QFI CORE Model Solutions
Spring 2016

1.

Learning Objectives:

1. The candidate will understand the fundamentals of stochastic calculus as they apply to option pricing.

Learning Outcomes:

(1a) Understand and apply concepts of probability and statistics important in mathematical finance.

(1b) Understand the importance of the no-arbitrage condition in asset pricing.

(1g) Demonstrate understanding of the differences and implications of real-world versus risk-neutral probability measures.

(1h) Define and apply the concepts of martingale, market price of risk and measures in single and multiple state variable contexts.

Sources:
Nefci Ch.2, 6

Frequently Asked Questions, Wilmott Q36, Q37, Q60

Commentary on Question:
This question tests candidate's understanding of a martingale, and no-arbitrage valuation, as well as market completeness.

Solution:

(a) Determine if each of the following is a martingale. Justify your answer.

(i) $\frac{S_t}{(1+r)^t}$ under the real-world measure.

(ii) $Z_t = E(S_2|I_t)$ under the real-world measure.

1. Continued

Commentary on Question:
This part tests the definition of a martingale. Candidates did generally well on part (i), but modestly well on part (ii) for which only a few received full points. Some candidates did not have clear understanding of the martingale characteristics and were especially confused about the Information Set I_t. Other common mistake includes trying to show $E[S_2|I_t] = E[S_2]$. To receive full points when using Approach 2 below, one needs to demonstrate all three expectations: $E[Z_{t+s}|I_t] = Z_t$ *for* $t = 0, 1,$ *and* 2.

(i)
The expected return using the real probablity is $\frac{E[S_{t+1}|S_t]}{S_t} - 1 = 0.3 * 1.4 + 0.7 * 0.9 - 1 = 5\%$, which is greater than the risk-free rate 2%. Thus S_t is a risky asset and there is a risk premium assocated with this asset in additional to the risk-free return. In other words, $\frac{S_t}{(1+r)^t}$ is not a martingale because $E\left[\frac{S_{t+1}}{(1+r)^{t+1}}|S_t\right] = \frac{E[S_{t+1}|S_t]}{S_t}\frac{S_t}{(1+r)^{t+1}} = \frac{1.05}{1.02}\frac{S_t}{(1+r)^t} > \frac{S_t}{(1+r)^t}$.

(ii)
Approach 1: for s = 1 or 2, under real-world measure,
$E[Z_{t+s}|I_t] = E[E[S_2|I_{t+s}]|I_t]$by substituting $Z_{t+s} = E(S_2|I_{t+s})$
Then since $E[E[S_2|I_{t+s}]|I_t] = E[S_2|I_t]$,
$E[Z_{t+s}|I_t] = E[S_2|I_t] = Z_t$ which satisfies the definition for the martingale.

Approach 2:

$$Z_0 = E(S_2|I_o) = S_0 * (1.4^2 * 0.3^2 + 0.9^2 * 0.7^2 + 2 * 1.4 * 0.9 * 0.3 * 0.7$$
$$= 1.05^2 S_0$$
$$Z_1 = E(S_2|I_1) = S_1 * (1.4 * 0.3 + 0.9 * 0.7) = 1.05S_1$$
$$Z_2 = E(S_2|I_2) = S_2$$
$$E[Z_1|I_0] = E[E(S_2|I_1)|I_0] = E[1.05S_1|I_0] = 1.05E[S_1|I_0]$$
$$= 1.05 * S_0 * (1.4 * 0.3 + 0.9 * 0.7) = 1.05^2 S_0 = Z_0$$
$$E[Z_2|I_0] = E[E(S_2|I_2)|I_0] = E[S_2|I_0] = Z_0$$
$$E[Z_2|I_1] = E[E(S_2|I_2)|I_1] = E[S_2|I_1] = Z_1$$

Since for s = 1 or 2, $E[Z_{t+s}|I_t] = Z_t$, Z_t is a martingale under the real-world measure

(b) Construct a portfolio to replicate the option, using the stock S_t, risk-free borrowing and lending, and assuming the absence of arbitrage.

1. Continued

Commentary on Question:
This part is a straight-forward calculation of the put option price and using stock and risk-free accounts to replicate the put value on each node of the binomial tree. Candidates did poorly on this part. Many candidates can describe briefly how to replicate but the majority cannot elaborate on that. Some candidates incorrectly applied real-world probability when calculating the put option price. Very few candidates noticed that re-balancing would be needed at time 1 too.

Under no-arbitrage, using risk-neutral measure to calculate the price of the put.
The Stock price will look like:

$t=0$	$t=1$	$t=2$
		196
	140	
100		126
	90	
		81

The risk-neutral probablity q that an up-node occurs satisfies $(1+r) = 1.4 * q + 0.9 * (1-q), q = 0.24$

At time 0, the put option price is then

$$[0 * q^2 + 0 * q(1-q) + 39 * (1-q)^2]/(1+r)^2 = 21.65$$

Assume Δ_0 units of Share and B_0 for risk-free account. The total value of the replicating portfolio

$$\Delta_0 * 100 + B_0 = 21.65 \qquad (1)$$

At time 1, if the stock price goes up, the put has no value, this gives

$$\Delta_0 * 140 + B_0 * (1+r) = 0 \qquad (2)$$

Solve for equation (1) & (2) gives $\Delta_0 = -0.58, B_0 = 79.76$

At time 1, if the stock price goes down, the put value is $(q * 0 + 39 * (1-q))/(1+r) = 29.06$. Assume Δ_1 units of Share and B_1 for risk-free account (after rebalancing). The replicating portfolio should be

$$\Delta_1 * 90 + B_1 = 29.06 \qquad (3)$$

At time 2, if the stock price goes up, the put has no value, this gives

$$\Delta_1 * 126 + B_1 * (1+r) = 0 \qquad (4)$$

Solve for equation (3) & (4) gives $\Delta_1 = -0.87, B_1 = 107.06$

So to replicate the put one needs to:
At time 0, short 0.58 shares of stock and invest $79.76 at risk-free
At time 1, if stock price goes up, do nothing and the portfolio is worthless. Use the investment in the risk-free account to close the short selling of the stock; if stock price goes down, short additional -0.29=-0.87-(-0.58) shares of stock and invest additional $27.30=107.06-79.76 at risk-free.

1. Continued

At time 2, if stock price goes up, the portfolio is worthless. Again, use the investment in the risk-free account to close the short selling of the stock; if stock price goes down, the option's payoff of $39 is replicated by the portfolio's value.

(c) Describe situations in which risk-neutral pricing is not valid.

Commentary on Question:
This part is intended to test candidates understanding of incomplete market. Candidates did moderately well on this part. Many candidates did identify incomplete market but failed to describe its characteristics and thus only received partial marks.

Risk-neutral pricing doesn't work perfectly in practice, especially when market is not complete (*Candidates answered incomplete market and accurately described two of the following characteristics would receive full points for this part*):

1) The underlying follows a jump-diffusion process;
2) The underlying or one of the variables governing the behavior of the underlying is random. For example, options on terrorist acts cannot be hedged since terrorist acts aren't traded.
3) If the volatility is a known time dependent function then market is still complete. But in reality, we don't know what volatility will be so markets are not complete.
4) When market is not complete, it is not always possible to find enough quantities of instruments traded to replicate the option. Since dynamic hedging is impossible, risk cannot be eliminated continuously;
5) There are transaction costs. Continuously dynamic hedging (or rebalancing) is impossible as there's always a cost to replicate the option;
6) Market is not continuous so that dynamic hedging is not continuously possible to eliminate the risk;

2. Learning Objectives:

1. The candidate will understand the fundamentals of stochastic calculus as they apply to option pricing.

Learning Outcomes:

(1d) Understand and apply Ito's Lemma.

(1f) Demonstrate understanding of option pricing techniques and theory for equity and interest rate derivatives.

(1g) Demonstrate understanding of the differences and implications of real-world versus risk-neutral probability measures.

Sources:

Netfci Ch. 10,

Wilmott Ch. 6

Commentary on Question:

Candidates did not do well on this question as many candidates struggled with the interpretation of certain parts. Candidates did the best on parts (a) and (c).Most of the candidates did not attempt part (d) or completely got it wrong. Candidates received partial credit for identifying some of the early steps in part (e) but very few got beyond that.

Solution:

(a) Derive the dynamics of N_t^B under the $\mathbb{P}$-measure.

$$dN_t = d[M_t^B X_t]$$
$$Apply\ Ito's\ Lemma\ and\ get$$
$$dN_t = N_t(\mu_X dt + \sigma_X dW_X^P) + N_t r_t^B dt$$
$$dN_t = N_t(\mu_X + r_t^B)dt + N_t \sigma_X dW_X^P$$

(b) Derive the dynamics of X_t under Country A's martingale measure ($\mathbb{Q}^A$)

N_t is another asset in Country A and markets are complete, therefore it must be that under Q:

$$dN_t = N_t r_t^A dt + N_t \sigma_X dW_X^{Q_A}$$

Apply Ito's Lemma again with $X_t = \frac{N_t}{M_t^B}$:

$$dX_t = d\left(\frac{N_t}{M_t^B}\right) = -\frac{N_t}{M_t^B} r_t^B dt + \frac{N_t}{M_t^B}(r_t^A dt + \sigma_X dW_X^{Q_A})$$
$$\frac{dX_t}{X_t} = (r_t^A - r_t^B)dt + \sigma_X\, dW_X^{Q_A}$$

2. Continued

(c) *Derive the dynamics of F_t under Country A's martingale measure ($\mathbb{Q}^A$)*

$$F_t = D_t X_t$$
$$dF_t = dD_t X_t + D_t dX_t + dD_t dX_t$$
$$dF_t = F_t(\mu_D dt + \sigma_D dW_D^P) + F_t(\mu_X dt + \sigma_X dW_X^P) + F_t \sigma_X \sigma_D \rho_{D,X} dt$$
$$\frac{dF_t}{F_t} = \left(\mu_D + \sigma_X \sigma_D \rho_{D,X} + \mu_X\right)dt + \sigma_D dW_D^P + \sigma_X dW_X^P$$

By the Girsanov Theorem, in a complete market there exists a risk-neutral measure Q_A such that:

$$\frac{dF_t}{F_t} = r_A dt + \sigma_D dW_D^{Q_A} + \sigma_X dW_X^{Q_A}$$

(d) Derive the dynamics of D_t under Country A's martingale measure ($\mathbb{Q}^A$)

$$dD_t = d\left(\frac{F_t}{X_t}\right)$$

by Ito's Lemma, we get

$$dD_t = -\frac{F_t}{X_t^2} dX_t + \frac{dF_t}{X_t} + \frac{1}{2}\frac{2\,F_t}{X_t^3}\ dX_t dX_t - \frac{1}{X_t^2} dF_t dX_t$$
$$\frac{dD_t}{D_t} = -\left((r_t^A - r_t^B)dt + \sigma_X\ dW_X^{Q_A}\right) + \left(r_A dt + \sigma_D dW_D^{Q_A} + \sigma_X dW_X^{Q_A}\right) + \ \sigma_X^2 dt$$
$$- \sigma_D\ \sigma_X \rho_{D,X} dt - \sigma_X^2 dt$$
$$\frac{dD_t}{D_t} = \left(r_t^B - \sigma_D\ \sigma_X \rho_{D,X}\right)dt + \sigma_D dW_D^{Q_A}$$

(e) Derive the fair strike price, K*, at time 0 of this forward contract.

The value of the contract is:

$$G_t = e^{-r_A \times (T-t)} \times E^{Q_A}[G_T | I_t] = \ e^{-r_A \times (T-t)} \times E^{Q_A}[X_0 \times (D_T - K) | I_t]$$
$$G_t = e^{r_A \times (T-t)} \times X_0 (E^{Q_A}[(D_T)|I_t] - K)$$

To find the fair strike, set $G_0 = 0$

$$0 = \ E^{Q_A}[(D_T)|I_0] - K$$
$$K^* = E_0^{Q_A}[D_T]$$

Then, let $V_t = Ln(D_t)$

$$\frac{\partial V_t}{\partial t} = 0\ ;$$
$$\frac{\partial V_t}{\partial D_t} = \frac{1}{D_t}$$
$$\frac{\partial^2 V_t}{\partial D_t^2} = -\frac{1}{D_t^2}$$

2. **Continued**

Apply Ito's Lemma:

$$dV_t = \left(r_t^B - \sigma_D \, \sigma_X \rho_{D,X} - \frac{\sigma_D^2}{2} \right) dt + \sigma_D \, dW_D^{Q_A}$$

$$D_t = D_0 e^{\left(r_t^B - \sigma_D \, \sigma_X \rho_{D,X} - \frac{\sigma_D^2}{2} \right) t + \sigma_D \, W_D^{Q_A}}$$

$$E_0^{Q_A}[D_T] = D_0 E_0^{Q_A} \left[e^{\left(r_t^B - \sigma_D \, \sigma_X \rho_{D,X} - \frac{\sigma_D^2}{2} \right) t + \sigma_D \, W_D^{Q_A}} \right]$$

The term inside the expectation is log-normal with mean $e^{(r_t^B - \sigma_D \, \sigma_X \rho_{D,X})t}$

$$E_0^{Q_A}[D_T] = D_0 e^{(r_t^B - \sigma_D \, \sigma_X \rho_{D,X})t}$$

3. **Learning Objectives:**

1. The candidate will understand the fundamentals of stochastic calculus as they apply to option pricing.

3. The candidate will understand the quantitative tools and techniques for modeling the term structure of interest rates and pricing interest rate derivatives.

Learning Outcomes:

(1b) Understand the importance of the no-arbitrage condition in asset pricing.

(1c) Understand Ito integral and stochastic differential equations.

(1d) Understand and apply Ito's Lemma.

(1f) Demonstrate understanding of option pricing techniques and theory for equity and interest rate derivatives.

(1i) Understand and apply Girsanov's theorem in changing measures.

(1j) Understand the Black Scholes Merton PDE (partial differential equation).

(3a) Understand and apply the concepts of risk-neutral measure, forward measure, normalization, and the market price of risk, in the pricing of interest rate derivatives.

(3c) Understand and apply popular one-factor interest rate models including Vasicek, Cox-Ross_Ingersoll, Hull-White, Ho-Lee, Black-Derman_Toy, Black-Karasinski.

Sources:

Neftci Ch. 6, 10, 12, 14, 15, 17

Wilmott – Introduces Quantitative Finance Ch. 16, 17

Commentary on Question:

Overall candidates did poorly on the question. Many candidates were able to derive some of the correct expressions in parts (a) and (b). Few candidates were able to adequately answer part (c). No candidates demonstrated the ability to apply Girsanov's Theorem to change measures in part (d). In addition, no candidates recognized the need for simulation or numerical methods where there was no closed form solution.

Solution:

(a) Derive $d\left(a_{t:\overline{n}|}\right)$ in terms of dt and dX_t.

3. Continued

Commentary on Question:
Many candidates did not recognize that the formula for $dP(t,T)$ was given in the Wilmott text and attempted to derive it. Many candidates were able to derive it by application of Ito's Lemma, but many were not careful in maintaining the term structure of volatility.

Note that pg. 407 of Wilmott – Introduces Quantitative Finance Ch. 19.5 indicate that

$$\mathrm{dP(t,T)} = \mathrm{P(t,T)}\left[\mathrm{r(t)dt} - \int_t^T \upsilon(\mathrm{t,s})\mathrm{ds\,dX_t}\right]$$

$$a_{\mathrm{t}:\overline{\mathrm{n}}|} = \sum_i^n \mathrm{P(t,T_i)}$$

$$\mathrm{d}a_{\mathrm{t}:\overline{\mathrm{n}}|} = \sum_i^n \mathrm{dP(t,T_i)} = \sum_i^n \mathrm{P(t,T_i)}\left[\mathrm{r(t)dt} - \int_t^{T_i} \upsilon(\mathrm{t,s})\mathrm{ds\,dX_t}\right]$$

$$= a_{\mathrm{t}:\overline{\mathrm{n}}|}\mathrm{r(t)dt} - \sum_i^n \mathrm{P(t,T_i)}\int_t^{T_i} \upsilon(\mathrm{t,s})\mathrm{ds\,dX_t}$$

(b) Derive expressions for A, B, and C.

Commentary on Question:
Most candidates were able to derive $d\big(log\big(S(t)\big)\big)$. Many candidates did not properly apply Ito's Lemma to $d\big(log\big(a_{t:\overline{n}|}\big)\big)$ and dropped the second order term. Partial credit was awarded for deriving individual pieces correctly.

Note that $d\left(\log\left(\frac{a_{\mathrm{t}:\overline{\mathrm{n}}|}}{S(t)}\right)\right) = d\big(\log\big(a_{\mathrm{t}:\overline{\mathrm{n}}|}\big)\big) - d\big(\log\big(S(t)\big)\big)$. We calculate each of the two components using Ito's Lemma.

First, since

$$\mathrm{dS(t)} = \mathrm{r(t)S(t)dt} + \sigma_s \mathrm{S(t)dW_t}$$

by Ito's lemma,

$$\mathrm{d\big(\log\big(S(t)\big)\big)} = \left(\mathrm{r(t)} - \frac{1}{2}\sigma_s^2\right)\mathrm{dt} + \sigma_s \mathrm{dW_t}$$

Next, denote $\mathrm{G} = \log\big(a_{\mathrm{t}:\overline{\mathrm{n}}|}\big)$.
Recall that from part (a) above $\mathrm{d}a_{\mathrm{t}:\overline{\mathrm{n}}|} = \mu(t)dt - \sigma_a(t)\mathrm{dX_t}$ where $\mu(t) = a_{\mathrm{t}:\overline{\mathrm{n}}|}\mathrm{r(t)}$ and $\sigma_a(t) = \sum_1^n \mathrm{P(t,T_i)}\int_t^{T_i}\upsilon(\mathrm{t,s})\mathrm{ds}$.

3. Continued

By Ito's lemma,

$$d\left(\log\left(a_{t:\overline{n}|}\right)\right) = \frac{\partial G}{\partial t}dt + \frac{\partial G}{\partial a_{t:\overline{n}|}}da_{t:\overline{n}|} + \frac{1}{2}\frac{\partial^2 G}{\partial a_{t:\overline{n}|}{}^2}\sigma_a^2(t)dt$$

$$= 0 + \mathrm{r(t)dt} - \frac{\sigma_a(t)}{a_{t:\overline{n}|}}\mathrm{dX_t} - \frac{1}{2}\left[\frac{\sigma_a(t)}{a_{t:\overline{n}|}}\right]^2 \mathrm{dt}$$

$$= \left\{\mathrm{r(t)} - \frac{1}{2}\left[\frac{\sigma_a(t)}{a_{t:\overline{n}|}}\right]^2\right\}\mathrm{dt} - \frac{\sigma_a(t)}{a_{t:\overline{n}|}}\mathrm{dX_t}$$

Therefore, noting that $X_t = \rho W_t + \sqrt{1-\rho^2}Z_t$, we have

$$d\left(\log\left(\frac{a_{t:\overline{n}|}}{S(t)}\right)\right) = -\frac{1}{2}\left\{\left[\frac{\sigma_a(t)}{a_{t:\overline{n}|}}\right]^2 - \sigma_s^2\right\}\mathrm{dt} - \frac{\sigma_a(t)}{a_{t:\overline{n}|}}\mathrm{dX_t} - \sigma_s \mathrm{dW_t}$$

$$= -\frac{1}{2}\left\{\left[\frac{\sigma_a(t)}{a_{t:\overline{n}|}}\right]^2 - \sigma_s^2\right\}\mathrm{dt} - \frac{\sigma_a(t)}{a_{t:\overline{n}|}}\left(\rho \mathrm{dW_t} + \sqrt{1-\rho^2}\mathrm{dZ_t}\right) - \sigma_s \mathrm{dW_t}$$

$$= -\frac{1}{2}\left\{\left[\frac{\sigma_a(t)}{a_{t:\overline{n}|}}\right]^2 - \sigma_s^2\right\}\mathrm{dt} - \left\{\frac{\sigma_a(t)}{a_{t:\overline{n}|}}\rho + \sigma_s\right\}\mathrm{dW_t} - \frac{\sigma_a(t)}{a_{t:\overline{n}|}}\sqrt{1-\rho^2}\mathrm{dZ_t}$$

(c) Derive $d\left(\frac{a_{t:\overline{n}|}}{S(t)}\right)$ based on your answer for part (b).

Commentary on Question:
Candidates generally did poorly on this question because many were not successful in deriving the solution to part (b).

Suppose that the solution is of the form $d\left(\frac{a_{t:\overline{n}|}}{S(t)}\right) = \frac{a_{t:\overline{n}|}}{S(t)}(D\ dt + E\ dW_t + F\ dZ_t)$.
We can derive the coefficients D, E, and F as below.

Applying Ito's lemma to the $\log\left(\frac{a_{t:\overline{n}|}}{S(t)}\right)$ yields

$$d\left(\log\left(\frac{a_{t:\overline{n}|}}{S(t)}\right)\right) = \left(D - \frac{E^2}{2} - \frac{F^2}{2}\right)dt + E\ dW_t + F\ dZ_t$$

Thus, equating the above to the $d\left(\log\left(\frac{a_{t:\overline{n}|}}{S(t)}\right)\right)$ obtained from part (b) gives us

3. Continued

$$E = -\left\{\frac{\sigma_a(t)}{a_{t:\overline{n}|}}\rho + \sigma_s\right\}$$
$$F = -\frac{\sigma_a(t)}{a_{t:\overline{n}|}}\sqrt{1-\rho^2}$$

$$D = -\frac{1}{2}\left\{\left[\frac{\sigma_a(t)}{a_{t:\overline{n}|}}\right]^2 - \sigma_s^2\right\} + \frac{1}{2}\left\{\frac{\sigma_a(t)}{a_{t:\overline{n}|}}\rho + \sigma_s\right\}^2 + \frac{1}{2}\left\{\frac{\sigma_a(t)}{a_{t:\overline{n}|}}\sqrt{1-\rho^2}\right\}^2 = \frac{\sigma_a(t)}{a_{t:\overline{n}|}}\rho\sigma_s + \sigma_s^2$$

(d)

(i) Derive the equivalent martingale measure for evaluating the expectation of the payoff of this option using Girsanov Theorem and all the previous results.

(ii) Describe how the expectation of this payoff can be evaluated using the results in part (d)(i).

Commentary on Question:
For part (i), partial marks were awarded for general commentary demonstrating an understanding of Girsanov Theorem. No candidate attempted to derive the equivalent martingale measure, with many only restating the definition of Girsanov's Theorem.

For part (ii), no candidates recognized that the solution is not of a closed form, and therefore simulation or numerical methods are necessary.

(i) By Girsanov Theorem we can find a drift-shift for $d\widetilde{W}_t$ such that the drift of $\frac{a_{t:\overline{n}|}}{S(t)}$ vanishes, that is, for some γ and ξ:

$$d\widetilde{W}_t = dW_t - \gamma dt$$
$$d\frac{a_{t:\overline{n}|}}{S(t)} = \xi\frac{a_{t:\overline{n}|}}{S(t)}d\widetilde{W}_t + \eta\sqrt{1-\rho^2}\frac{a_{t:\overline{n}|}}{S(t)}dZ_t$$

, where $\xi = \eta\rho - \sigma_s$ and $\eta = \frac{\sum_1^n P(t,T_i)\int_t^{T_i}\upsilon(t,s)ds}{a_{t:\overline{n}|}}$

Applying Ito's Lemma on the preceding sets of equations above, we find that $d(\log\frac{a_{t:\overline{n}|}}{S(t)}) = -\frac{1}{2}(\xi^2 + \eta^2(1-\rho^2))dt + \xi\, d\widetilde{W}_t + \eta\sqrt{1-\rho^2}dZ_t = -\left(\gamma\xi + \frac{\xi^2}{2} + \frac{1}{2}\eta^2(1-\rho^2)\right)dt + \xi\, dW_t + \eta\sqrt{1-\rho^2}dZ_t$

3. Continued

Comparing this last equation with the $d(\log\frac{a_{t:\overline{n}|}}{S(t)})$ solved in part b) above, we'll be able to solve the parameters ξ, η and γ .

Denote $\max(a_{T:\overline{n}|} - S(T),0) = (a_{T:\overline{n}|} - S(T))_+$ and price of the option as $\mathcal{H}$, which is

$$\mathcal{H} = \mathbb{E}^{\mathbb{Q}}[e^{-\int_t^T r(s)ds}(a_{T:\overline{n}|} - S(T))_+]$$

By Girsanov Theorem,
$\frac{\mathcal{H}}{S(0)} = \mathbb{E}^{\mathbb{Q}^S}[(\frac{a_{T:\overline{n}|}}{S(T)} - 1)_+]$, where the expectation is taken with respect to the equivalent martingale measure (EMM) that has the tradeable equity S(t) as the numeraire asset.

(ii) From part (d) (i), we want to evaluate the the expectation $\mathbb{E}^{\mathbb{Q}^S}[(\frac{a_{T:\overline{n}|}}{S(T)} - 1)_+]$. Given that there is no closed-form solution, we are resorted to obtain the solution using either Monte Carlo simulation or PDE numerical methods.

If solving by PDE numerical methods, the following steps can be followed (Neftci Ch. 13.7.2):

- Derive the PDE equation satisfied by the option price, which is a multivariate version of the Black-Scholes PDE;
- The grid sizes for ΔS, ΔF, Δt must be selected as minimum increments in the price of the security, forward interest rate and time;
- Determine the range of possible values for S and F, namely $S_{min} \le S_t \le S_{max}$ and $F_{min} \le F(t,T) \le F_{max}$;
- The boundary conditions must be determined, namely $\mathcal{H}(S_{max}, F_{max}, t) = 0$ and $\mathcal{H}(S_{min}, F_{min}, t) = S(0)(\frac{a_{T:\overline{n}|}^{F_{min}}}{S_{min}} - 1)$

Assuming the for small but noninfinitesimal $\Delta S, \Delta F, \Delta t$ the same PDE is valid, the value of option $\mathcal{H}(S, F, t)$ at the grid points should be determined.

4. **Learning Objectives:**

1. The candidate will understand the fundamentals of stochastic calculus as they apply to option pricing.

Learning Outcomes:

(1c) Understand Ito integral and stochastic differential equations.

(1d) Understand and apply Ito's Lemma.

Sources:

An Introduction to the Mathematics of Financial Derivatives, Neftci, Salih, 3rd Edition, Ch 8, 10

Commentary on Question:

Commentary listed underneath question component.

Solution:

(a) Derive the stochastic differential equation for F in terms of Z_t.

Commentary on Question:

There are a few ways to express 2-variable Ito's lemma. All correct expressions are acceptable. Partial credit is given for each correct partial differentiation if the final answer is incorrect

Using Ito's Lemma (Source: Neftci 3rd Ed., Section 10.7.1)

$F_x = e^{Y_t}$

$F_{xx} = 0$

$F_y = X_t e^{Y_t}$

$F_{yy} = X_t e^{Y_t}$

$F_{xy} = e^{Y_t}$

$F_t = 1$

$$\begin{aligned}
dF &= [F_t + F_x Y_t + F_y X_t + \tfrac{1}{2} F_{xx}(tX_t)^2 + \tfrac{1}{2} F_{yy} t^2]dt \\
&\quad + F_x\, tX_t dW_t + F_y\, tdV_t \\
&= [1 + e^{Y_t} Y_t + X_t^2 e^{Y_t} + 0 + \tfrac{1}{2} e^{Y_t} X_t t^2]dt \\
&\quad + e^{Y_t} tX_t dW_t + X_t e^{Y_t} tdV_t \\
&= \left[1 + e^{Y_t} Y_t + X_t^2 e^{Y_t} + \tfrac{1}{2} e^{Y_t} X_t t^2\right] dt + e^{Y_t} tX_t (dW_t + dV_t) \\
&= \left[1 + e^{Y_t} Y_t + X_t^2 e^{Y_t} + \tfrac{1}{2} e^{Y_t} X_t t^2\right] dt + \sqrt{2} e^{Y_t} tX_t dZ_t
\end{aligned}$$

4. Continued

(b) Calculate the mean and variance of $\left(Z_{t+s} - Z_t\right)$ for $t \geq 0$ and $s > 0$.

Commentary on Question:
Most candidates did very well in this part, full mark is also given for arguing Z is also a Weiner process

$$E(Z_{t+s} - Z_t) = \frac{1}{\sqrt{2}}E(W_{t+s} - W_t) \; + \; \frac{1}{\sqrt{2}}E(V_{t+s} - V_t)$$
$$= \; 0 \; + \; 0 = \; 0$$
$$VAR(Z_{t+s} - Z_t) = \frac{1}{2}VAR[(W_{t+s} - W_t) \; + \; (V_{t+s} - V_t)]$$
$$= \frac{1}{2}[VAR(W_{t+s} - W_t) \; + \; VAR(V_{t+s} - V_t)]$$ (increments of W and V are independent)
$$= \frac{1}{2}[s + s] = s$$

(c) Calculate the mean of $Z_t\left(Z_{t+s} - Z_t\right)$ for $t \geq 0$ and $s > 0$.

Commentary on Question:
There are a number of ways to complete this part. Full mark is given for arguing Z is also a Weiner process, therefore Z_t and $(Z_{t+s}$-$Z_t)$ are independent. Points are taken off if candidate argues independence without given any reasoning. Full mark is also given for expanding $Z_t\left(Z_{t+s} - Z_t\right)$ to a combination of W and V, then find independent relationships there.

Let I_t be the information set at time t

$$E[(Z_{t+s} - Z_t)Z_t] = E(Z_{t+s}Z_t - Z_t^2)$$
$$= E(Z_{t+s}Z_t) - E(Z_t^2)$$
$$= \; \frac{1}{2}E[(W_{t+s} + V_{t+s})\,(W_t + V_t)] - VAR(Z_t)$$
$$= \; \frac{1}{2}E[W_{t+s}\,W_t + V_{t+s}V_t + \; W_{t+s}V_t + V_{t+s}W_t] - t$$
$$= \; \frac{1}{2}[E(W_{t+s}\,W_t) + E(V_{t+s}V_t) + E(W_{t+s}V_t) + E(V_{t+s}W_t)] - t$$
$$= \; \frac{1}{2}\{E[E(W_{t+s}\,W_t|I_t)] + E[E(V_{t+s}V_t|I_t)] + \; 0 + \; 0\} - t$$ (W and V are independent)
$$= \; \frac{1}{2}\{E[W_tE(W_{t+s}|I_t)] + E[V_tE(V_{t+s}|I_t)]\} - t$$
$$= \; \frac{1}{2}\{E[W_t^2] + E[V_t^2]\} - t$$

(Since W_t and V_t are martingale, $E(W_{t+s}|\, I_t) = \; W_t$ and $E(V_{t+s}|\, I_t) = \; V_t$)

$$= \; \frac{1}{2}\{t + t\} - t = 0$$

5. Learning Objectives:

1. The candidate will understand the fundamentals of stochastic calculus as they apply to option pricing.

Learning Outcomes:

(1c) Understand Ito integral and stochastic differential equations.

(1d) Understand and apply Ito's Lemma.

Sources:

Nefci Ch. 9

Commentary on Question:

This question tests the candidates' understanding of the fundamentals of stochastic calculus, such as Ito integrals and mean square convergence. The key to solving this question is to understand and apply the properties of Wiener process, specifically the independence of increments and their means and variances, and knowing the definition of mean square convergence. The majority of the candidates struggle with parts (a) to (c), and part (d) is done relatively well.

Solution:

(a) Calculate $E\left[\left(\sum_{i=0}^{n-1} e^{W_{t_i}-\frac{t_i}{2}} \Delta W_{t_i}\right)^2\right]$.

Commentary on Question:

Most Candidates struggled with parts (a) and (b), as they did not recognize the independence structures within the expressions that they were trying to simplify. Partial marks are given for steps done towards the final answer.

5. Continued

$$E\left(\sum_{i=0}^{n-1} e^{W_{t_i}-\frac{t_i}{2}}\Delta W_{t_i}\right)^2$$

$$= E\left(\sum_{i=0}^{n-1} e^{2W_{t_i}-t_i}\,(\Delta W_{t_i})^2\right) + 2E\left(\sum_{i<j} e^{W_{t_i}-\frac{t_i}{2}+W_{t_j}-\frac{t_j}{2}}\Delta W_{t_i}\Delta W_{t_j}\right)$$

All terms in the second summation are 0 since ΔW_{t_j} is independent from the rest and has a mean of 0.

Therefore,

$$E\left(\sum_{i=0}^{n-1} e^{W_{t_i}-\frac{t_i}{2}}\Delta W_{t_i}\right)^2 = E\left(\sum_{i=0}^{n-1} e^{2W_{t_i}-t_i}\,(\Delta W_{t_i})^2\right) = \sum_{i=0}^{n-1} E\left(e^{2W_{t_i}-t_i}\right)E\left((\Delta W_{t_i})^2\right)$$

Since $W_{t_i} \sim N(0, t_i)$, thus $2W_{t_i} - t_i \sim N(-t_i, 4t_i)$, and so

$$E\left(e^{2W_{t_i}-t_i}\right)E\left(\Delta W_{t_i}{}^2\right) = e^{-t_i+\frac{4t_i}{2}}h = he^{t_i}$$

Or

$$E\left(e^{2W_{t_i}-t_i}\right)E\left(\Delta W_{t_i}{}^2\right) = e^{-t_i}E\left(e^{2W_{t_i}}\right)E\left(\Delta W_{t_i}{}^2\right) = e^{-t_i}e^{\frac{4t_i}{2}}h = he^{t_i}$$

Therefore,

$$E\left(\sum_{i=0}^{n-1} e^{W_{t_i}-\frac{t_i}{2}}\Delta W_{t_i}\right)^2 = \sum_{i=0}^{n-1} he^{t_i} = h\frac{1-e^{nh}}{1-e^{h}} = (1-e^{T})\frac{h}{1-e^{h}}$$

(b) Calculate $E\left[\left(\sum_{i=0}^{n-1} e^{W_{t_i}-\frac{t_i}{2}}\Delta W_{t_i}\right)\left(e^{W_T-\frac{T}{2}}-1\right)\right]$.

$$E\left[\left(\sum_{i=0}^{n-1} e^{W_{t_i}-\frac{t_i}{2}}\,\Delta W_{t_i}\right)\left(e^{W_T-\frac{T}{2}}-1\right)\right] = \sum_{i=0}^{n-1} E\left[e^{W_{t_i}-\frac{t_i}{2}}\Delta W_{t_i}\left(e^{W_T-\frac{T}{2}}-1\right)\right]$$

For each term in the summation, denote $A = W_{t_i}$, $B = \Delta W_{t_i}$, $C = W_T - W_{t_{i+1}}$, then

$$E\left[e^{W_{t_i}-\frac{t_i}{2}}\Delta W_{t_i}\left(e^{W_T-\frac{T}{2}}-1\right)\right] = E\left[e^{A-\frac{t_i}{2}}B\left(e^{(A+B+C)-\frac{T}{2}}-1\right)\right]$$

$$= E\left[e^{2A+B+C-\frac{t_i}{2}-\frac{T}{2}}B\right] - E\left[e^{A-\frac{t_i}{2}}B\right]$$

5. **Continued**

Due to independence of A, B, and C

$$E\left[e^{W_{t_i}-\frac{t_i}{2}}\Delta W_{t_i}\left(e^{W_T-\frac{T}{2}}-1\right)\right]=e^{-\frac{t_i}{2}-\frac{T}{2}}E(e^{2A})E(e^C)E(Be^B)$$
$$=e^{-\frac{t_i}{2}-\frac{T}{2}}e^{2t_i}e^{\frac{T-t_{i+1}}{2}}he^{\frac{h}{2}}=he^{t_i}$$

Therefore,

$$E\left[\left(\sum_{i=0}^{n-1}e^{W_{t_i}-\frac{t_i}{2}}\Delta W_{t_i}\right)\left(e^{W_T-\frac{T}{2}}-1\right)\right]=\sum_{i=0}^{n-1}he^{t_i}=h\frac{1-e^{nh}}{1-e^h}=(1-e^T)\frac{h}{1-e^h}$$

(c) Show that $\int_0^T e^{W_s-\frac{s}{2}}dW_S=e^{W_T-\frac{T}{2}}-1$ by proving that $\sum_{i=0}^{n-1}e^{W_{t_i}-\frac{t_i}{2}}\Delta W_{t_i}$ converges to $e^{W_T-\frac{T}{2}}-1$ in mean square convergence.

Commentary on Question:
Partial marks are given for understanding the definition of mean square convergence.

To show that the Ito sum converges in mean square convergence, we need to show:

$$\lim_{n\to\infty}E\left[\left(\sum_{i=0}^{n-1}e^{W_{t_i}-\frac{t_i}{2}}\Delta W_{t_i}\right)-\left(e^{W_T-\frac{T}{2}}-1\right)\right]^2=0$$

$$E\left[\left(\sum_{i=0}^{n-1}e^{W_{t_i}-\frac{t_i}{2}}\Delta W_{t_i}\right)-\left(e^{W_T-\frac{T}{2}}-1\right)\right]^2$$
$$=E\left[\left(\sum_{i=0}^{n-1}e^{W_{t_i}-\frac{t_i}{2}}\Delta W_{t_i}\right)^2\right]$$
$$-2E\left[\left(\sum_{i=0}^{n-1}e^{W_{t_i}-\frac{t_i}{2}}\Delta W_{t_i}\right)\left(e^{W_T-\frac{T}{2}}-1\right)\right]+E\left[\left(e^{W_T-\frac{T}{2}}-1\right)^2\right]$$
$$=(1-e^T)\frac{h}{1-e^h}-2(1-e^T)\frac{h}{1-e^h}+e^T-1$$
$$=-(1-e^T)\frac{h}{1-e^h}+e^T-1$$

5. Continued

Since $n \to \infty$ is equivalent to $h \to 0$, we calculate the limit of the right hand side as $h \to 0$.
To calculate $\lim_{h\to 0} \frac{h}{1-e^h}$ we need to employ L'Hospital's Rule; using it we see the limit is 1.

$$\lim_{n\to\infty} E\left[\left(\sum_{i=0}^{n-1} e^{W_{t_i}-\frac{t_i}{2}}\Delta W_{t_i}\right)-\left(e^{W_T-\frac{T}{2}}-1\right)\right]^2 = -(1-e^T)+(e^T-1)=0$$

(d) Show that $\int_0^T e^{W_s-\frac{s}{2}}dW_S = e^{W_T-\frac{T}{2}}-1$ by proving that $d\left(e^{W_t-\frac{t}{2}}\right)=e^{W_t-\frac{t}{2}}dW_t$ using Ito's Lemma.

Commentary on Question:
This part is done relatively well, as most candidates were able to apply Ito's Lemma properly and demonstrate how $d\left(e^{W_t-\frac{t}{2}}\right)=e^{W_t-\frac{t}{2}}dW_t$ *leads to proving* $\int_0^T e^{W_s-\frac{s}{2}}dW_S = e^{W_T-\frac{T}{2}}-1$.

By Ito's Lemma,

$$d\left(e^{W_t-\frac{t}{2}}\right)=\frac{\partial\left(e^{W_t-\frac{t}{2}}\right)}{\partial t}dt+\frac{\partial\left(e^{W_t-\frac{t}{2}}\right)}{\partial W_t}dW_t+\frac{1}{2}\frac{\partial^2\left(e^{W_t-\frac{t}{2}}\right)}{\partial W_t^2}dt$$
$$=-\frac{1}{2}e^{W_t-\frac{t}{2}}dt+e^{W_t-\frac{t}{2}}dW_t+\frac{1}{2}e^{W_t-\frac{t}{2}}dt$$
$$=e^{W_t-\frac{t}{2}}dW_t$$

Therefore, integrating both sides

$$\int_0^T d(e^{W_t-\frac{t}{2}})=\int_0^T e^{W_t-\frac{t}{2}}dW_t$$
$$e^{W_T-\frac{T}{2}}-e^{0-\frac{0}{2}}=\int_0^T e^{W_t-\frac{t}{2}}dW_t$$
$$\int_0^T e^{W_t-\frac{t}{2}}dW_t=e^{W_T-\frac{T}{2}}-1$$

6. **Learning Objectives:**

2. The candidate will understand how to apply the fundamental theory underlying the standard models for pricing financial derivatives. The candidate will understand the implications for option pricing when markets do not satisfy the common assumptions used in option pricing theory such as market completeness, bounded variation, perfect liquidity, etc. The Candidate will understand how to evaluate situations associated with derivatives and hedging activities.

Learning Outcomes:

(2a) Identify limitations of the Black-Scholes pricing formula

(2g) Describe and explain some approaches for relaxing the assumptions used in the Black-Scholes formula.

Sources:

QFIC-103-13: How to Use the Holes in Black-Scholes

Frequently Asked Questions in Quantitative Finance, Wilmott, Paul, 2nd Edition
Ch. 2: Q53

Commentary on Question:

This question tests candidates' understanding of the Black-Scholes formula assumptions in a practical framework (various market facts were given in the question) and different types of volatilities. Most candidates were able to identify and describe the assumptions made in Black-Scholes formula and their limitations. In terms of volatilities, most candidates were able to identify the skewedness of the volatility and apply the skewedness function given in the questions.

Solution:

(a) Describe five differences in the actual market conditions above from the Black-Scholes assumptions.

Commentary on Question:

Candidates were expected to relate the assumptions of the Black-Scholes formula to the reality given in the question, compare and contrast 'reality' against 'theory'. Considerable portion of candidates failed to mention the facts given and simply stated the limitations, in which case could only earned partial credits.

Any five of the following:

i) **Non-constant risk-free rate:**
The Black-Scholes formula assumes a constant risk-free interest rate. However the treasury yield curve in the market is not flat cross all maturities – the interest rate increases from 1.92% to 3.24% as maturity increases. This implies that market expects the rate to change, which is different from the constant risk-free rate assumption.

6. Continued

ii) **Non-constant volatility:**
The Black-Scholes formula assumes the volatility of the stock does not change over the life of the option.
However two facts in the market data imply that there is no single constant volatility:
1) The at-the-money volatility increases from 20% to 24% as maturity of options increases;
2) There is a volatility smile. The volatility is a function of moneyness of the option: the farther the option is from at-the-money, the greater the volatility is.

iii) **Borrowing penalty:**
Black-Scholes formula assumes that market participants can borrow or lend at a single rate.
However, in the real market the borrowing rate (50bps + treasury rate) is 20 basis points higher than the lending rate (30bps + treasury rate).

iv) **Tax:**
Black-Scholes formula assumes no tax impact in the trades.
However, in the reality above, there is tax applied on the investment income.

v) **Trading cost:**
Black-Scholes formula assumes no trading cost.
However in the real market trading cost exists in the format of 1% bid-ask spread. The Black-Scholes assumption is violated.

vi) **Short-selling penalty:**
Black-Scholes formula assumes no barrier in short selling assets.
However in the reality above, investors face the barrier that does not allow them to short sell the asset if the market moves downward.

vii) **Early termination:**
Black-Scholes formula assumes no takeovers or other business interruption that can end the option’s life early.
However in the reality, the ongoing merger negotiation implies that there is possibility that the option on XYZ stocks might be terminated before expiry because all XYZ stock shares might be bought back after the merger.

(b) Calculate the volatility under the above two assumptions, respectively.

6. Continued

Commentary on Question:
Most candidates were able to apply the skewedness function given to derive the skewedness-adjusted volatility for terms 1-month, 3-month and 1-year. However considerable portion of candidates failed to identify the correct ratio k to use in the skewedness function. The strike price is determined at issue of the option; as underlying stock price evolves, the ratio k also changes by time. Those candidates who failed to note this used the original k (1.21) for the volatility calculation, in which case they earned full credits only for the calculation steps.

The strike price is determined at issue and calculated as
Underlying stock price * ratio k determined at issue
= 100 * 1.21 = 121

The current ratio k is thus calculated as
Exercise Price / Current underlying stock price
= 121 / 110 = 1.1

We should use this ratio to derive the proper volatility assumption.

i) The skewness-adjusted one-year volatility is
V(1year) = 22% + (1.1 – 1) * 13% = 22% + 1.3% = 23.3%

ii) Since term to maturity is 1 year, the volatility change within the next one year is of our concern.
The skewness-adjusted volatilities are
V(1mon) = 20% + (1.1 – 1) * 11% = 20% + 1.1% = 21.1%
V(3mon) = 21% + (1.1 – 1) * 12% = 21% + 1.2% = 22.2%
V(1year) = 23.3% as calculated in part i)

Take arithmetic average of the three volatilities: The volatility assumption to use would be (21.1% + 22.2% + 23.3%) / 3 = 22.2%

Note:
If candidate uses 1.21 as the ratio k for the volatility calculation, full credits were given in the following steps:

The skewness-adjusted one-year volatility is
V(1year) = 22% + (1.21 – 1) * 13% = 24.73%

Since term to maturity is 1 year, the volatility change within the next one year is of our concern.
The skewness-adjusted volatilities are
V(1mon) = 20% + (1.21 – 1) * 11% = 22.31%
V(3mon) = 21% + (1.21 – 1) * 12% = 23.52%

6. Continued

Take arithmetic average of the three volatilities: The volatility assumption to use would be (24.73% + 22.31% + 23.52%) / 3 = 23.52%

Note:
The question explicitly asked for arithmetic average of the three volatilities. So time-weighted-average should **not** be taken, i.e. the volatilities do not need to be weighted by the corresponding term.

(c) Determine which volatility should be used to price the option and state why.

Commentary on Question:
Candidates did not generally do well in this part. To achieve the conclusion that the 1-year implied volatility should be used, candidates need to understand the concept of the implied volatility.

The one-year volatility should be used.
The volatility given is implied volatility, not local time-dependent volatility. Implied volatility is the volatility with which Black-Scholes formula can give the option price that matches the market value of the option. So to properly price the option, we should use the implied volatility for one-year option.

7. **Learning Objectives:**

1. The candidate will understand the fundamentals of stochastic calculus as they apply to option pricing.

2. The candidate will understand how to apply the fundamental theory underlying the standard models for pricing financial derivatives. The candidate will understand the implications for option pricing when markets do not satisfy the common assumptions used in option pricing theory such as market completeness, bounded variation, perfect liquidity, etc. The Candidate will understand how to evaluate situations associated with derivatives and hedging activities.

Learning Outcomes:

(1a) Understand and apply concepts of probability and statistics important in mathematical finance.

(2d) Understand the different approaches to hedging.

Sources:

Paul Wilmott Introduces Quantitative Finance, Wilmott, Paul, 2nd Edition Ch. 2, 8

Commentary on Question:

This question focuses on the Theta of Black Scholes options, but more in general aims to test the candidates' understanding of the fundamentals of derivatives properties (i.e. relationship between call and put and the Black Scholes PDE), as well as their skills in performing standard differentiation. In general, the question is well done. However, some candidates copied the solutions as-is from the formula sheet and received no points because the question specifically asks for a proof ("Show that…") and copying formulas is clearly not the purpose of this question.

Solution:

(a) Show that, at time t, the theta θ_c of a call option is:

$$\theta_c = -S_t \frac{\varnothing(d_1)\sigma}{2\sqrt{T-t}} - rKe^{-r(T-t)}N(d_2)$$

Commentary on Question:

In general, this part was well done. Some candidates copied the formula for Theta call directly from the formula sheet. This would have earned them no marks.

7. **Continued**

Solution 1)

$$Price\ of\ a\ call\ option\ C_t = S_t N(d_1) - Ke^{-r(T-t)}N(d_2)$$

$$\theta_c = \frac{\partial C_t}{\partial t} = S_t \emptyset(d_1)\left(\frac{\partial d_1}{\partial t}\right) - Ke^{-r(T-t)}\emptyset(d_2)\left(\frac{\partial d_2}{\partial t}\right) - rKe^{-r(T-t)}N(d_2)$$

$$d_1 = \frac{ln\left(\frac{S_t}{K}\right) + \left(r + \frac{\sigma^2}{2}\right)(T-t)}{\sigma\sqrt{T-t}}$$

$$d_2 = \frac{ln\left(\frac{S_t}{K}\right) + \left(r - \frac{\sigma^2}{2}\right)(T-t)}{\sigma\sqrt{T-t}}$$

$$\left(\frac{\partial d_1}{\partial t}\right) = \frac{ln\left(\frac{S}{K}\right)}{2\sigma(T-t)^{\frac{3}{2}}} - \frac{r + \frac{\sigma^2}{2}}{2\sigma(T-t)^{\frac{1}{2}}}, \qquad \left(\frac{\partial d_2}{\partial t}\right) = \frac{ln\left(\frac{S}{K}\right)}{2\sigma(T-t)^{\frac{3}{2}}} - \frac{r - \frac{\sigma^2}{2}}{2\sigma(T-t)^{\frac{1}{2}}}$$

$$Ke^{-r(T-t)}\emptyset(d_2) = S_t\emptyset(d_1)$$

Substituting these 3 expressions into the theta equation will give

$$\theta_c = -S_t\frac{\emptyset(d_1)\sigma}{2\sqrt{T-t}} - rKe^{-r(T-t)}N(d_2)$$

Solution 2)

Alternativly, you could notice that

$$d_2 = d_1 - \sigma\sqrt{(T-t)}$$

And thus,

$$\frac{\partial d_2}{\partial t} = \frac{\partial d_1}{\partial t} - \frac{1}{2}\frac{\sigma}{\sqrt{(T-t)}}$$

So that,

$$\begin{aligned}\theta_c = \frac{\partial C_t}{\partial t} &= S_t\emptyset(d_1)\left(\frac{\partial d_1}{\partial t}\right) - Ke^{-r(T-t)}\emptyset(d_2)\left(\frac{\partial d_2}{\partial t}\right) - rKe^{-r(T-t)}N(d_2)\\ = \frac{\partial C_t}{\partial t} &= S_t\emptyset(d_1)\left(\frac{\partial d_1}{\partial t}\right) - Ke^{-r(T-t)}\emptyset(d_2)\left(\frac{\partial d_1}{\partial t} - \frac{1}{2}\frac{\sigma}{\sqrt{(T-t)}}\right)\\ &- rKe^{-r(T-t)}N(d_2)\\ &= -S_t\frac{\emptyset(d_1)\sigma}{2\sqrt{T-t}} - rKe^{-r(T-t)}N(d_2)\end{aligned}$$

7. Continued

Solution 3)

Or you could have started with the Black Scholes PDE:

$$\frac{\partial C}{\partial t}+\frac{1}{2}\sigma^2 S^2\frac{\partial^2 C}{\partial S^2}+rS\frac{\partial C}{\partial S}-rC=0$$

Notice that $\frac{\partial C}{\partial t}$ is the theta of the option, we can rewrite the equation as:

$$\frac{\partial \mathrm{C}}{\partial \mathrm{t}}=\mathrm{r}\left[\mathrm{S_t N(d_1)}-\mathrm{Ke^{-r(T-t)}N(d_2)}\right]-\mathrm{rSN(d_1)}-\frac{1}{2}\sigma^2\mathrm{S}^2\frac{\partial \mathrm{N(d_1)}}{\partial \mathrm{S}}$$

$$=-r\,Ke^{-r(T-t)}N(d_2)-\frac{1}{2}\sigma S\frac{\phi(d_1)}{\sqrt{T-t}}$$

(b) Derive an expression for the theta θ_p of a put option.

Commentary on Question:

The easiest way to solve for this part is to use the Put-Call parity as shown below. Some candidates took the same approach as what they used to solve for part a). This is also acceptable but not recommended because this makes their part a) and b) scores highly correlated. Using the put-call parity would not have relied on the work in part a) (θ_c was given in part a).

$$Put\ call\ parity: P_t=C_t+\ K\ e^{-r(T-t)}-S_t$$

Differentiate the equation with respect to t

$$\frac{\partial P_t}{\partial t}=\frac{\partial C_t}{\partial t}+\ rK\ e^{-r(T-t)}$$

$$\frac{\partial \mathrm{P_t}}{\partial \mathrm{t}}=-\mathrm{S_t}\frac{\emptyset(\mathrm{d_1})\sigma}{2\sqrt{\mathrm{T-t}}}-\ \mathrm{rKe^{-r(T-t)}N(d_2)}$$
$$+\,rK\ e^{-r(T-t)}$$

$$\theta_\mathrm{p}=\frac{\partial \mathrm{P_t}}{\partial \mathrm{t}}=-\mathrm{S_t}\frac{\emptyset(\mathrm{d_1})\sigma}{2\sqrt{\mathrm{T-t}}}+\ \mathrm{rKe^{-r(T-t)}(1-N(d_2))}$$

$$\theta_\mathrm{p}=\frac{\partial \mathrm{P_t}}{\partial \mathrm{t}}=-\mathrm{S_t}\frac{\emptyset(\mathrm{d_1})\sigma}{2\sqrt{\mathrm{T-t}}}+\ \mathrm{rKe^{-r(T-t)}(N(-d_2))}$$

(c) Calculate $\lim_{S_t\to 0}\theta_c$ and $\lim_{S_t\to 0}\theta_p$

Commentary on Question:

In general, this part was well done. Some algebraic mistakes were carried over from previous parts but would not be penalized. Some candidates forgot that there is an S term in N(d₂) as well and thus failed to complete the limits.

Call, out of the money limiting value

$\mathrm{S}\to 0 \Rightarrow\ \theta_c\to\ 0$

Put, in the money limiting value

$\mathrm{S}\to 0 \Rightarrow\ \theta_p\to\ rKe^{-r(T-t)}$

8. **Learning Objectives:**

1. The candidate will understand the fundamentals of stochastic calculus as they apply to option pricing.

2. The candidate will understand how to apply the fundamental theory underlying the standard models for pricing financial derivatives. The candidate will understand the implications for option pricing when markets do not satisfy the common assumptions used in option pricing theory such as market completeness, bounded variation, perfect liquidity, etc. The Candidate will understand how to evaluate situations associated with derivatives and hedging activities.

3. The candidate will understand the quantitative tools and techniques for modeling the term structure of interest rates and pricing interest rate derivatives.

Learning Outcomes:

(1b) Understand the importance of the no-arbitrage condition in asset pricing.

(1c) Understand Ito integral and stochastic differential equations.

(1d) Understand and apply Ito's Lemma.

(1f) Demonstrate understanding of option pricing techniques and theory for equity and interest rate derivatives.

(2b) Compare and contrast the various kinds of volatility, (eg actual, realized, implied, forward, etc.).

(3a) Understand and apply the concepts of risk-neutral measure, forward measure, normalization, and the market price of risk, in the pricing of interest rate derivatives.

(3c) Understand and apply popular one-factor interest rate models including Vasicek, Cox-Ross_Ingersoll, Hull-White, Ho-Lee, Black-Derman_Toy, Black-Karasinski.

(3e) Understand and differentiate between the classical approach to interest rate modelling and the HJM modeling approach, including the basic philosophy, arbitrage conditions, assumptions, and practical implementations.

Sources:
Neftci Ch 19 page 426-447

Commentary on Question:
Commentary listed underneath question component.

8. **Continued**

Solution:

(a) Derive the formula for $F(t,T)$

The arbitrage bond price using forward rate is shown

$$P(t,T) = exp\left(-\int_t^T F(t,s)ds\right)$$
$$exp\left(-\int_t^T F(t,s)ds\right) = exp\big(A(t,T) - B(t,T) * r(t)\big),$$
$$-\int_t^T F(t,s)ds = A(t,T) - B(t,T) * r(t),$$

Differentiating with respect to T,

$$F(t,T) = -\frac{\partial A(t,T)}{\partial T} + \frac{\partial B(t,T)}{\partial T} * r(t),$$
$$\frac{\partial A(t,T)}{\partial T} = -a(T-t) + \frac{\sigma^2}{2}(T-t)^2$$
$$\frac{\partial B(t,T)}{\partial T} = 1$$
$$F(t,T) = a(T-t) - \frac{\sigma^2}{2}(T-t)^2 + r(t)$$

(b) Show that $dF(t,T) = \sigma^2(T-t)dt + \sigma dW$

Commentary on Question:
Many candidates didn't apply Ito's lemma in the proof but used total differentiation or total derivative. This is acceptable if they had derived the linear relationship between the forward rate $F(t,T)$ *and short rate* $r(t)$ *as shown in part (a).*

Applying Ito's lemma for forward rate,

$$dF(t,T) = \frac{\partial F(t,T)}{\partial t}dt + \frac{\partial F(t,T)}{\partial r}dr + \frac{1}{2}\frac{\partial^2 F(t,T)}{\partial r^2}(dr)^2$$

From (a),

$$\frac{\partial F(t,T)}{\partial t} = -a + \sigma^2(T-t),$$
$$\frac{\partial F(t,T)}{\partial r} = 1,$$

8. Continued

$$dF(t,T) = [-a + \sigma^2(T-t)]dt + adt + \sigma dW = \sigma^2(T-t)dt + \sigma dW$$

Forward rate volatility is found

$$\sigma_F = \sigma$$

(c) Show that $\frac{dP}{P} = r\,dt - \sigma(T-t)dW$

Let $P = e^y$ and from the Ito's lemma,

$$dP = \frac{\partial P}{\partial y}dy + \frac{1}{2}\frac{\partial^2 P}{\partial y^2}(dy)^2 = Pdy + \frac{1}{2}P(dy)^2,$$

$$\frac{dP}{P} = dy + \frac{1}{2}(dy)^2,$$

Applying Ito's lemma to $y(t,T) = ln[P(t,T)]$,

$$y(t,T) = A(t,T) - B(t,T) * r(t)$$

$$dy(t,T) = \frac{\partial y(t,T)}{\partial t}dt + \frac{\partial y(t,T)}{\partial r}dr + \frac{1}{2}\frac{\partial^2 y(t,T)}{\partial r^2}(dr)^2,$$

$$\frac{\partial y(t,T)}{\partial t} = \frac{\partial A(t,T)}{\partial t} - \frac{\partial B(t,T)}{\partial t}r(t) = a(T-t) - \frac{\sigma^2}{2}(T-t)^2 + r(t),$$

$$\frac{\partial y(t,T)}{\partial r} = -B(t,T), \quad \frac{\partial^2 y(t,T)}{\partial r^2} = 0,$$

so

$$dy(t,T) = \frac{\partial y(t,T)}{\partial t}dt - B(t,T)dr.$$

And

$$(dy)^2 = [B(t,T)dr]^2 = (T-t)^2\sigma^2 dt,$$

Therefore

$$\begin{aligned}\frac{dP}{P} = \frac{\partial y(t,T)}{\partial t}dt - B(t,T)dr + \frac{1}{2}(T-t)^2 dt \\ &= \left[a(T-t) - \frac{\sigma^2}{2}(T-t)^2 + r(t)\right]dt \\ &+ \left[-a(T-t) + \frac{1}{2}(T-t)^2\sigma^2\right]dt - \sigma(T-t)dW \\ &= r(t)dt - \sigma(T-t)dW\end{aligned}$$

Hence the volatility of bond price return is $\sigma\big(t,T,B(t,T)\big) = \sigma(T-t)$

8. Continued

Alternatively, Ito's lemma can be directly applied to the bond price such that

$$dP = \frac{\partial P}{\partial t}dt + \frac{\partial P}{\partial r}dr + \frac{1}{2}\frac{\partial^2 P}{\partial r^2}(dr)^2, \qquad (1)$$

$$\frac{\partial P}{\partial t} = P\left(\frac{\partial A(t,T)}{\partial t} - \frac{\partial B(t,T)}{\partial t}r(t)\right)$$

$$= P\left(a(T-t) - \frac{\sigma^2}{2}(T-t)^2 + r(t)\right).$$

$$\frac{\partial P(t,T)}{\partial r} = -B(t,T)P = -(T-t)P,$$

And

$$\frac{\partial^2 P(t,T)}{\partial r^2} = [B(t,T)]^2 P = (T-t)^2 P, \qquad (dr)^2 = \sigma^2 dt,$$

From (1) and the short rate process

$$\frac{dP}{P} = \left(a(T-t) - \frac{\sigma^2}{2}(T-t)^2 + r(t)\right)dt - (T-t)dr + \frac{\sigma^2}{2}(T-t)^2 dt$$

$$= a(T-t)dt + r(t)dt - a(T-t)dt - \sigma(T-t)dW$$

$$= r(t)dt - \sigma(T-t)dW$$

(d) Prove $F(t,T)$ satisfies the condition relating diffusion to drift in the HJM forward rate model.

Commentary on Question:
The question tests candidates' ability to relate HJM and arbitrage free bond price through the sequence of sub questions that include the bond price volatility for the HJM framework.

8. Continued

From the (b), we obtained

$$dF(t,T) = [\sigma^2(T-t)]dt + \sigma dW$$

Also from the HJM model in Neftci,

$$dF(t,T) = \sigma\big(t,T,B(t,T)\big)\left[\frac{\partial\sigma\big(t,T,B(t,T)\big)}{\partial T}\right]dt + \left[\frac{\partial\sigma\big(t,T,B(t,T)\big)}{\partial T}\right]dW.$$

The volatility and drift should be calculated from (c)

$$\sigma\big(t,T,B(t,T)\big) = \sigma(T-t),$$

$$\sigma_F = \frac{\partial\sigma\big(t,T,B(t,T)\big)}{\partial T} = \frac{\partial\sigma(T-t)}{\partial T} = \sigma,$$

$$\begin{aligned}\sigma\big(t,T,B(t,T)\big)\left[\frac{\partial\sigma\big(t,T,B(t,T)\big)}{\partial T}\right] &= \sigma\big(t,T,B(t,T)\big)\sigma_F \\ &= \sigma(T-t)\sigma \\ &= \sigma^2(T-t),\end{aligned}$$

which is proved.

9. **Learning Objectives:**

1. The candidate will understand the fundamentals of stochastic calculus as they apply to option pricing.

3. The candidate will understand the quantitative tools and techniques for modeling the term structure of interest rates and pricing interest rate derivatives.

Learning Outcomes:

(1a) Understand and apply concepts of probability and statistics important in mathematical finance.

(3b) Apply the models to price common interest sensitive instruments including: callable bonds, bond options, caps, floors, swaptions, caption, floortions.

Sources:

Nefci Ch.16

Commentary on Question:

Commentary listed underneath question component.

Solution:

(a) Show that for $t < T$,

$$r_T = \left(r_t - \frac{\eta}{\gamma}\right)e^{-\gamma(T-t)} + \frac{\eta}{\gamma} + \beta^{\frac{1}{2}}e^{-\gamma T}\int_t^T e^{\gamma s}dX_s$$

Commentary on Question:

Candidates did fairly well in this question. Candidates who recognized $\frac{\partial}{\partial t}e^{\gamma t}r_t$ *from the partial differentiation were usually able to get full (or near full) credit. Since this is a proof question, candidates need to show clearly how they go from one step to the next in order to receive the full score, which include rearrangement of the SDE, integral from t to T and multiplication of* $e^{-\gamma T}$.

9. Continued

Rearrange the SDE $dr_t = (\eta - \gamma\, r_t)dt + \beta^{\frac{1}{2}} dX_t$ to
$dr_t + \gamma r_t dt = \eta dt + \beta^{\frac{1}{2}} dX_t$.

Multiply both sides by $e^{\gamma t}$:
$e^{\gamma t}(dr_t + \gamma r_t dt) = \eta e^{\gamma t} dt + \beta^{\frac{1}{2}} e^{\gamma t} dX_t$
The left-hand side equals $\frac{\partial}{\partial t}(e^{\gamma t} r_t)$ from partial differentiation.

Then taking the integral from $t\ to\ T$:

$$\int_t^T \frac{\partial}{\partial s}[e^{\gamma s} r_s] ds = \eta \int_t^T e^{\gamma s}\, ds + \beta^{\frac{1}{2}} \int_t^T e^{\gamma s} dX_s$$

Then this becomes:

$$e^{\gamma s} r_s|_{s=t}^{s=T} = \frac{\eta}{\gamma} e^{\gamma s}|_{s=t}^{s=T} + \beta^{\frac{1}{2}} \int_t^T e^{\gamma s} dX_s$$

$$e^{\gamma T} r_T - e^{\gamma t} r_t = \frac{\eta}{\gamma} e^{\gamma T} - \frac{\eta}{\gamma} e^{\gamma t} + \beta^{\frac{1}{2}} \int_t^T e^{\gamma s} dX_s$$

Multiply both sides by $e^{-\gamma T}$

$$r_T - e^{-\gamma(T-t)} r_t = \frac{\eta}{\gamma} - \frac{\eta}{\gamma} e^{-\gamma(T-t)} + \beta^{\frac{1}{2}} e^{-\gamma T} \int_t^T e^{\gamma s} dX_s$$

$$r_T = e^{-\gamma(T-t)} r_t + \frac{\eta}{\gamma} - \frac{\eta}{\gamma} e^{-\gamma(T-t)} + \beta^{\frac{1}{2}} e^{-\gamma T} \int_t^T e^{\gamma s} dX_s$$

$$r_T = \left(r_t - \frac{\eta}{\gamma}\right) e^{-\gamma(T-t)} + \frac{\eta}{\gamma} + \beta^{\frac{1}{2}} e^{-\gamma T} \int_t^T e^{\gamma s} dX_s.$$

Proved.

(b) Calculate the payoff of the forward contract at its maturity, assuming $\gamma = 0.1, \eta = 0.01, \beta = 0.004,$ and the prevailing short rate 10 years from now is 2%.

Commentary on Question:
Candidates did poorly on this part of the question. Most candidates failed to identify the right formula to use. However, those candidates who were able to do so had no difficulty in arriving at the correct answer.

9. Continued

The bond price after 10 years:

$$B = \frac{1}{\gamma}\left(1 - e^{-\gamma(T-t)}\right) = \frac{1}{0.1}(1 - e^{-0.1*10}) = 6.32121$$

$$A = \frac{1}{\gamma^2}(B(t,T) - T + t)(\eta\gamma - 0.5\beta) - \frac{\beta B(t;T)^2}{4\gamma}$$

$$= \frac{1}{0.1^2}(6.32121 - 20 + 10)(0.01 * 0.1 - 0.5 * 0.004) - \frac{0.004 * 6.32121^2}{4 * 0.1}$$
$$= -0.0317$$

With initial short rate $r = 2\%$, Zero-coupon bond price at the end of 10 years is:

$$100e^{A(t,T)-rB(t,T)} = 100e^{-0.0317-6.32121*2\%} = 85.3746$$

With the forward contract, the bond can be delivered at price of 58. Hence payoff of the forward contract is therefore $85.3746 - 58 = 27.3746$

10. Learning Objectives:

7. The candidate will understand how to develop an investment policy including governance for institutional investors and financial intermediaries.

Learning Outcomes:

(7a) Explain how investment policies and strategies can manage risk and create value.

(7b) Identify a fiduciary's obligations and explain how they apply in managing portfolios.

(7c) Determine how a client's objectives, needs and constraints affect investment strategy and portfolio construction. Include capital, funding objectives, risk appetite and risk-return trade-off, tax, accounting considerations and constraints such as regulators, rating agencies, and liquidity.

(7d) Incorporate financial and non-financial risks into an investment policy, including currency, credit, spread, liquidity, interest rate, equity, insurance product, operational, legal and political risks.

Sources:

Maginn and Tuttle Chapter 3

Commentary on Question:

Candidates were expected to know the main IP differences between DB plans and endowments – mostly recall items from the study note. Knowing these main differences would have received most of credits given. Part (e) tested how well candidates could apply those recall items in a case study situation; it had no single right answer and credits were given to a logical answer with reasons.

Solution:

(a) Compare and contrast your fiduciary responsibilities for each fund.

Commentary on Question:

The key point of this question was to determine if the candidates knew the fiduciary responsibility for DB plans is greater than that of an endowment. Many candidates did not describe this key difference, so most received partial credit.

DB: a pension plan trustee is a fiduciary, legally responsible to ensure assets are managed solely for interests of beneficiaries. Subject to legal standards, and can be sued. Governed by ERISA
Endowment: governing board must exercise ordinary business care and prudence. Follows UMIFA legislation. Less accountability than pension plans

(b) Compare considerations in setting return objectives for DB plan assets and for endowments.

10. Continued

Commentary on Question:
Most candidates were able to explain the return objectives, although few wrote anything about education inflation.

DB: minimize amount of future pension contributions, maintain or increase pension income impact of bottom line

Endowment: balance stable flow of income vs. maintaining principal, must keep up with education inflation, which has been higher than CPI

(c) Compare considerations in setting risk objectives for DB plan assets and for endowments.

Commentary on Question:
Most candidates were able to point out the key differences.

DB: must consider plan status, sponsor financial status, common risk exposures, plan features (early retirement and lump sum distributions), workforce characteristics, active vs. retired lives

Endowment: spending policy, and long term objective of providing significant, stable, and sustainable stream of spending distributions

(d) Identify the following investment policy constraints for each fund:

I. Liquidity requirements

II. Time horizon

III. Tax concerns

IV. Legal and regulatory requirements

V. Unique circumstances

Commentary on Question:
Liquidity – points given for explaining that endowments typically have lower liquidity needs. Many candidates missed this point
Time Horizon – most candidates had this right.
Tax Concerns – many candidates did not know this
Legal and Regulatory – many candidates missed that DB Plans are highly regulated vs. endowments. Of those that noted the difference, many did not mention ERISA.

10. Continued

Unique – There were no unique circumstances mentioned in the question lead-in, but candidates were expected to list possible items common to any DB plan or endowment. Few did.

Liquidity
DB: greater number of retired lives, smaller corporate contributions relative to benefit disbursements, early retirement and lump sum payout options all mean higher liquidity is needed
Endowment: limited need for liquidity, except to meet spending targets. Well suited to illiquid, non-marketable securities

Time Horizon
DB: depends whether going concern or plan termination is expected, age of the workforce and proportion of active lives
Endowment: extremely long term

Tax Concerns
DB: usually exempt from taxation
Endowments: not a major concern for endowments in general

Legal and Regulatory
DB: All retirement plans governed by laws and regulations. In US, ERISA. If unions, Taft-Hartley also
Endowments: Few laws and regulations in the US. Most states have adopted UMIFA

Unique Circumstances
DB: human and financial resources for smaller plans, complex due diligence, self-imposed constraints against investing in certain industries
Endowments: could unsophisticated board, very small staff, if large enough, can be Qualified Purchasers for privately placed assets

(e) Assess whether this is an acceptable strategy.

Commentary on Question:
Minimal credit was given for just stating that this is unacceptable. Few candidates gave enough reasons to receive full credit on this part.

No, this is not an acceptable strategy. It does not address the underlying problem of why the DB is underfunded. It violates the purpose of the endowment. Giving less money to underprivileged students. If this were made public, you could expect a lawsuit from the donors, and lots of bad PR for the University.

11. Learning Objectives:

5. The candidate will understand and identify the variety of fixed instruments available for portfolio management. This section deals with fixed income securities. As the name implies the cash flow is often predictable, however there are various risks that affect cash flows of these instruments. In general the candidates should be able to identify the cash flow pattern and the factors affecting cash flow for commonly available fixed income securities. Candidates should also be comfortable using various interest rate risk quantification measures in the valuation and managing of investment portfolios. Candidates should also understand various strategies of managing the portfolio against given benchmark.

Learning Outcomes:

(5g) Demonstrate understanding of cash flow pattern and underlying drivers and risks of mortgage-backed securities and collateralized mortgage obligations.

Sources:

The Handbook of Fixed Income Securities, Fabozzi, Frank, 8th Edition, Ch. 31-32

Commentary on Question:

Most candidates did well on the retrieval parts of the syllabus reading, i.e. parts (a) and (b). But when it came to the analytical part of the question, i.e. part (c), few earned full credit.

Partial credit was given on all three parts when candidates provided relevant explanations.

Solution:

(a) Compare the non-agency RMBS portfolio and the CMBS portfolio on the relative basis in terms of the following risks.

(i) Concentration risk

(ii) Interest rate risk

(iii) Prepayment risk

Commentary on Question:

Candidates who earned full credit on this part were those who systematically compared each risk in both the non-agency RMBS and the CMBS.

11. Continued

(i) Concentration risk: the CMBS portfolio has higher concentration risk than the RMBS portfolio:

CMBS: are larger-sized loans with fewer loans in the portfolio. There is higher concentration risk with top 15 loans comprising 30-50% of the portfolio. The loans that serve as CMBS collateral commonly are secured by commercial real estate such as apartment building, shopping malls, warehouse facilities, etc.

RMBS: portfolio has a large number of loans. Loans are small in size and individual loans represent a small portion of the portfolio. Given large number of loans, collateral is fairly well-diversified and concentration risk is lower than CMBS portfolio.

(ii) Interest rate risk:

CMBS: portfolio typically has shorter-term loans, with 5-, 7-, or 10-year balloon maturities. Most loans are fixed-rate. Duration is shorter and less sensitive to rate changes on the long-end of yield curve but more sensitive to rate changes on the short-term and medium-term.

RMBS: typically has 15- and 30-year fixed rate and floating rate loans. Duration is longer. The RMBS portfolio is more sensitive to rate changes on the long-end of the yield curve.

(iii) Prepayment risk:

CMBS: Commercial mortgage loans typically prohibit prepayment until a few months prior to the maturity date, so the prepayment risk is lower than CMBS portfolio.

RMBS: many residential mortgage loans are freely pre-payable, so the RMBS portfolio has much higher prepayment risk.

(b) Describe how the following two measures can be used in assessing risk of CMBS.

(i) Debt Service Coverage Ratio (DSCR)

(ii) Loan-to-Value (LTV)

11. Continued

Commentary on Question:
Candidates who earned full credit on this part not only correctly defined DSCR and LTV, but also described how the two measures are used in assessing risk of CMBS.

Unlike most residential mortgage loans, commercial mortgage loans often do not provide recourse, so lenders and investors look to the collateral, not the borrower, for ultimate repayment in case of default. Credit analysis is performed on a loan-by-loan basis with the following two main, relevant risk measures:

(i) Debt service coverage ratio (DSCR) is net operating income divided by debt service. Loans with DSCR above 1.0 has a lower likelihood of default because they have a built-in excess cash flow buffer available which would have to erode before the borrower would experience losses and consider defaulting.

(ii) Loan-to-value (LTV) is the ratio of loan amount to the value of the collateral property. A lower LTV loan is considered more credit worth due to its better default protection.

(c) Describe a deterministic, rule-based default modeling framework for the CMBS portfolio, projecting DSCR and LTV to determine one of the three possible outcomes of the loan:

- Term default
- Timely pay-off
- Maturity default/loan extension

Since loan-level commercial mortgage performance datasets are not readily available, it is typical to use a deterministic, rule-based modeling approach to determine the projected property performance and valuation. Two risk measures, DSCR and LTV, are projected and deterministically calculated, based on other variable and economic terms such as net operating income (NOI) and cap rate.

Each mortgage loan's projected DSCR and LTV are compared through time to a set of trigger levels to discretely determine one of the three possible loan outcomes: (1) term default, (2) timely pay-off, or (3) maturity default/loan extension.

11. Continued

1) Term default: during the term of the mortgage loan, periodic tests are run to determine if the loan's DSCR drops below a pre-fixed DSCR default trigger. The loan is immediately defaulted and enters a workout period if the DSCR trigger fails. At the end of the workout period, the property's terminal value is determined using NOI and cap rate, less a workout or disposition fee. If the final property value is less than the outstanding loan amount, the deficit is recorded as a loss. If the property's NOI never drops below the DSCR trigger, no default is assumed during the loan's term.

2) Timely pay-off: At the loan's maturity date, property value is once again determined by applying a cap rate to projected NOI. The loan is assumed to pay-off on time if the property value is sufficiently above the outstanding loan amount (e.g., LTV<80%)

3) Maturity defaults/loan extension: If the loan does not pay-off as scheduled at maturity, it either enters into a one-time term extension or defaults. Deterministic models assumes that the loan is extended as long as it is not underwater (LTV<100%). Under-collateralized loans (LTV>100%) are assumed to default at maturity. In case of maturity default, the property value is calculated at the end of the workout period to determine the loss amount. If the loan is extended, the pay-off versus default calculation is repeated at the end of the extension period.

12. **Learning Objectives:**

5. The candidate will understand and identify the variety of fixed instruments available for portfolio management. This section deals with fixed income securities. As the name implies the cash flow is often predictable, however there are various risks that affect cash flows of these instruments. In general the candidates should be able to identify the cash flow pattern and the factors affecting cash flow for commonly available fixed income securities. Candidates should also be comfortable using various interest rate risk quantification measures in the valuation and managing of investment portfolios. Candidates should also understand various strategies of managing the portfolio against given benchmark.

Learning Outcomes:

(5i) Construct and manage portfolios of fixed income securities using the following broad categories.

(i) Managing funds against a target return

(ii) Managing funds against liabilities.

Sources:

Managing investment portfolios: A dynamic process, Maginn & Turtle chapter 6

Commentary on Question:

It was an easy question, but many candidates did poorly on part (b).

Solution:

(a) Calculate the tracking risk of this portfolio based on the past 4 years of returns.

<u>We have first to calculate:</u>
Active return = Portfolio's return – Benchmark returns
Average active return per period = Summation of active returns/4
Tracking risk = Standard deviation of the active returns
It will give the following calculation matrix:

Year	Portfolio Return	Benchmark Return	Active return (AR)	(AR-Avg. AR)^2
Y-4	8.5%	8%	0.5%	0.0019141%
Y-3	5%	5.10%	-0.10%	0.0002641%
Y-2	-4%	-3.80%	-0.20%	0.0006891%
Y-1	3.50%	3.45%	0.05%	0.0000016%
Total			0.25%	0.0028688%
Avg. AR			0.0625%	

12. Continued

Tracking risk is: Square root [(Sum of the squared deviations)/3]
Square root (0.0028688%/3) = 3.092% = near 309 bps = tracking risk
[Using n=4 the tracking risk is = 2.68%]

The calculation could also have been done using:
Variance = E [E(x^2) – [E(x)] ^2]

(b) Describe the limitations of using variance to measure the sensitivity of a portfolio with many bonds.

Commentary on Question:
Part (b) was a recall question but many candidates did poorly.

Limitations:
The number of variances and covariance factors to be estimated increases dramatically with the number of bonds;

Accurately estimating the variances and covariance is difficult;
The characteristics may change as time change;

Embedded options change the characteristics of bonds over time;

The normal distribution assumption may not be descriptive of the distribution.

(c) Describe the following three measures that can be used in risk measurement and the limitations that each measure may have in risk quantification:

- Semi Variance
- Shortfall Risk
- Value at Risk (VaR)

Commentary on Question:
Surprisingly many candidates received no credit in describing semi-variance even though reasonable variations were accepted. It appears that the semi-variance concept is less well known to the candidates generally than the other measures (shortfall risk and VaR).

12. Continued

Measures:

Semi- variance: Dispersion of the returns that are below the target return;
Limitation: It contains no additional information if investment returns are symmetric;
Otherwise, it is very difficult to forecast.

Shortfall risk: Probability of not achieving the return target;
Limitation: Does not account for the magnitude of losses in money terms.

Value at risk (VaR): Estimate of the loss (in money terms) expected to be exceeded at a given level of probability over a specific time period;
Limitation: VaR does not indicate the magnitude of the very worst possible outcomes.

13. **Learning Objectives:**

5. The candidate will understand and identify the variety of fixed instruments available for portfolio management. This section deals with fixed income securities. As the name implies the cash flow is often predictable, however there are various risks that affect cash flows of these instruments. In general the candidates should be able to identify the cash flow pattern and the factors affecting cash flow for commonly available fixed income securities. Candidates should also be comfortable using various interest rate risk quantification measures in the valuation and managing of investment portfolios. Candidates should also understand various strategies of managing the portfolio against given benchmark.

Learning Outcomes:

(5c) Demonstrate understanding of the different characteristics of securities issued by government agencies or government-sponsored enterprise.

(5e) Describe the cash flow of various corporate bonds considering underlying risks such as interest rate, credit and event risk.

(5f) Demonstrate an understanding of the characteristics of leveraged loans.

(5i) Construct and manage portfolios of fixed income securities using the following broad categories.
- (i) Managing funds against a target return
- (ii) Managing funds against liabilities.

Sources:

The Handbook of Fixed Income Securities, Fabozzi, Frank, 8th Edition
Ch. 1-2,9 (background only), 10-13, 17-18, 21, 24-26, 31-32

Managing Investment Portfolios: A Dynamic Process, Maginn & Tuttle, 3rd Edition
Ch. 6, Fixed Income Portfolio Management

Commentary on Question:

Candidates generally did well on this question except for part (c). Comments will be made under the appropriate section.

Solution:

(a) Assess whether a non-callable bond will outperform or underperform a callable bond during the following rate environment.

- Rising interest rate environment
- Falling interest rate environment

13. **Continued**

Rising interest rate environment

- The non-callable bond will underperform;
- The probability of callable bond being called falls;
- The investor benefits from the higher yield/coupon.

Falling interest rate environment

- The non-callable bond will outperform;
- The probability of callable bond being called increases;
- The borrower will have to invest money at lower interest rate (reinvestment risk);
- Prices of callable bonds do not rise as much when rates fall due to negative convexity.

(b) Explain the impact of each of the following provisions to the risk of a callable bond.

- Deferment period
- Nonrefundable
- Make-whole call provision

Commentary on Question:
Most candidates couldn't address well of non-refundable provision.

Deferment period

Definition

- Limitation on the borrower's right to call (period of call protection);
- Specified number of years during which the issuer may not call the debt.

Impact on the risk

- It offers protection against reinvestment risk during that period (the bond cannot be called).

Non-refundable

Definition

- May be called during that period if the fund used to retire the bond issue comes from internally generated funds (cash-flow from operations, sale of property of equipment, or from no debt funding such as common stock).

Impact on the risk

- It offers some protection against reinvestment risk but less than a non-callable bond.

13. Continued

Make-whole call provision

Definition

- Make-whole call price is the sum of the present values of the remaining coupon payments and principal discounted at a yield on a bond that matches the bond's remaining maturity plus a spread;
- The redemption price is the greatest of (1) 100% of the principal amount plus accrued interest, or (2) the make-whole redemption amount plus accrued interest.

Impact on the risk

- Make-whole price varies inversely with the level of interest rates;
- The price increases as the interest rates decrease so if the issuer then exercises the call provision, the bond holder receives a higher call price;
- Thus it provides investors with some protection against reinvestment risk.

(c) Explain how the prices of floating rate bonds move over time before and at the reset dates.

Commentary on Question:
Candidates didn't do well with this part.

At initial offering and at reset dates the price is at or near par because the issuer will set the quoted margin based on market conditions.
Subsequently, if the market requires a higher/lower margin, the price will decrease/increase the current margin required.
As to the movements in the interest rates, the floater moves in the same direction so that its price shows very low sensitivity to changes in market rates.

(d) Calculate the effective duration of the portfolio described above.

Commentary on Question:
Most candidates did well of this part except for determining effective duration of a Floating Rate Bond.

Effective duration of floating rate bond (semi-annual) = 0.25/2 = 0.125 (0.25 can be used as well).
Calculation of market values or use of the correct market values is as follows:
UST: 20*120/100 = $24mm
IG bonds: 30*110/100 = $33mm
Floating rate: $60mm
Cash: $10mm

[(24*7) + (33*5) + (60*0.125) + (10*0)]/ (24+33+60+10) =2.68
[Or 2.74 using 0.25 effective duration of floating rate]

13. Continued

(e) Demonstrate how the manager can achieve this goal using Treasury futures. Show your calculations.

Calculate target duration

$2mm = (30*110/100)*(1.50%)*X
X = 4.04 = target duration

Recognize needing to lower duration, therefore need to sell Treasury futures:

The prices of an interest rate futures contract are negatively correlated with the change in the interest rate;
Buying a futures contract will increase the sensitivity to interest rates and the portfolio duration will increase;
Selling a futures contract will lower a portfolio's sensitivity to interest rates and the duration will decrease;
An interest rate strategy commonly involves reducing the portfolio duration when the expectation is that interest rates will rise.

Number of contracts

$(D_T - D_I)*P_I/ (D_{CTD}P_{CTD})$ * CTD conversion factor
(4.04 – 5)*(30*110/100)/ (8.5*0.1)*1.3 = - 48 contracts (selling 48 contracts)

(f) Describe how the manager can also achieve this goal using swaps.

An interest rate swap is a contract between two parties to exchange periodic interest payments between two parties based on a specified dollar amount of principal (notional principal amount).

To manage the risk of an increase in the interest rates, one should enter into a 'pay fixed, receive floating' swap.

D (Pay-Fixed) = D (Floating) - D (Fixed) is a negative number since D (Floating) is very small, so it will always reduce the duration of the non-callable bullet bond.

(g) Recommend two methods that the manager can use to hedge against interest rate increases while retaining upside potential.

Interest rate call
Interest rate cap
Bond put

14. **Learning Objectives:**

4. The candidate will understand the concept of volatility and some basic models of it.

Learning Outcomes:

(4b) Understand and apply various techniques for analyzing conditional heteroscedastic models including ARCH and GARCH.

Sources:

Tsay Chapter 3.8

Commentary on Question:

Commentary listed underneath question component.

Solution:

(a) Describe the principal disadvantage of GARCH(1,1) model.

Commentary on Question:
Candidates did generally well on this part, which is the easiest one of the question.

The GARCH(1,1) model has the following form:

$$a_t = \sigma_t \epsilon_t, \quad \sigma_t^2 = \alpha_0 + \alpha_1 a_{t-1}^2 + \sigma_{t-1}^2$$

The principle disadvanatge of GARCH model is that it does not allow asymetric effects between positive and negative gains.

(b) Describe briefly how the EGARCH(1,1) model addresses the disadvantage identified in part (a).

Commentary on Question:
Most candidates just wrote EGARCH(1,1) model allowed asymmetric effects between positive and negative gains, so only partial points were given. They should substantiate the statement by using the EGARCH model formula.

EGARCH(1,1) model as given simplifies to

$$\ln(\sigma_t^2) = \begin{cases} \alpha_0 + \alpha_1(1+\gamma_1)\epsilon_{t-1} + \beta_1 \ln(\sigma_{t-1}^2) & if\ e_{t-1} \geq 0 \\ \alpha_0 - \alpha_1(1-\gamma_1)\epsilon_{t-1} + \beta_1 \ln(\sigma_{t-1}^2) & if\ e_{t-1} < 0 \end{cases}$$

14. Continued

and

$$\sigma_t^2 = \sigma_{t-1}^{2\beta_1}\exp(\alpha_0)\begin{cases} exp\left(\alpha_1(1+\gamma)\frac{a_{t-1}}{\sigma_{t-1}}\right) & if\ a_{t-1} \geq 0 \\ exp\left(-\alpha_1(1-\gamma)\frac{a_{t-1}}{\sigma_{t-1}}\right) & if\ a_{t-1} < 0 \end{cases}$$

As it can be seen from the last equation the coefficient $(1-\gamma)$ and $(1+\gamma)$ show the asymmetry in response to positive and negative values of a_t.

(c) Compute the ratio of the conditional variance σ_t^2 given that $\varepsilon_{t-1} = -2$ to the conditional variance given that $\varepsilon_{t-1} = 2$

Commentary on Question:
Most candidates used the wrong formula and arrived different answers, thus received only partial points. But a number of them correctly applied the formula from part (b).

Candidates need to calculate the ratio of

$$\frac{\sigma_t^2(\epsilon_{t-1} = -2)}{\sigma_t^2(\epsilon_{t-1} = 2)} = \frac{\sigma_{t-1}^{2\beta_1}\exp(\alpha_0)\exp(\alpha_1(1+\gamma)\frac{a_{t-1}}{\sigma_{t-1}})}{\sigma_{t-1}^{2\beta_1}\exp(\alpha_0)\exp\left(-\alpha_1(1-\gamma)\frac{a_{t-1}}{\sigma_{t-1}}\right)}$$

$$= \frac{\sigma_{t-1}^{2\beta_1}\exp(\alpha_0)\exp(\alpha_1(1+\gamma)\frac{\sigma_{t-1}\times -2}{\sigma_{t-1}})}{\sigma_{t-1}^{2\beta_1}\exp(\alpha_0)\exp\left(-\alpha_1(1-\gamma)\frac{\sigma_{t-1}\times 2}{\sigma_{t-1}}\right)}$$

$$= \frac{\exp(\alpha_1(1-\gamma)2)}{\exp(\alpha_1(1+\gamma)2)}$$

$$= \frac{\exp(0.22(1-0.264)2)}{\exp(0.22(1+0.264)2)} = 1.2615$$

The ratio is 1.2615; so the impact is 26% more.

15. **Learning Objectives:**

6. The candidate will understand the variety of equity investments and strategies available for portfolio management.

Learning Outcomes:

(6c) explain the basic active equity selection strategies including value, growth and combination approaches.

(6d) Demonstrate an understanding of equity indices and their construction, including distinguishing among the weighting schemes and their biases.

(6g) Recommend and justify, in a risk-return framework, the optimal portfolio allocations to a group of investment managers.

(6h) Describe the core-satellite approach to portfolio construction with a completeness fund to control overall risk exposures.

(6i) Explain alpha and beta separation as an approach to active management and demonstrate the use of portable alpha.

Sources:

Managing Investment Portfolios: A Dynamic Process, Maginn & Tuttle, 3rd Edition
Ch. 7 Equity Portfolio Management

Commentary on Question:

Commentary listed underneath question component.

Solution:

(a) Assess the chief actuary's belief regarding equities being a good inflation hedge.

Commentary on Question:
Candidates were expected to provide an assessment. To earn full credit, candidates need to include both arguments for and against the chief actuary's belief.

Yes, the chief actuary is correct.
There is historical evidence that equities are good inflation hedge compared to bonds, since bonds have fixed cash flows.
However, corporate income and capital gains tax rates are not indexed to inflation, therefore inflation can reduce the stock return.
The ability of individual stock to hedge inflation depends on its industry and competitive position

15. Continued

(b) Describe the biases of the following weighting schemes for stock indices.

(i) Price-weighted index

(ii) Market capitalization-weighted index

(iii) Equal-weighted index

Commentary on Question:
Most candidates did fairly well in this part. The question asks for the biases for the weighting schemes. Credit is not rewarded for simply providing definitions of the indices.

(i) The price-weighted index is biased towards higher priced stocks.
The price of a stock is arbitrary and will change through time as companies split stocks, repurchase stocks, or distribute dividends.
The index also assumes investor purchases one share of each stock, which is rarely the case in practice.

(ii) The value-weighted index is biased towards stocks with bigger capitalization, and therefore is biased towards large, mature firms that may be overvalued.
Institutional investors who are subject to maximum holdings may not be able to mimic the index.
The index may also be less diversified if it's overrepresented by large firms.

(iii) The equal-weighted index is biased towards small cap firms.
It requires rebalancing, therefore higher transaction costs will incur.
It emphasizes on small cap stocks in which investors may not be able to find liquidity.

(c) Evaluate whether active or passive management is more suitable for the equity portfolio, based on the above information.

Commentary on Question:
Candidates generally did well in this part. Successful candidates provided arguments for both passive and active management based on the given information.

15. **Continued**

BB insurance company is a taxable company. Passive management is favored because active management requires more turnover and therefore taxes will be realized more frequently with active management.

Efficient market favors passive management since it's difficult to earn active return in an efficient market.

Preference for small cap markets favors active management since small cap markets are more likely to have mispriced stocks.

Preference for international markets favors passive management since managers may lack information that local investors have.

(d)

(i) Recommend whether Manager X or Manager Y should be selected.

(ii) Describe how increasing λ_A would impact your recommendation in part (d)(i).

Commentary on Question:
In part (i), some candidates failed to use the correct return and/or risk. To obtain full credit, total active return and total active risk must be used. Part (ii) does not require calculation.

(i) $U_A = R_A - \lambda_A \sigma^2_A$
Alternative formula: $U_A = R_A - 0.005\lambda_A \sigma^2_A$

R_A = total active return = total return – investor's benchmark return
σ_A = total active risk

Manager X:
Total active return = 10% - 9% = 1%
Total active risk = 4%
$U_A = (10\% - 9\%) - (3)(0.04^2) = 0.0052 = 0.52\%$
Manager Y:
Total active return = 11% - 9% = 2%
Total active risk = 6%
$U_A = (11\% - 9\%) - (3)(0.06^2) = 0.0092 = 0.92\%$
Recommend Manager Y since 0.52% < 0.92%

(ii) With a higher λ (risk aversion score), the manager with lower active risk and active return will more likely be selected. Increasing λ will increase the likelihood for selecting Manager X.

15. Continued

(e)

(i) Describe the sub-styles of value-oriented investing.

(ii) Explain concerns an investor should have regarding value-oriented managers.

Commentary on Question:
Candidates were expected to describe the sub-styles instead of general characteristics of value-oriented investing.

(i) High dividend yield: Expect stocks will maintain their dividend yield in the future.
Low price multiple: Believe that once the economy, industry, or firm improves, their stocks will increase in value.
Contrarian investors: Look for stocks that they believe are temporarily depressed; frequently invest in firms selling at less than book value.

(ii) Managers might misinterpret a stock's cheapness. The stock may be cheap for a very good economic reason that the manager does not fully appreciate.
There is a risk that perceived undervaluation will not be corrected within the investor's investment time horizon.
Specifically, certain catalysts (triggering events) might need to happen to make the price rise.
With value-oriented investing, extra return may be concentrated in a small portion of equities by market value, which is an illiquid segment, so the return premium might just be liquidity premium.

(f) Recommend and explain for each of the strategies below, how a portfolio can be constructed to capitalize on the higher growth rate in the broad European economy and the managers' stock-picking skills.

(i) Core-satellite approach

(ii) Alpha and beta separation approach

Commentary on Question:
Candidates were expected to apply the portfolio construction approaches to the specific case, instead of providing general descriptions of the approaches.
Some candidates recommended of using active managers in Europe, which is incorrect answer because the active managers' expertise is in the U.S. stock market instead of European stock market. To capitalize on the high growth rate in the broad European economy, passive investment should be used in Europe.
Candidates generally performed better in (i) than (ii).

15. Continued

(i) Core-satellite approach:
Invest the "core" in a passive and/or enhanced European broad market index.
Use the active managers as satellites to invest actively in the U.S. market.
The active risk is mitigated by the core since the passive strategy has low tracking risk.
Active return (alpha) is added by the satellites by capitalizing on the equity managers' excellent stock picking skills.

(ii) Alpha and beta separation approach:
Gain systematic risk exposure (beta) using European index fund or ETF to capitalize on the high growth rate in the European economy.
The beta can also be earned using European equity futures.
Use a long-short strategy to gain active return and to cancel out the beta exposure in the U.S.
Invest in the U.S. using the active managers to gain alpha.
Short U.S. stock index or sell U.S. futures to achieve market neutral in the U.S. market.

(g) Outline the limitations of the alpha and beta separation approach.

Commentary on Question:
Candidates who were successful in part (f)(ii) generally did well in this part as well. Candidates who didn't understand alpha and beta separation did poorly.

It may be difficult or costly to implement short positions.
Some long-short strategies are not truly market neural and some systematic risk may remain.
Some investors may not be able to use long-short strategy.
In this case, insurance regulations may prohibit using long-short strategy

16. **Learning Objectives:**

5. The candidate will understand and identify the variety of fixed instruments available for portfolio management. This section deals with fixed income securities. As the name implies the cash flow is often predictable, however there are various risks that affect cash flows of these instruments. In general the candidates should be able to identify the cash flow pattern and the factors affecting cash flow for commonly available fixed income securities. Candidates should also be comfortable using various interest rate risk quantification measures in the valuation and managing of investment portfolios. Candidates should also understand various strategies of managing the portfolio against given benchmark.

8. The candidate will understand the theory and techniques of portfolio asset allocation.

Learning Outcomes:

(5f) Demonstrate an understanding of the characteristics of leveraged loans.

(8a) Explain the impact of asset allocation, relative to various investor goals and constraints.

(8b) Propose and critique asset allocation strategies.

Sources:

Managing Investment Portfolios (MIP): A Dynamic Process, Maginn & Tuttle, 3rd edition, Chapter 5

Risk Factors as Building Blocks for Portfolio Diversification: The Chemistry of Asset Allocation

The Handbook of Fixed Income Securities, Fabozzi, Frank, 8th edition, Chapter 13

Commentary on Question:

Many candidates did well on parts (a), (b) and (d) of this question that required the use of formulas and calculations. For the remaining parts (c), (e), (f), and (g) there were very few candidates who earned full credit. Partial credit was given on all parts of this question when candidates showed correct formulas and calculations, or provided sufficient explanations or descriptions pertaining to what was asked in the question.

Solution:

(a) Determine the tangency portfolio within portfolios 1-6.

Commentary on Question:

Most candidates were able to determine the tangency portfolio within portfolios 1-6 by identifying the highest-Sharpe ratio efficient portfolio.

16. **Continued**

The Sharpe ratio is $(E(R_P) - R_F)/\sigma_P$, where:

- $E(R_P)$ is the expected return of portfolio P
- R_F is risk-free rate of return
- σ_P is standard deviation of return of the portfolio P

Determine the risk-free rate of return; (using portfolio #4, or any portfolio 1-4):
R_F = (7.75 - 0.468 x 11.75) = 2.25

Determine Sharpe ratio for portfolios 5 and 6:
Portfolio #5: x = (6.00 - 2.25)/ (10.5) = 0.357
Portfolio #6: y = (5.50 - 2.25)/ (9.00) = 0.361

Corner Portfolio No.	Sharpe Ratio
1	0.431
2	0.426
3	0.435
4	0.468
5	0.357
6	0.361

Corner portfolio #4 has higher expected Sharpe ratio than neighboring corner portfolios. So it is a tangency portfolio (corner portfolio with the highest Sharpe ratio.

(b) Create the most appropriate asset allocation based on the above information and the result in part (a).

Commentary on Question:
Candidates that earned full credit on this part were able to determine the portfolio weights for the tangency portfolio and U.S. T-bills, which is then used to create the most appropriate asset allocation.

To minimize standard deviation of return without lowering Sharpe ratio and still meeting the return objectives, we can combine the tangency portfolio with U.S. T-Bills to choose a portfolio on RMC's capital allocation efficient frontier line.

RMC's investment policy return objective is earning at least 5.5% annually.

5.5% = 7.75%*w + 2.25 %*(1 – w)
w = 0.591; where w = the portfolio weight of corner portfolio #4.
1-w = 0.409 is the portfolio weight of the U.S. T-bills

16. Continued

Asset Class	Portfolio Weight	Tangency Portfolio
A. U.S. Equities	29.55%	= 50% x 0.591
B. International Equities	11.82%	= 20% x 0.591
C. U.S. Corporation short-term Bonds	5.91%	= 10% x 0.591
D. U.S. Corporation intermediate Bonds	5.91%	=10% x 0.591
E. International Bonds	0.00%	= 0% x 0.591
F. Real Estate	5.91%	=10% x 0.591
G. U.S. T-Bill	40.90%	
Total	100.00%	

(c)

(i) Describe key characteristics of leverage loans and their advantages from a borrower's perspective.

(ii) Describe aspects that mitigate the risk of a leveraged loan for the investors.

Commentary on Question:
This question tested candidates understanding of the characteristics of leverage loans. There were very few candidates who earned full credit on this part, although the solution is straight from the course reading.

(i) Key characteristics are:
- A loan made to a company whose credit rating below investment-grade BBB,
- Loans that are broadly syndicated (to 10 or more bank / nonbank investors) leverage loans,
- Senior secured loans are at the top of the borrower's capital structure,
- Larger loans to larger companies.

The advantages from the borrower's perspective:
- Negotiate loan terms only once yet gaining access to multiple lenders
- Avoids conflict in priority; all lenders in the syndicate share equal rights under the credit agreement.

16. Continued

(ii) Aspects that mitigate the risk of a leveraged loan for the investors:

- Covenants that must be satisfied by the loan issuer (maintenance and incurrence requirements in the loan terms)
- Based on a study of defaulted loans and bonds of nonfinancial U.S. corporations, the ultimate recovery rates were higher for loans in comparison to bonds.
- Senior secured loans are at the top of the borrower's capital structure.
- Broad syndication spreads the risk to multiple lenders.

(d) Determine the condition needed for a leveraged loan portfolio to improve the efficiency of the portfolio in part (b) under the MVO framework.

Adding a new asset class to the portfolio is optimal if the following condition is met:
$(E(R_{new}) - R_F)/(\sigma_{new})) > ((E(R_p)- R_F)/\sigma_p)*Corr(R_{new}, R_p)$

Using the information from part c:
$(3.0\% - 2.25\%)/(8.0\%) > (7.75\% - 2.25\%)/(11.75\%)*Corr(R_{new}, R_p)$
$Corr(R_{new}, R_p) < 0.210$

Correlation between portfolio 4 and leveraged loan need to be less than 0.21.

It is irrelevant whether one uses the portfolio in part c, or portfolio 4 (both have the same Sharpe ratio and correlation)

(e) Describe the limitations of using the MVO approach in strategic asset allocation process.

Mean-Variance Optimization (MVO) approach is highly sensitive to small changes in input and estimation error.

The most important inputs into MVO are the expected returns. Unfortunately, mean returns are the most difficult input to estimate.

With different capital market expectations and risk-free rates, the most appropriate strategic asset allocation may not be optimal.

Based on the faulty premise that portfolio variance is a complete measure of risk.

(f) Explain how risk factor based portfolio optimization may be able to address the CRO's concerns.

16. Continued

The factor based optimization singles out the risk factors which have very low correlation with each other. It allows the optimization tool to better utilize the diversification benefits generated by each additional risk factors. Factors are building blocks of asset classes and a source of common risk exposures across asset classes. Factors are the smallest systematic (or nonidiosyncratic) units that influence investment return and risk characteristics. They include such elements as inflation, GDP growth, currency, and convexity of returns. Thus factors help explain the high level of internal correlation between asset classes.

(g) Identify challenges in implementing risk factor based portfolio optimization result.

It is not immediately obvious what the investable portfolio should be after risk factor allocation is determined. There is no theoretical opportunity set that encompasses all of the significant factors. Many risk factors have poor proxies, and the investable indices may contain more than 1 risk factor. Frequent rebalancing is required when the factor weightings change within the investments. Forward-looking assumptions are hard to develop. Long and short exposures, often via derivatives, are used to implement.

17. Learning Objectives:

6. The candidate will understand the variety of equity investments and strategies available for portfolio management.

Learning Outcomes:

(6f) compare techniques for characterizing investment style of an investor.

Sources:

QFIC-110-15: Liquidity as an Investment Style

Commentary on Question:

Commentary listed underneath question component.

Solution:

(a) Describe how liquidity is proposed to be measured in Ibbotson's article, "Liquidity as an Investment Style."

Commentary on Question:
Most candidates did well on this question, however, some candidates misunderstood the question and answered it as "how Liquidity can be treated as an investment style"

Liquidity can be measured by the following methods

- the annual share turnover, which is the sum of 12 months volume divided by each month's share outstanding
- the ask/bid spread, lower the spread, more liquid
- the price impact of a unit trade size

(b) Explain why liquidity meets the following criteria identified by William Sharpe to characterize an investment style.

(i) Not easily beaten

(ii) A viable alternative

Commentary on Question:
Most candidates received partial credit by answering part (i) as "low liquidity outperforms others" only without explaining how. Candidates general did well on part (ii)

17. **Continued**

(i) Not easily beaten

(a) When we put the long-term cumulative portfolio returns in quartiles according to different styles such as Size, Value, Momentum, and Liquidity, the Q1 portfolios all outperform the equally weighted universe portfolio.

(b) The low-liquidity quartile portfolio clearly outperforms both the microcap portfolio and the high-momentum portfolio

(ii) A viable alternative (one of the following three answers is acceptable)

(a) When we compare double-quartile portfolios that combine liquidity with each of other styles such as size, value, and momentum, the other style does not fully capture liquidity (i.e., the liquidity premium holds regardless). This indicates liquidity and other styles are distinctly different ways of picking stocks

(b) Also acceptable: When we examine double-sorted portfolios, comparing liquidity with size, value, and momentum in four-by-four grid shows the impact of liquidity on returns is stronger than that of size and momentum and roughly comparable to that of value.

(c) Also acceptable: When we express liquidity as a factor (for example, constructing monthly returns of a long-short portfolio in which the returns of the most liquid quartile are subtracted from the returns of the least liquid quartile) and regress the factor over other styles-> positive alpha. This additive to each style demonstrates liquidity is a viable alternative.

(c) Explain why liquidity is an economically significant indicator of long-run returns.

Commentary on Question:
Most candidates received partial credits by only specifying that low liquidity offers higher return without explaining why.

(i) Investors clearly want more liquidity and are willing to pay for it in all asset classes, including stocks.

(ii) Less liquidity comes with costs: It takes longer to trade less liquid stocks, and the transaction costs tend to be higher

(iii) In equilibrium, these costs must be compensated by less liquid stocks earning higher gross returns.

(iv) The liquidity style rewards the investor who has longer horizons and is willing to trade less frequently.