio.

6. Continued

The no-arbitrage price of D at time 0 should be:

$$0.7 \times 0 + 0.1 \times 80 + 0.1 \times 90 = 17$$

Therefore it's currently underpriced at $D_0 = 15$.

An arbitrage portfolio can be constructed by buying 1 unit of D and selling a replicating portfolio (assume θ_A, θ_B, and θ_C positions in the assets A, B and C respectively).

$$\begin{cases} \theta_A + 80\theta_B = 0 \\ \theta_A + 100\theta_B = 80 \\ \theta_A + 120\theta_B + 140\theta_C = 90 \end{cases}$$

Solving the equations:

$$\theta_A = -320$$
$$\theta_B = 4$$
$$\theta_C = -0.5$$

Therefore, the arbitrage portfolio consists of:

- 320 units of A
- -4 units of B
- 0.5 unit of C
- 1 unit of D

The portfolio has a value of 2 at time 0 and a guaranteed value of 0 at time 1.

7. **Learning Objectives:**

2. The candidate will understand how to apply the fundamental theory underlying the standard models for pricing financial derivatives. The candidate will understand the implications for option pricing when markets do not satisfy the common assumptions used in option pricing theory such as market completeness, bounded variation, perfect liquidity, etc. The Candidate will understand how to evaluate situations associated with derivatives and hedging activities.

Learning Outcomes:

(2a) Identify limitations of the Black-Scholes pricing formula

(2d) Understand the different approaches to hedging.

(2e) Understand how to delta hedge and the interplay between hedging assumptions and hedging outcomes.

(2f) Appreciate how hedge strategies may go awry.

Sources:

QFIC-103-13: How to Use the Holes in Black-Scholes

Frequently Asked Questions in Quantitative Finance, Wilmott, Paul, 2nd Edition Ch. 2: Q53

Paul Wilmott Introduces Quantitative Finance, Wilmott, Paul, 2nd Edition Ch. 8

Commentary on Question:

Candidates generally did poorly in this question. The question intended to test candidates' understanding of the option pricing and hedging in a practical setting, where liability and asset modeling were involved in an integrated framework.
Most of candidates were able to identify the violations of Black-Scholes assumptions in part (a) and attempted to calculate the liability delta in part (b) (i). However, most candidates did poorly for part (b) (ii) and (b) (iii).

Solution:

(a) Explain your statement for three assumptions that you have identified.

Commentary on Question:
Most of candidates were able to identify the violations of Black-Scholes assumptions. Candidates did not get full marks if they did not clearly state, for each violation, what Black-Scholes formula assumes, and what on contrary is observed with respect to FJM in the question.
One common mistake about the violation is the 'Trading Cost': the trading cost is not explicitly stated in the question. But it can be inferred from the rebalancing threshold adopted by FJM – as an attempt to control the trading cost, FJM rebalances the position only if the mismatch is greater than certain level.

7. Continued

Candidates need to state clearly their reasoning by observing this rebalancing threshold to get full marks if they identify trading cost as a violation.
Another common mistake about the violation is the 'Trading limit': the trading limit FJM has is for the consideration of cost reduction, not a limitation in the market. Therefore no mark was given if one identified 'trading limit in market' as a violation.
Some candidates listed examples that were not explicitly stated in the question, such as mentioning 'Real-life stock prices have jumps', which in reality may be correct but was never mentioned in the question.

FJM's hedge strategy does not strictly follow the Black-Scholes assumptions because of the limiations of the formula in reality:

1. Black-Scholes formula assumes that hedging is **continuous**. However FJM rebalances its hedge position discretely (daily rebalance).
2. Black-Scholes formula assumes **constant interest rate**. However FJM's hedge strategy refers to the yield curve, which is a function of the term.
3. Black-Scholes formula assumes **consant volatility**. However FJM's hedge strategy refers to the implied volatility smile, which is a function of different strikes of the option.
4. Black-Scholes formula assumes that there is no **transaction cost**. However FJM' hedge strategy imposes a trading limit to avoid hedge asset transaction to lower the cost.

(b) Calculate the following using Black-Scholes:

(i) GMAB liability delta on date 1

(ii) Equity forward delta after rebalancing on date 1

(iii) Equity forward delta after rebalancing on date 2

Commentary on Question:
Candidates did poorly in this part.
Most candidates attempted to calculate the liability delta in (i). Many lost marks by using the wrong Black-Scholes parameters including strike, volatility and risk-free rate corresponding to the correct term-to-maturity.
Candidates failed to apply the rebalancing threshold correctly by checking how much the delta mismatch is relative to the target. Candidates also struggled to get the correct delta-neutral formula in order to calculate the correct amount of delta mismatch. For those who did attempt (ii) and (iii), we did not penalizing carry-over mistakes if the liability delta was not correctly calculated in (i).

7. Continued

(i)
Current contract value = 105,000
Current underlying index value = 1,050
So the position is shorting put option on 105,000/1,050 = 100 units of the underlying index

The paramters needed to calculate liability delta are:
S_0 = 1050 or 105,000
K = 1200 or 120,000
T = 10 – 5 = 5
r = 5-year risk-free rate = 4%
σ = 5-year equity option = 26%

$$d_1 = \frac{log\left(\frac{S}{K}\right) + \left(r + \frac{\sigma^2}{2}\right) * T}{\sigma\sqrt{T}}$$

$$= \frac{log\left(\frac{1050}{1200}\right) + \left(0.04 + \frac{0.26^2}{2}\right) * 5}{0.26 * \sqrt{5}}$$

= 0.40502

The delta of a put can be calculated as
100 * (N(-d_1)) = 100 * (0.34247) = 34.247 or 34.25

(ii)
The target is to have zero net position, i.e.
Delta(Liability) + Delta(Option) + Delta(Needed) = 0
So Delta(Needed) = -(Delta(Liability) + Delta(Option)) = - (34.25 + (-19.05)) = -15.2
The absolute value of percentage gap is 15.2/34.25 = 44.3% > 5%

So a rebalancing transaction will be triggered, equity forwards will be traded to cover the delta gap. The number of equity forwards is thus -15.2 / 1.0 = -15.2 ~ -16, i.e. need to short 16 units of equity forward contracts

The asset delta is thus Delta(Equity option) + Delta(Equity forward) = -19.05 - 16 = -35.05

(iii)
Now there are four pieces of delta contributions in the total net position:
Delta(Liability) + Delta(Option) + Delta (Forward traded on date 1) + Delta(Needed) = 0

7. Continued

It is given that
Delta(Option) + Delta (Forward traded on date 1) = 37.49 +16 = 53.49

Delta(Needed) = -(Delat(Liability) + Delta(Option) + Delta (Forward traded on date 1))
=-(-51.49 +53.49)
= -2

Absolute value of percentage gap is 2/ 51.49 = 3.88% < 5%

The delta gap is within tolerance, so there won't be a rebalancing transaction at this date. The delta of equity forward after rebalancing is the same as before rebalancing: -16.

(c) Assess if your intern made any mistakes in the strategy described above.

Commentary on Question:
Only a few candidates were able to identify the mistake of not including delta introduced by new option into the delta-neutral formula in order to calculate correct mismatch amount.
Most candidates incorrectly stated that the delta of forward was not 1 as a mistake; equity futures do not have delta of 1.

The intern did not calculate the correct number of equity forwards to trade.

The new equity option also contributes some delta to the hedge asset portofolio.

So after the new equity option is added to the portfolio, the total asset delta will be Delta(PutOption) + Delta(NewOption).
The correct delta gap should be
-(Delta(Liability) + Delta(PutOption) + Delta(NewOption))

Since equity forward has delta equal to 1, above will be the correct number of equity forwards to trade for the portofolio to reach Gamma-Delta neutral.

(d) Describe the benefit a Delta-Gamma hedge strategy can bring to FJM.

Commentary on Question:
Candidates generally did well in this part. Most candidates were able to demonstrate their understanding of the Gamma neutral position and identify the benefit of a Gamma-Delta hedge strategy.

7. Continued

The rehedging results from change in delta between the rebalancing dates.

FJM implements a trading limit to reduce the frequency of rehedging (rebalancing), which indicates that transaction cost is a concern to FJM.

Gamma is the sensitivity of delta to the movement of the underlying. Therefore a Gamma-neutral position can partially offset the fluctuation of the delta with respect to the change in the underlying index.
With more stablized delta measure, there will be less need to rebalance the hedging position, which can further save FJM's hedging cost.

8. Learning Objectives:

2. The candidate will understand how to apply the fundamental theory underlying the standard models for pricing financial derivatives. The candidate will understand the implications for option pricing when markets do not satisfy the common assumptions used in option pricing theory such as market completeness, bounded variation, perfect liquidity, etc. The Candidate will understand how to evaluate situations associated with derivatives and hedging activities.

Learning Outcomes:

(2b) Compare and contrast the various kinds of volatility, (eg actual, realized, implied, forward, etc.).

(2c) Compare and contrast various approaches for setting volatility assumptions in hedging.

(2d) Understand the different approaches to hedging.

(2e) Understand how to delta hedge and the interplay between hedging assumptions and hedging outcomes.

(2f) Appreciate how hedge strategies may go awry.

Sources:

Paul Wilmott Introduces Quantitative Finance, Wilmott, Paul 2nd Edition, 2007 Ch 8, 10

Frequently Asked Questions in Quantitative Finance, Wilmott, Paul, 2nd Edition, 2009, Ch 2: Q38

"Which Free Lunch Would You Like Today, Sir?: Delta Hedging, Volatility Arbitrage and Optimal Portfolios", Ahmad and Wilmott, Wilmott Magazine

Commentary on Question:

The question mainly tested the candidate's understanding of implied vs actual volatility difference, and how to solve for simple volatility arbitrage. The candidates generally did well on this question. Some candidates lost partial marks for not explaining their stated answer in part a, b, and c; More emphasis on the candidate's arguments for the correct answer rather than the answer itself.

Solution:

(a) Recommend which volatility, actual volatility or implied volatility, should be used for a company that is required to report hedge positions on a mark-to-market basis.

Commentary on Question:

Candidates lost partial marks for not explaining their stated answer; Emphasis on the candidate's arguments for the correct answer rather than the answer itself.

8. **Continued**

When hedging using actual volatility, the guaranteed profit at expiry is known. However on a mark-to-market basis, the P&L over the period can be fluctuating. Therefore if the company has requirement to report the hedge position on a mark-to-market basis, the management may not like the volatile P&L over the period. In that case, hedge using implied volatility might be preferred.

(b) Assess the statements by identifying any mistakes your colleague made and explain why.

Commentary on Question:
Candidates lost partial marks for not explaining their stated answer; Emphasis on the candidate's arguments for the correct answer rather than the answer itself.

1) Definition of volatility is not correct.
The volatility described is the definition of historical volatility or realized volatility.
The actual volatility is the instantaneous measure of the randomness. It is unknown and not observable.

2) Statement is correct.

3) Arbitrage opportunity is not identified correctly.
The implied volatility is the volatility that is plugged into the Black-Scholes formula to give the market price of the equity option.
By definition, the implied volatilities are three dates are:
Date 1: 20%
Date 2: 25%
Date 3: 30%

If the colleague believes the actual volatility is 20%. The option is correctly priced on Date 1, so there will not be any arbitrage opportunities on that date.

(c) Identify and explain any mistakes your colleague made.

Commentary on Question:
Candidates lost partial marks for not explaining their stated answer; Emphasis on the candidate's arguments for the correct answer rather than the answer itself.

If we hedge using the implied volatility, we only know that the P&L throughout the period is positive. The actual positive amount of P&L is unknown and highly path dependent.
The situation where a guaranteed profit amount can be calculated is when we hedge using the actual volatility.

8. Continued

(d) Describe how you would execute the arbitrage strategy by answering the following three parts (i)-(iii) if you hedged using actual volatility and you believed the actual volatility would be 23% on Day 2.

(i) Recommend whether to buy or sell the option.

(ii) Calculate the number of shares to trade.

(iii) Determine the guaranteed profit if your belief is proven to be correct.

Commentary on Question:
Most candidates knew how to approach this problem. Some made numerical mistakes in the process, but partial marks were given and the candidates were not penalized for carry-over mistakes; some candidates did confuse the direction of the arbitrage, where they stated that implied volatility > actual volatility → buy the option, which is incorrect.

On Date 2, implied volatility is 25%, and actual volatility is believed to be 23%.

(i) Implied volatility > actual volatility, indicating that the option is overpriced.

So we should short sell the option and delta hedge (replicate the option) the position.

(ii) The delta of the call option calculated using the actual volatility is the number of shares that we should buy.

$$d_1 = \frac{\log\left(\frac{S}{K}\right) + \left(r + \frac{\sigma^2}{2}\right) * T}{\sigma\sqrt{T}}$$

$$= \frac{\log\left(\frac{1100}{1000}\right) + \left(0.03 + \frac{0.23^2}{2}\right) * 1}{0.23 * \sqrt{1}}$$

= 0.6598

The delta of the short call option is $-N(d_1) = -0.7453$

So the numer of shares in the delta hedging position is (long) 0.7453. (if a fraction of shares are allowed)

8. **Continued**

(iii) The guaranteed profit over at expiry if the difference between the call option values calculated using actual volatility and implied volatility

V(σ=25%) = 180.34 as given

$d_2 = d_1 - \sigma\sqrt{T}$ = 0.6598 – 0.23 = 0.4298

V(σ=23%) = $S_0N(d_1) - Ke^{-rT}N(d_2)$ = 173.20

The guaranteed profit is
V(σ=25%) - V(σ=23%) = 180.34 – 173.20 = 7.14

9. **Learning Objectives:**

3. The candidate will understand the quantitative tools and techniques for modeling the term structure of interest rates and pricing interest rate derivatives.

Learning Outcomes:

(3c) Understand and apply popular one-factor interest rate models including Vasicek, Cox-Ross-Ingersoll, Hull-White, Ho-Lee, Black-Derman-Toy, Black-Karasinski.

(3d) Understand the concept of calibration and describe the issues related to calibration, including yield curve fitting.

(3e) Understand and differentiate between the classical approach to interest rate modelling and the HJM modeling approach, including the basic philosophy, arbitrage conditions, assumptions, and practical implementations.

Sources:
Nefci Ch.16 Ch.19, Wilmott Introduces Quant. Finance Ch.16, Ch.17

Commentary on Question:
This question tests the candidates' understanding on interest rate modelling. Most candidates received partial marks.

Solution:

(a) Derive the SDE for the short rate r_t given that the forward rate volatility $\sigma(t,T)=\sigma$ is a positive constant.

Commentary on Question:
Candidates only get partial marks if no detailed steps are shown and directly use the formula sheet.

$dF(t,T) = \sigma \int_t^T \sigma\, ds + \sigma dW_t = \sigma * \sigma(T-t)dt + \sigma dW_t = \sigma^2(T-t)dt + \sigma dW_t$

Take integrals for both sides:

The left-hand side: $\int_o^t dF(s,T) = F(t,T) - F(0,T)$

The right-hand side equals:

$\int_0^t \sigma^2(T-s)ds + \int_0^t \sigma dW_s = \sigma^2 Tt - \frac{\sigma^2}{2}t^2 + \sigma W_t = \sigma^2\left(tT - \frac{1}{2}t^2\right) + \sigma W_t$

Hence: $F(t,T) = F(0,T) + \sigma^2\left(tT - \frac{1}{2}t^2\right) + \sigma W_t$

Since the short rate $r_t = F(t,t)$

Substitute $T = t$ gives:

$r_t = F(0,t) + \sigma^2\left(t^2 - \frac{1}{2}t^2\right) + \sigma W_t = F(0,t) + \frac{1}{2}\sigma^2 t^2 + \sigma W_t$

Take differentials gives: $dr_t = \left(\frac{\partial F(0,t)}{\partial t} + \sigma^2 t\right)dt + \sigma dW_t$

9. **Continued**

(b) Compare the Black-Scholes model with respect to arbitrage conditions to the following:

(i) The HJM

(ii) The classical approach (for interest rate modeling)

Commentary on Question:
Most candidates received partial marks for this part.

In the Black-Scholes world, the arbitrage restrictions were directly and explicitly incorporated into the model by replacing the unknown drift of the underlying process by the known spot rate. There was no need to model the drift term of the stock price process. As a result, Black-Sholes reduced the problem to one of volatility modeling.

i)
The HJM approach uses the second arbitrage relation between the bond price and the instantaneous forward rates. It involves no drift modeling, but volatilities need to be calibrated.

Under the HJM approach, the risk-adjusted drift of instantaneous forward depends only on the volatility parameters. Also, the arbitrage relations can be directly built into the forward rate dynamics. This is similar to the Black-Sholes environment.

ii)
The classical approach uses the first arbitrage relation between the bond price and the spot rate under the risk-neutral measure. It involves modeling the drift of the spot rate dynamics, as well as calibration to observed volatilities.

If one pursues the classical approach, arbitrage restrictions will be incorporated into the model indirectly, through fitting initial yield curves. Hence, the classical approach appears to be fundamentally different methodology from the arbitrage-free pricing Black-Sholes model.

(c) Calculate the price of 1-year bond with a principal of 100 and 4% semiannual coupon, given that $\eta = 0.01, \gamma = 0.1, \ \beta = 0.0004,$ and initial short rate $r_0 = 2\%$.

Commentary on Question:
Partial credits were given if the candidate was not able to get the final answer but knew how to decompose the bond and use the formula to calculate A, B and P.

The bond can be decomposed into two zero-coupon bonds: $P(0, 0.5)$ with payoff 2 and $P(0,1)$ with payoff 102.

9. Continued

Use the formula sheet to calculate these two bonds.

$P(0, 0.5) = 2e^{A(0,0.5)-0.02B(0,0.5)}$

$P(0, 1) = 102e^{A(0,1)-0.02B(0,1)}$

$B(t, T) = \frac{1}{\gamma}\left(1 - e^{-\gamma(T-t)}\right)$

$B(0,0.5) = \frac{1}{0.1}(1 - e^{-0.1*0.5}) = 0.48771$

$B(0,1) = \frac{1}{0.1}(1 - e^{-0.1*1}) = 0.95163$

$A(t, T) = \frac{1}{\gamma^2}(B(t, T) - T + t)\left(\eta\gamma - \frac{1}{2}\beta\right) - \frac{\beta B(t,T)^2}{4\gamma}$

$A(0,0.5) = \frac{1}{0.1^2}(B(0, 0.5) - 0.5)(0.01 * 0.1 - 0.5 * 0.0004) - \frac{0.0004*B(0,0.5)^2}{4*0.1} =$
-0.00122

$A(0,1) = \frac{1}{0.1^2}(B(0, 1) - 1)(0.01 * 0.1 - 0.5 * 0.0004 - \frac{0.0004*B(0,1)^2}{4*0.1} =$
-0.00478

$P(0,0.5) = 2e^{-0.00122-0.02*0.48771} = 1.97817$ (or 0.989 for $1)

$P(0,1) = 102e^{-0.00478-0.02*0.95163} = 99.60026$ (or 0.976 for $1)

Hence the price of the coupon bond is $1.97817 + 99.60026 = 101.57843$

10. Learning Objectives:

5. The candidate will understand and identify the variety of fixed instruments available for portfolio management. This section deals with fixed income securities. As the name implies the cash flow is often predictable, however there are various risks that affect cash flows of these instruments. In general the candidates should be able to identify the cash flow pattern and the factors affecting cash flow for commonly available fixed income securities. Candidates should also be comfortable using various interest rate risk quantification measures in the valuation and managing of investment portfolios. Candidates should also understand various strategies of managing the portfolio against given benchmark.

7. The candidate will understand how to develop an investment policy including governance for institutional investors and financial intermediaries.

Learning Outcomes:

(5c) Demonstrate an understanding of the characteristics of leveraged loans.

(7b) Identify a fiduciary's obligations and explain how they apply in managing portfolios.

(7c) Determine how a client's objectives, needs and constraints affect investment strategy and portfolio construction. Include capital, funding objectives, risk appetite and risk-return trade-off, tax, accounting considerations and constraints such as regulators, rating agencies, and liquidity.

Sources:

Managing Investment Portfolios: A Dynamic Process, Maginn & Tuttle, 3rd Edition, 2007, Ch. 3

Managing Investment Portfolios: A Dynamic Process, Maginn & Tuttle, 3rd Edition, 2007, Ch. 6

Commentary on Question:

The question was trying to test candidates' understanding about objectives, risks and constraints based on the information provided and determination of corresponding investment strategy and portfolio construction. Furthermore, the question tested candidates' knowledge on repurchase agreement as well as the calculation of the duration. Most candidates did well for part (a) and (b). For part (d), many candidates did not give further descriptions rather just list the definition and factors.

Solution:

(a) Determine for each of the following three attributes whether ABC's ability to take risk is above or below the telecommunication industry average:

10. Continued

(i) sponsor financial condition

(ii) plan provisions

(iii) participant characteristics

Commentary on Question:
Candidates did well for this part. .

(i) Below industry because ABC's debt/asset ratio of .60 is higher than the industry average of .40.

(ii) Below industry because ABC employees over age 50 are allowed to retire early, while most industry employees are not allowed to retire early. The early retirement feature increases the present value of ABC's benefit payments compared to the industry.

(iii) Below industry because the proportion of retired lives in
ABC is (35%) above the industry average, resulting in a shorter duration for ABC's liabilities than that of the industry.

(b) Critique X's statement in light of the pension plan liabilities.

Commentary on Question:
Very few candidates give the correct conclusion. Most of candidates did identify the increase of the fund shortfall due to the low interest earned by investing the short-term risk-free securities. Less than half of candidates managed to explain the duration mismatch between assets and liabilities,

Statement is incorrect

Mismatch between the short-term risk-free securities and 15-year pension liabilities duration => change in asset values will not correlate closely to change in pension liabilities => increase volatility of fund shortfall. Risk-free securities earn less than actuarial discount rate

(c)

(i) Determine the amount ABC needs to borrow to increase the return from 4% to 5% assuming all invested funds earn 4% per annum.

(ii) Calculate the duration of the portfolio after borrowing.

10. Continued

Commentary on Question:
Most of candidates did not give any formula. Credits were given based on the correct answer and steps.

(i)
The return on the total funds invested (initial plus borrowed) equals the return on the borrowed funds less borrowing costs, plus the return on the initial funds, divided by the size of the fund.
$RP = [B \times (rF - k) + E \times rF] / E$
Where:
rF = return on invested funds = 4%
k = cost of borrowing = 3%
E = initial (or Equity) funds = 150,000,000

B = borrowed funds
RP = Required return on initial (equity) funds, after leveraging = 5%
Solving the above equation for the B:
$B = (RP \times E - E \times rF) / (rF - k) = E\,(RP - rF) / (rF - k)$
$= 150{,}000{,}000 \times (0.05 - 0.04) / (0.04 - 0.03)$
$= 150{,}000{,}000$

(ii)
Let DE = duration of the initial (equity) funds (not dollar duration)
DA = dollar duration of the assets (the bond portfolio)
DL = dollar duration of the liabilities (borrowed funds)
A = value of bond portfolio (initial funds plus borrowed funds)
L = value of liabilities (borrowed funds)
E = A – L = value of equity
Therefore:
$DE = (DA - DL)/E$
$= [15 \times (150{,}000{,}000 + 150{,}000{,}000) - 1 \times 150{,}000{,}000] / 150{,}000{,}000$
$= 29$

(d) Describe repurchase agreements and the six factors that will affect repo rates.

Commentary on Question:
Most of candidates only give the definition without further explanations.

Repurchase agreement
- contract involving the sale of securities such as Treasury instruments coupled with an agreement to repurchase the same securities on a later date.
- It functions like a collateralized loan.
- The difference is selling price and purchase price is the interest
- Low cost way to borrow funds
- Enable lenders to earn excess return without sacrificing liquidity

10. Continued

Quality of the collateral – higher quality, lower repo rate
Term of the repo rate – longer term, higher repo rate
Delivery requirement – greater control over the collateral, lower the repo rate
Availability of collateral – the more difficult to obtain the securities, the lower the repo rate
Prevailing interest rates in the economy – higher the federal fund rate, higher the repo rate
Seasonal factors – some institution's supply and demand of funds influenced by seasonal factors

(e) Calculate the duration of the call option.

Commentary on Question:
There is discrepancy on page 384 of Managing Investment Portfolios between the formula and the text immediately followed, and full credit is given to answers pertaining to either of the descriptions.

Duration for an option = Duration of underlying instrument × Option Delta × Price of underlying instrument / Price of option instrument

Option Delta was not provided in the question. Full credit is given to either

=17 × 1,000,000 / 30,000 = 566.66 or approximately 567

OR

=17 × 1,000,000 × Option Delta / 30,000

11. **Learning Objectives:**

4. The candidate will understand the concept of volatility and some basic models of it.

Learning Outcomes:

(4b) Understand and apply various techniques for analyzing conditional heteroscedastic models including ARCH and GARCH.

Sources:

Tsay Chapter 3.3, 3.4, 3.5

QFIC-109-15 9.7

Commentary on Question:

Understand the features of the conditional heteroscedastic models (i.e. ARCH and GARCH)
Forecast volatility using GARCH model

Solution:

(a) Describe the features of ARCH and GARCH models.

Commentary on Question:
Most candidates attempted this questions and were able to list some features of ARCH and GARCH models. However, few candidates had thorough understanding of the models and most candidates were awarded partial marks for the question.

ARCH model

The shock a_t of an asset return is serially uncorrelated but dependent.

$$a_t = \sigma_t \epsilon_t$$
$$\sigma_t^2 = \alpha_0 + \alpha_1 a_{t-1}^2 + \cdots + \alpha_m a_{t-m}^2$$

where ϵ_t is a sequence of independent and identically distributed (iid) random variables with mean zero and variance 1, $\alpha_0 > 0, \alpha_i \geq 0 \; for \; i > 0$.

GARCH model

GARCH model is the generalized ARCH model. For a log return series $r_t, let\; a_t = r_t - \mu_t$ be the innovation at time t.
Then a_t follows a GARCH (m,s) model if

$$a_t = \sigma_t \epsilon_t$$
$$\sigma_t^2 = \alpha_0 + \sum_{i=1}^{m} \alpha_i a_{t-i}^2 + \sum_{j=1}^{s} \beta_j \sigma_{t-j}^2$$

where ϵ_t is a sequence of independent and identically distributed (iid) random variables with mean zero and variance 1, $\alpha_0 > 0, \alpha_i \geq 0,\ \beta_j \geq 0 \; and \; \sum_{i=1}^{\max(m,s)} (\alpha_i + \beta_i) < 1$.

11. Continued

ARCH model

- The model assumes that positive and negative shows have the same effects on volatility because it depends on the square of the previous shocks. In practice, it is well known that the price of a financial asset responds differently to positive and negative shocks.
- The ARCH model is rather restrictive. For instance, α_1^2 of an ARCH(1) model must be in the interval [0, 1/3] if the series has a finite fourth moment. The constraint becomes complicated for higher order ARCH models. In practice, it limits the ability of ARCH model with Gaussian innovations to capture excess kurtosis.
- The ARCH model does not provide any new insight for understanding the source of variations of a financial time series. It merely provides a mechanical way to describe the behavior of the conditional variance. It gives no indication about what causes such behavior to occur.
- ARCH models are likely to over predict volatility because they respond slowly to large isolated shocks to the return series.

GARCH model

- A large σ_{t-1}^2 $gives\ rise\ to\ a\ large\ \sigma_t^2$. This is the well-known behavior of volatility clustering in financial time series.
- It does not allow asymmetric effects between positive and negative gains.
- If $If\ 1 - 2\alpha_i^2 - (\alpha_1 + \beta_1)^2 > 0$, then
  $$\frac{E(a_t^4)}{[E(a_t^2)]^2} = \frac{3[1-(\alpha_1+\beta_1)^2]}{1-2\alpha_i^2-(\alpha_1+\beta_1)^2} > 3$$
- Similar to ARCH models, the tail distribution of a GARCH process is heavier than that of a normal distribution.

(b) Derive the l-step ahead variance forecast $\sigma_t^2(l)$ and find its limit as l goes to infinity.

Commentary on Question:

Very few candidates are able to obtain full marks for this question. Many candidates knew the conclusion (i.e. long-term variance goes to $\frac{\alpha_0}{1-\alpha_1-\beta_1}$*) but failed to calculate the one-step-ahead formula, therefore could not correctly calculate the l-step-ahead formula. Some candidates were able to identify the correct formula for l-step-ahead from the formula sheet but were not able to calculate the limit properly.*

11. Continued

For GARCH(1,1) model

$$\sigma_{t+1}^2 = \alpha_0 + \alpha_1 a_t^2 + \beta_1 \sigma_t^2$$

As

$$a_t^2 = \sigma_t^2 \epsilon_t^2$$

$$\sigma_{t+1}^2 = \alpha_0 + (\alpha_1 + \beta_1)\sigma_t^2 + \alpha_1 \sigma_t^2(\epsilon_t^2 - 1)$$

$As\ E(\epsilon_t^2 - 1|F_t) = 0$ *(1GP)*

$\sigma_t^2(l) = \alpha_0 + (\alpha_1 + \beta_1)\sigma_t^2(l-1), l > 1$

…

$$\sigma_t^2(l) = \frac{\alpha_0(1 - (\alpha_1 + \beta_1)^{l-1})}{1 - \alpha_1 - \beta_1} + (\alpha_1 + \beta_1)^{l-1}\sigma_t^2(1)$$

Therefore, $\sigma_t^2(l) \rightarrow \frac{\alpha_0}{1 - \alpha_1 - \beta_1}\ as\ l \rightarrow \infty\ provided\ that\ \alpha_1 + \beta_1 < 1$

(c) Calculate the forecasted variance in ten days, i.e., the 10-step ahead forecasted variance.

Commentary on Question:

Around a quarter of the candidates were able to correctly calculate the result. Many candidates attempted this question. Candidates who calculated the l-step-ahead formula wrong in part (b) tend to use the wrong result from part (b). Another common mistake is to substitute "10" to the l-step-ahead formula sheet (include $(\alpha_1 + \beta_1)^{l-1}$ term and calculate the l-step-ahead volatility using 1-step-ahead volatility), so instead, these candidates calculated the 9-step-ahead volatility.

The long term variance (V_L)

$$= \frac{\alpha_0}{1 - \alpha_1 - \beta_1}$$

$$= \frac{0.0008}{1 - 0.7 - 0.1} = 0.004$$

Given $\sigma_t^2 = 0.00136,$

As $E(\sigma_{t+l}^2) = V_L + (\alpha_1 + \beta_1)^l(\sigma_t^2 - V_L)$

The expected future volatility in ten days, t + 10,

$$E(\sigma_{t+10}^2) = V_L + (\alpha_1 + \beta_1)^{10}(\sigma_t^2 - V_L)$$

$$= 0.004 + (0.7 + 0.1)^{10}(0.00136 - 0.004)$$

$$= 0.0037$$

12. Learning Objectives:

5. The candidate will understand and identify the variety of fixed instruments available for portfolio management. This section deals with fixed income securities. As the name implies the cash flow is often predictable, however there are various risks that affect cash flows of these instruments. In general the candidates should be able to identify the cash flow pattern and the factors affecting cash flow for commonly available fixed income securities. Candidates should also be comfortable using various interest rate risk quantification measures in the valuation and managing of investment portfolios. Candidates should also understand various strategies of managing the portfolio against given benchmark.

Learning Outcomes:

(5a) Demonstrate an understanding of par yield curves, sport curves, and forward curves and their relationship to traded security prices; and understanding of bootstrapping and interpolation.

Sources:

Wilmott Introduces Quantitative Finance Chapter 14

Commentary on Question:

This question was testing candidates' ability to understand the relationship between Treasury curves and spot rates, corporate spreads, duration and hedging.

Solution:

(a) Calculate the yield and the modified duration of the zero-coupon bond.

Commentary on Question:
Many candidates did not first calculate spot rates for time 2 and time 3.

First, calculate spot rate at t=2. **(1 point)**
100 = 2/(1.015) + 102/(1 + s)^2
s = 2.005%

Then, calculate spot rate at t = 3. **(2 points)**
100 = 2.5/(1.015) + 2.5/(1.02005)^2 + 102.5/(1 + s)^3
s = 2.517%

Then, add the corporate spread for BBB-rated bond. **(1 point)**
2.517% + 0.99% = 3.507%

Modified duration of a 3-year zero-coupon bond
= 3/(1+i)
= 3/(1.03507)
= 2.898

12. **Continued**

(b) Calculate the amount that you need to short sell in order to hedge the interest rate risk.

Commentary on Question:
Most candidates received some credit. Few candidates followed all the steps and arrived at the correct answer.

Time	**1**	**2**	**3**	**4**	**5**	**Total**
Cash flow	3.5	3.5	3.5	3.5	103.5	
Discounted cash flow	3.382	3.267	3.157	3.050	87.144	100.00
Time x Discounted CF	3.382	6.535	9.470	12.200	435.721	467.31
Sum of time x discount CF / Price						4.67
Modified Duration (4.67/1.035)						4.52

Second, apply formula to calculate hedge ratio.

Hedge ratio = Duration of zero-coupon / Duration of 5-year Treasury
= 2.898 / 4.52
= 0.642

Amount needed to short sell
= 0.642 x $20 million
= $12.8 million

(c) Describe the simplifications assumed in the hedge of part (b).

Commentary on Question:
Most candidates received some credit

Assume parallel shift
There may be twisting of the yield curve
Can always rebalance
Short sale allowed with no penalty or limit
No default in between

(d) Explain the impact on the amount of short-selling needed for each of the alternatives below. No calculation is needed.

(i) 2-year, annual coupon, A-rated corporate bond

(ii) 5-year, annual coupon, BBB-rated corporate bond

12. Continued

Commentary on Question:
Most candidates got this part correct.

(i) 2-year, annual coupon, A-rated corporate bond
The duration of the 2-year bond is lower. Therefore, more short selling amount is needed.

(ii) 5-year, annual coupon, BBB-rated corporate bond
The corporate bond has higher yield than the Treasury bond, so it has lower duration, Therefore, more short selling amount is needed.

13. **Learning Objectives:**

5. The candidate will understand and identify the variety of fixed instruments available for portfolio management. This section deals with fixed income securities. As the name implies the cash flow is often predictable, however there are various risks that affect cash flows of these instruments. In general the candidates should be able to identify the cash flow pattern and the factors affecting cash flow for commonly available fixed income securities. Candidates should also be comfortable using various interest rate risk quantification measures in the valuation and managing of investment portfolios. Candidates should also understand various strategies of managing the portfolio against given benchmark.

6. The candidate will understand the variety of equity investments and strategies available for portfolio management.

8. The candidate will understand the theory and techniques of portfolio asset allocation.

Learning Outcomes:

(5c) Demonstrate an understanding of the characteristics of leveraged loans.

(5d) Demonstrate understanding of cash flow pattern and underlying drivers and risks of mortgage-backed securities, and commercial mortgage-backed securities.

(6a) Explain the nature and role of equity investments within portfolios that may include other asset classes.

(8b) Propose and critique asset allocation strategies.

Sources:

Fabozzi (8th Edition) Ch: 12

Fabozzi (8th Edition) Ch: 13

High-Yield bond Market Primer. S&P

Managing Investment Portfolios: A Dynamic Process, Maginn & Tuttle, 3rd Edition Ch. 5

Managing Investment Portfolios: A Dynamic Process, Maginn & Tuttle, 3rd Edition Ch. 7

13. Continued

Commentary on Question:
The question was testing candidates' understanding about roles played by various investment vehicles, in particular, commonalities and differences between high-yield bonds and leveraged loans. Furthermore, the question was also testing candidates' ability to evaluate portfolio asset allocation as well as to make recommendations based on their knowledge. Most candidates did well for parts (a) and (b). For part (c), many candidates did not realize that they could combine the corner portfolio with the risk-free asset (i.e. make a leveraged investment) to satisfy both return and risk objectives.

Solution:
(a) Explain the role of equity investments in this portfolio.

Commentary on Question:
Candidates did well for this part.

- Equity offers diversification benefits since it has different characteristics compared to other asset types shown and it can also cover multiple markets.

- Equity provides an inflation hedge since its nominal returns tend to be highly correlated with inflation rates.

- Equity plays a growth role since its comparatively high historical long-term real rates of return are important for the portofolio to meet its return objectives.

(b) List two arguments for the switch and two arguments against the switch.

Commentary on Question:
Most candidates used arguments based on the particular situation given in the question set-up, e.g. leveraged loans have higher yields. These answers also received partial credits if a reasonable explanation was given.
However, it seemed that most candidates lack good grasp of the fundamental differences between the two securities as given below.

Arguments supporting switch:
1. <u>Coverant protection</u>: Leveraged loans commonly contain covenants or requirements that must be satisfied by borrowers, such as maintenance requirements and incurrence requirements. These requirements offer additional protection to investors.

13. Continued

High-yield bond issues are generally unsecured obligations of the issuing entity, and covenants are looser than on bank loans.

Leveraged loans are typically senior secured loans that at the top-most rank in the borrower's capital structure.

2. Liquidity: Given that as an endowment, ABC EF has a long investment horizon, the need for liquidity is less and switching to leveraged loans will more likely provide a higher return.

Arguments for not to switch:

1. Liquidity: High-yield bonds tend to have higher liquitidy compared to leveraged loans. Secondary trading for high-yield bonds is a well-established and active market place. After issuance, leveraged loans can change hands in only two ways: by assignment or by participation.

2. No additional diversification benefit: Both high-yield bonds and leveraged loans fall into the less-than-investment-grade credit ratings. Switching to leveraged loans does not necessarily provide additional diversification.

3. Expertise & Experience: Current investment managers may lack additional expertise required to switch to leverage loans.

4. Flexibility & investment universe: High-yield bonds have many different characteristics such as call protection, equity warrants, etc. that can better suit various needs of the investors.

(c) Determine and justify the overall most appropriate strategic asset allocation for ABC EF using the mean-variance analysis.

Commentary on Question:
This is a semi open-ended question, meaning to give the candidate the opportunity to apply their learnings of asset allocation strategies. In order to obtain all grading points, the candidate should state a firm recommendation and complete the relevant calculation to support their answer.
Besides the sample solution below, many candidates ended up recommending a combined portfolio of C and D. However, based on the existing question setup, these portfolios are correlated and they would only be able to approximate the standard deviation of the portfolio to show that it is within the risk objective. These candidates received almost full marks.

13. Continued

Sample solution:
Mean-variance
Sharpe ratio:
Back out the risk free rate used using the Sharpe ratio provided for corner portforlio A or D. For example, using portfolio A: $0.379 = \frac{6.9\% - r}{0.145}$
The risk-free rate is 1.4%.

Use the risk free rate of 1.4% to calculate the Sharpe ratio for the current portfolio and corner portfolio C. Sharpe ratio = (E(r) – r_f)/std.dev of portfolio

Corner portfolio C has a Sharpe ratio of $0.392 = \frac{4.3\% - 1.4\%}{0.074}$; Corner portfolio B has a Sharpe ratio of $0.387 = \frac{6.0\% - 1.4\%}{0.119}$.

Return objective:
Calculate ABC EF's return objective based on information given on ABC EF's spending rate, expected inflation, and cost for earning investment returns.
(1+2.5%)(1+1.4%)(1+0.2%) = 4.1%
<Additive returns are accepted as an alternative solution>
Note that the recommended portfolio has an expected return above the Endowment Fund's return objective 4.1%. Corner portfolios D is below is return objective, and should not be picked.

Risk tolerance:
Note that corner portfolios A-C and the current portfolio have risk objectives above the standard deviation limit.

Tangency portfolio and T-bills.
Corner portfolio C has the highest sharpe ratio, and is thus the tangency portfolio. The most appropriate allocation should be a combination of this portfolio and T-bills, to lower the portfolio standard deviation.

Assume weight in tangency portfolio as w, then weight in T-bills is "1-w"
Given T-bills have 0 standard deviation, (w*0.074)^2 ≤ 0.07^2
w = 0.9459

0.9459 * 4.3% + (1-0.9459) * 1.4% = 4.2%

This portfolio satisfies the return objective as well.

Resulting portfolio is approximately 94.59% of corner portfolio C and 5.41% in T-bills.

14. Learning Objectives:

5. The candidate will understand and identify the variety of fixed instruments available for portfolio management. This section deals with fixed income securities. As the name implies the cash flow is often predictable, however there are various risks that affect cash flows of these instruments. In general, the candidates should be able to identify the cash flow pattern and the factors affecting cash flow for commonly available fixed income securities. Candidates should also be comfortable using various interest rate risk quantification measures in the valuation and managing of investment portfolios. Candidates should also understand various strategies of managing the portfolio against given benchmark.

Learning Outcomes:

(5f) Construct and manage portfolios of fixed income securities using the following broad categories.
  (i) Managing funds against a target return
  (ii) Managing funds against liabilities.

Sources:

Managing Investment Portfolios: A Dynamic Process, Maginn & Tuttle, 3rd Edition, 2007 Ch. 6, Fixed Income Portfolio Management.

Commentary on Question:

The calculation of the tracking risk was well done in general. The recommendation of the appropriate index resulting from the situation describe was also successful. However, when the candidates were asked some retrieval/comprehension concepts as in part (a) (factors affecting the tracking risk), and part (c) (the risks associated with selecting a benchmark index), the results were clearly not as good as expected. The answers did come directly from the reading material and are not difficult concepts. It was an opportunity for the candidate to earn some additional grading points which may become crucial in some cases.

Solution:

(a) List two factors that might affect tracking risk.

Tracking risk is a measure of the variability with which a portfolio's return tracks the return of a benchmark index.

The following can potentially affect the tracking risk:

1. Portfolio duration
2. Key rate duration and present value distribution of cash flows
3. Sector and quality percent
4. Sector duration contribution
5. Quality spread duration contribution
6. Sector/coupon/maturity cell weights
7. Issuer exposure

14. **Continued**

(b) Calculate the tracking risk of the portfolio.

Active Returns = Portfolio's Return – Benchmark index's Return
Tracking risk = Standard deviation of the Active Returns.

Period	Portfolio Return	Benchmark Return	Active Return	(AR-Avg. AR) ^2
1	12.90%	12.60%	0.30%	0.000225%
2	6.80%	6.50%	0.30%	0.000225%
3	0.80%	1.20%	-0.40%	0.003025%
4	-4.60%	-5.00%	0.40%	0.000625%
		Average of Active Return =	0.15%	
		Sum of (AR-Avg. AR)^2 =		0.0041%
		Tracking risk (sample periods-1) $\sqrt{(sum/(4-1))}$ =		0.369685%
		Tracking risk (sample periods) $\sqrt{(sum/4)}$ =		0.320156%

(c)

(i) Identify three risks associated with selecting a benchmark index.

(ii) Evaluate each risk in part (c)(i) for ABC Club.

Market value risk:
- Portfolio and benchmark index should be comparable; therefore, ABC Club should choose a fixed-income (e.g. bond index).
- Since ABC Club is risk-averse short- or intermidiate-term index may be more appropriate as a benchmark index than a long index.

Income risk:
- Portfolio and benchmark index should provide comparable assured income stream.

Liability framework risk:
- The choice of an appropriate benchmark index should reflect the nature of the liabilities.
- Uncertain cash flows make it difficult to find a benchmark index with similar characteristics of the liabilities.

14. **Continued**

(d) Recommend the most suitable benchmark for ABC Club from the following and justify your recommendation.

- Short-Term Corporate Bond Index
- Short-Term Corporate High-Yield Bond Index
- Short-Term Emerging Market Bond Index
- Long-Term Corporate Bond Index
- Long-Term Corporate High-Yield Bond Index
- Long-Term Emerging Market Bond Index

<u>Short-Term Corporate Bond Index</u>
- Since ABC Club is risk-averse, an index with a short or intermediate duration would be appropriate to limit market value risk.
- High-yield bond index is not suitable since it invests in less-than-investment-grade bonds
- Emerging market bond index is not suitable because it tends to carry higher risks.

15. **Learning Objectives:**

3. The candidate will understand the quantitative tools and techniques for modeling the term structure of interest rates and pricing interest rate derivatives.

5. The candidate will understand and identify the variety of fixed instruments available for portfolio management. This section deals with fixed income securities. As the name implies the cash flow is often predictable, however there are various risks that affect cash flows of these instruments. In general the candidates should be able to identify the cash flow pattern and the factors affecting cash flow for commonly available fixed income securities. Candidates should also be comfortable using various interest rate risk quantification measures in the valuation and managing of investment portfolios. Candidates should also understand various strategies of managing the portfolio against given benchmark.

7. The candidate will understand how to develop an investment policy including governance for institutional investors and financial intermediaries.

Learning Outcomes:

(3b) Apply the models to price common interest sensitive instruments including: callable bonds, bond options, caps, floors, swaptions, caption, floortions.

(5b) Describe the cash flow of various corporate bonds considering underlying risks such as interest rate, credit and event risk.

(7a) Explain how investment policies and strategies can manage risk and create value.

(7c) Determine how a client's objectives, needs and constraints affect investment strategy and portfolio construction. Include capital, funding objectives, risk appetite and risk-return trade-off, tax, accounting considerations and constraints such as regulators, rating agencies, and liquidity.

(7d) Incorporate financial and non-financial risks into an investment policy, including currency, credit, spread, liquidity, interest rate, equity, insurance product, operational, legal and political risks.

Sources:

Maginn & Turtle Chapter 3

Maginn & Turtle Chapter 6

Commentary on Question:

Commentary listed underneath question component.

15. Continued

Solution:

(a) Describe the main difference between the active lives and retired lives portion of liabilities for each of the following:

(i) Inflation sensitivity

(ii) Duration

Commentary on Question:
Quite a few candidates listed but did not explain why active/retired lives are more/ less sensitive to inflation.

(i) Inflation sensitivity

- Benefit payment obligations in the retired-lives pool are exposed to less inflation risk because,
  - unlike the active-lives pool, payments are fixed in nominal terms and do not adjust for inflation.
- Benefit payment obligations in the active-lives pool are exposed to more inflation risk than in the retired-lives pool because,
  - unlike the retired-lives pool, active employees accrue pension benefits based on salary increases, which include inflation as a component.

(ii)

- Liabilities in the active-lives pool will have a relatively *longer* average duration than liabilities in the retired lives pool, reflecting the time remaining before active employees retire.
- Active employees tend to be younger than retired employees.

(b) Propose the following investment policy constraints for KL:

(i) Return objective

(ii) Liquidity Requirement

(iii) Time Horizon

Commentary on Question:
Most candidates did well on this question

15. Continued

(i) Return objective

- Goal is to earn a rate of return in excess of rate needed to maintain real purchasing power. = 1.04 (annual spending) * 1.0025 (investment expense) * 1.02 (inflation) – 1 = 6.345%
- additive solution is also acceptable (i.e. 6.25%)

(ii) Liquidity Requirement

- Planned spending + expenses – contribtutions = Asset Value * (4% + 0.25%)

(iii) Time Horizon

- KL has a single stage long term investment time horizon because
- KL was established with the intent of lasting into perpetuity;
- KL plans to maintain a 4% annual spending rate.

(c) Determine which of the HIJ pension plan and KL:

(i) Has greater ability to take risks.

(ii) Has greater willingness to take risks.

Commentary on Question:
Most candidates did fairly well on this question. However, most candidates did not point out KL's board seeks for additional return, or HIJ's assets are all invested in investment grade corporate bonds.

(i) Has greater ability to take risks.

- KL has greater ability to take risk because
  1. KL has a spending goal that is supported by objective of minimizing tax
- HIJ's defined benefit pension plan constitute a legal liability (must pay defined benefits. HIJ has less ability to take risk as it needs to takes underfunding into account.

(ii) Has greater willingness to take risks.

- KL has greater willingness to take risks because
  1. KL's board has chosen to seek additional return to maintain the real purchasing power of the portfolio.
  2. HIJ pension plan's asset allocation is conservative (currently completely invested in bonds) indicating a low willingness to take risk.
- HIJ could have more willingness to take risks if it is underfunded.

15. Continued

(d) Assess the expected effects (positive or negative) on the pension portfolio's value over the next 2 weeks for each potential trade given the strategist's market expectations.

Commentary on Question:
Candidates had difficulty assessing the effects of Trade 1. Most of them could not figure out the impact of the widened spread, some of them did have the correct assessment, but came up with incorrect conclusion.

Most candidates did well assessing the impact of Trade 2.

- Trade 1: negative impact, lower quality bond spreads widens more than higher quality bond spread in a weakening economy environment due to higher risk of default.

- Trade 2: Negative impact, negative convexity of callable bonds limit price appreciation relative to non-callable bonds in a falling rate environment.

16. **Learning Objectives:**

6. The candidate will understand the variety of equity investments and strategies available for portfolio management.

Learning Outcomes:

(6b) Demonstrate an understanding of the basic concepts surrounding passive, active, and semi active (enhanced index) equity investing, including managing exposures.

(6c) explain the basic active equity selection strategies including value, growth and combination approaches.

Sources:
Maginn & Tuttle, chp. 7

Commentary on Question:
The candidates had less success in identifying the Float-weighted index biases and in calculating the return on that index based on the given data.

Solution:

(a) Explain the rationale of using a passive management approach rather than an active management approach, based on the information given above.

The firing of a manager for tracking risk exceeding 1% indicates that the Board considers this amount of risk to be excessive. A passive approach provides low tracking risk.

Committee members believe the stock market is efficient. Active approaches can be expected to be more successful when markets are inefficient.

The information ratio of the passive approach is 0. The Board found a low ratio to be acceptable.

Minimize investment fees and minimize turnover also is an argument for passive approach.

(b) Describe biases of each of the following four basic index weighting methods:

(i) Price-weighted

(ii) Value-weighted

(iii) Equal-weighted

(iv) Float-weighted

16. Continued

Commentary on Question:
Most candidates did well on this retrieval question.

Describe the biases of the four basic index weighting methods:

1.Price-weighted: each stock is weighted according its absolute share price
Bias:toward the highest-price share.

2.Value- weighted/floating: each stocks weighted according to its market cap
Bias:toward the shares of the companies with the largest market cap.

3.Equal-weighted index: each stock in the index is weighted equally
Bias: toward small cap firms as the weight for a small cap firm is treated the same as a large cap firm; should rebalance periodically.

4.Float-weighted index: same as Value-weighted method, except the market value is adjusted by the float factor to reflect the numbers of shares avalaible to invertors)
Bias:toward large matured company and overvalued company.

(c) Calculate, for each of the four methods in part (b), the return on the index consisting of the above three stocks for the period from Dec 31, 2014 to Dec 31, 2015.

Commentary on Question:
Most candidates did well on this part of the question.

1.Price-weighted:

As of Dec31, 2014: 380/3=126.66
As of Dec 31, 2015: 385/3=128.33
Return is (128.33/126.66) - 1=1.32 %

2.Value- weighted/floating:

Index value as of Dec 31, 2015
(166,000/137,000)=1.21
1.21-1=21 %

3.Equal-weighted index:

(0 % + 50% -2%)/3=16 %

4.Float-weighted index:

16. Continued

Dec, 31 2014: (25,000*0.6 + 60,000*0.8 + 52,000)=115,000
Dec 31, 2015: (25,000*0.6 + 90,000*0.8 + 51,000)=138,000

138,000/115,000-1=20%

(d) Recommend the most appropriate index weighting method for the Company ABC and justify your recommendation.

Commentary on Question:
Most candidate picked the appropriate index weighting method, but only a small portion gave a convincing justification.

Float-weighted method should be used.

- Since there are 3 assets, we do not want to bias towards the highest price share so price weighted approach is not appropriate.
- Similarly we do not want to bias towards the shares with the largest cap.
- Equal weighted will bias towards small cap.
  Minimize turnover provides an argument for float-weighted.

(e) Recommend an alternative equity investment management approach to the committee.

Commentary on Question:
Most candidates got partial credits on this question.

Recommandation: semi-active (enhanced indexing) appoach.

1. The committee would like to outperform market and thus alpha from active management is expected. At the same time, the committee still control the tracking risk within a certain tolerance level; semi-active approaches seeks alpha from active management but at the same time also focus on controling tracking risk.

2.Under the semi-active approach, the portfolio return will have limited volatility around the benchmark return.

3. This approach still provides exposure to the desired equity market.

17. **Learning Objectives:**

6. The candidate will understand the variety of equity investments and strategies available for portfolio management.

Learning Outcomes:

(6b) Demonstrate an understanding of the basic concepts surrounding passive, active, and semi active (enhanced index) equity investing, including managing exposures.

(6g) Recommend and justify, in a risk-return framework, the optimal portfolio allocations to a group of investment managers.

(6h) Describe the core-satellite approach to portfolio construction with a completeness fund to control overall risk exposures.

Sources:

Managing Investment Portfolios: A Dynamic Process, Maginn & Tuttle, 3rd Edition, 2007, Ch. 7 Equity Portfolio Management

Commentary on Question:

This question is testing candidates' fundamental knowledge in equity investment approaches (passive, semi-active, active) and portfolio managers optimization. Candidates should understand the basic concepts and calculations, as well as expanding in solving real life problems.

Solution:

(a) Identify the investment approach of each manager.

Commentary on Question:

Majority of candidates can correctly identify investment approaches, however, not a lot would come up with a clear justification (e.g. Return compared to benchmark, as well as tracking risk)

The expected active return for manager A is E(A) – E(Ben) = 2% - 2% =0%, with tracking risk 0%, manager A's investment approach is indexing/passive management.

Manager B has expected active return of 1%: E(B) – E(Ben) = 3% - 2% = 1%, expected tracking risk is positive but relatively small. Therefore manager B's investment approach is Semi-active management.

Manager C has expected active return of 3%: E(C) – E(Ben) = 5% - 2% =3%, Manager D has expected active return of 4%: E(D) – E(Ben) = 6% -2% =4%; tracking risks are high for manager C and D. Thus manager C and D's investment approach is Active management.

17. Continued

(b) Identify the best option based on the specified expected utility function and describe the type of portfolio constructed under this option.

Commentary on Question:
Some candidates calculated the utility of each manager and then applied weighting to obtain aggregate utility for options A/B/C, which is incorrect. Some candidates did not calculate the tracking risk correctly. Another common mistake is to apply gross return instead of active return in calculating the utility function.

The utility funciton of active return for the manager mix is:
$U_A = r_A - 0.1\ \sigma_A^2$
Option 1:
r_A = 3% - 2% = 1%
Tracking risk = 2%
$U_A = 1\% - 0.1 \times 2\%^2 = 0.996\%$
Option 2:
r_A = 60% *0% + 40%*4% = 1.6%
Tracking risk = $\sqrt{(60\%^2 \times 0\%^2 + 0\%^2 \times 2\%^2 + 0\%^2 \times 8\%^2 + 40\%^2 \times 10\%^2)} = 4\%$
$U_A = 1.6\% - 0.1 \times 4\%^2 = 1.584\%$

Option 3:
r_A = 30% *0% +20%*1% + 20%*3% +30%*4%= 2%
Tracking risk = $\sqrt{(30\%^2 \times 0\%^2 + 20\%^2 \times 2\%^2 + 20\%^2 \times 8\%^2 + 30\%^2 \times 10\%^2)} = 3.423\%$
$U_A = 2\% - 0.1 \times 3.423\%^2 = 1.988\%$

The utility is the highest under option 3, so option 3 should be chosen
However, the information ratio for option 3 =2%/3.423% = 58.4%, which does not meet the information ratio requirement of 60%.

The portfolio type constructed under option 3 is Core-Satelline portfolio: 50% of the asset are held in passive or semi-active management style (core), and the rest 50% is held actively to gain active return by active managers (satellines).

(c) Recommend the actions for the committee to take to meet this requirement based on all information described above.

Commentary on Question:
Overall candidates did poorly on this question. Many candidates got stuck with the provided manager mix and did not consider more managers.

17. Continued

1. The investment officer should first check for other mix of managers A,B ,C and D to obtain the optimal mix so that utility is the highest with tracking risk less than 4%.

With this optimal mix, if the information ratio is higher than 60%, then investment officer should allocate asset based on this optimal mix structure.

2. If under the optimal mix the information ratio is still lower than 60%, the investment officer may want to research for other portfolio managers with higher active return but lower tracking risk.

3. Under option 3, 50% of portfolio is managed actively solely by 2 managers C and D. The investment officer may want to include more active managers with different investment style to diversify the active management risk.

18. Learning Objectives:

8. The candidate will understand the theory and techniques of portfolio asset allocation.

Learning Outcomes:

(8a) Explain the impact of asset allocation, relative to various investor goals and constraints.

(8b) Propose and critique asset allocation strategies.

(8c) Evaluate the significance of liabilities in the allocation of assets.

(8d) Incorporate risk management principles in investment policy and strategy, including asset allocation.

(8e) Understand and apply the concept of risk factors in the context of asset allocation.

Sources:
Litterman Chapter 10

Commentary on Question:
Overall candidates can correctly do some formula calculations and draw graphs. But only a few can correctly do both, under different funding ratios.

Solution:
(a)

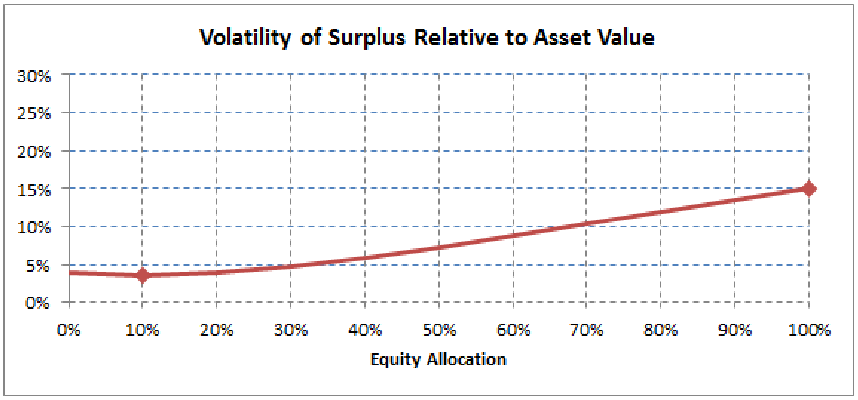
(i) Sketch a curve in the following chart to show the sensitivity of the surplus risk relative to the equity allocation.

(ii) Determine the equity allocation that minimizes the surplus risk on your curve.

(iii) Determine the equity allocation that maximizes the surplus risk on your curve.

Commentary on Question:
Majority of the candidates can perform the calculation (choosing right formula and parameters). Only a few candidates can sketch the graph correctly. Also, some candidates address the 100% equity allocation, but only few address the equity allocation at 0%.

18. Continued



Equity allocation to minimize surplus risk given a funding ratio:

$$\alpha = \frac{(1 - \beta \frac{L_t}{A_t})(\sigma_B^2 - \rho\sigma_E\sigma_B)}{\sigma_E^2 + \sigma_B^2 - 2\rho\sigma_E\sigma_B}$$

Given the funding ratio assumed at 2.0, we have
$L_t/A_t = 0.5$
Beta = pension fund liability duration/liability index duration = 12/10 = 1.2
$\sigma_E = 15\%$
$\sigma_B = 10\%$
Correlation(Equity, Bond) = 20%

Plug in all these parameter values, we get
α = 11% (around 10%), which is the local minima

Intuitively, the more equity risk, the higher the surplus risk. However, for the situation where the fund is overfunded (funding ratio 2.0), it is not optimal to allocate 0 equity to minimize the surplus risk.

0% equity allocation actually produces a higher surplus risk. Therefore, the curve drawn should be downwards sloping as the equity allocation increases from 0% to 10%.

After that, as equity allocation increases from 10% to 100%, the curve should be upward sloping, as the higher equity risk, the higher surplus risk.

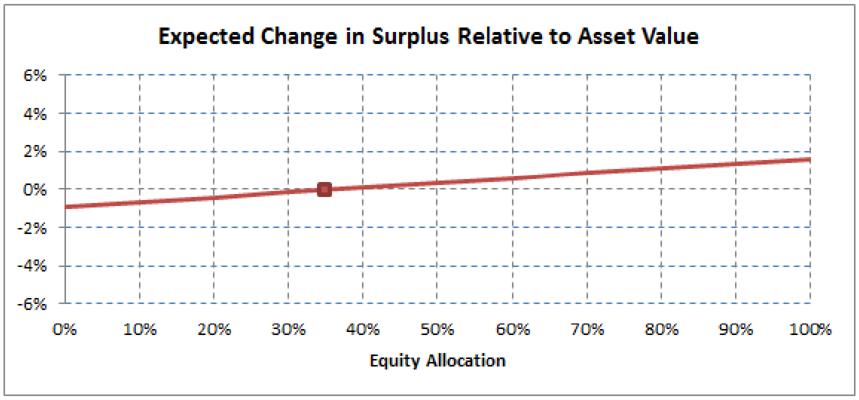
(b)

(i) Sketch a curve in the following chart to show the expected change in surplus relative to asset value.

(ii) Determine the equity allocation level at which the expected future surplus starts to decrease.

18. Continued

Commentary on Question:
Only a few candidates can sketch the graph correctly (only a small portion of candidates recognize that it should be a straight line). Some candidates can perform the calculation using the correct formula.



Minimum equity allocation needed for a fund to prevent the surplus from shrinking:

$$\frac{\mu_B\left(\beta\frac{L_t}{A_t}-1\right)+\frac{L_t}{A_t}R_f(1-\beta)}{\mu_E-\mu_B}$$

Given the funding ratio assumed at 1.0, we have
L/A = 1.0, which means the pension fund is exactly funded.

Beta = pension fund liability duration/liability index duration = 12/10 = 1.2

Expected return(Equity) = 8%
Expected return (Bond) = 5.5%
Risk free rate = 1%

Plug in all these parameter values, we get
α = 36% , which is the minimum equity allocation needed for a fund to prevent the surplus from shrinking

(c) Evaluate whether the pension plan is expected to be underfunded, exactly funded, or overfunded 10 years from now.

Commentary on Question:
The majority of the candidates can perform the calculation correctly and get full credits.

18. Continued

Because the asset investment portfolio has a 40/60 split between U.S. Equity and the bond index, the expected portfolio annual return = 40% * 8% + 60% * 5.5% = 6.5%

The excess return = 2.45%
Payout ratio p = 7.5%
Initial funding ratio = 0.9
t = 10

Plug the parameters value into the following formula

$$E_0[F_t] = \left[\frac{1 + E[R_x]}{1 - p}\right]^t F_0 + p\,\frac{1 - \left[\frac{1 + E[R_x]}{1 - p}\right]^t}{E[R_x] + p}$$

The future expected funding ratio = 1.16 > 1.0, therefore the pension plan will be overfunded 10 years from now.