SOCIETY OF ACTUARIES
Quantitative Finance and Investment Core

Exam QFICORE

MORNING SESSION

Date: Wednesday, May 4, 2016
Time: 8:30 a.m. – 11:45 a.m.

INSTRUCTIONS TO CANDIDATES

General Instructions

1. This examination has a total of 100 points. It consists of a morning session (worth 60 points) and an afternoon session (worth 40 points).

   a) The morning session consists of 11 questions numbered 1 through 11.

   b) The afternoon session consists of 6 questions numbered 12 through 17.

   The points for each question are indicated at the beginning of the question.

2. Failure to stop writing after time is called will result in the disqualification of your answers or further disciplinary action.

3. While every attempt is made to avoid defective questions, sometimes they do occur. If you believe a question is defective, the supervisor or proctor cannot give you any guidance beyond the instructions on the exam booklet.

Written-Answer Instructions

1. Write your candidate number at the top of each sheet. Your name must not appear.

2. Write on only one side of a sheet. Start each question on a fresh sheet. On each sheet, write the number of the question that you are answering. Do not answer more than one question on a single sheet.

3. The answer should be confined to the question as set.

4. When you are asked to calculate, show all your work including any applicable formulas. When you are asked to recommend, provide proper justification supporting your recommendation.

5. When you finish, insert all your written-answer sheets into the Essay Answer Envelope. Be sure to hand in all your answer sheets because they cannot be accepted later. Seal the envelope and write your candidate number in the space provided on the outside of the envelope. Check the appropriate box to indicate morning or afternoon session for Exam QFICORE.

6. Be sure your written-answer envelope is signed because if it is not, your examination will not be graded.

Tournez le cahier d'examen pour la version française.

1. (*5 points*) Consider the stock price that

$$S_t = \begin{cases} 1.4S_{t-1} \ with \ probability = 0.3 \\ 0.9S_{t-1} \ with \ probability = 0.7 \end{cases}$$

where $t = 1, 2$, with $S_0 = 100$.

The risk free rate r for each period is 2%.

(a) (*2 points*) Determine if each of the following is a martingale. Justify your answer.

    (i) $\frac{S_t}{(1+r)^t}$ under the real-world measure.

    (ii) $Z_t = E\left(S_2 | I_t\right)$ under the real-world measure.

Consider a put option on this stock with a strike price 120 and expiration at $t = 2$.

(b) (*2 points*) Construct a portfolio to replicate the option, using the stock S_t, risk-free borrowing and lending, and assuming the absence of arbitrage.

(c) (*1 point*) Describe situations in which risk-neutral pricing is not valid.

2. (*6 points*) Consider an economy of two countries, A and B from the perspective of an investor in country A.

The currency exchange spot rate X_t (expressed in number of units of A's currency per unit of B's currency) follows:

$$\frac{dX_t}{X_t} = \mu_X dt + \sigma_X dW_X^P.$$

A non-dividend-paying stock D_t which is traded in country B (expressed in B's currency) follows:

$$\frac{dD_t}{D_t} = \mu_D dt + \sigma_D dW_D^P,$$

where μ_x, σ_x, μ_D and σ_D are constants with $\sigma_x > 0$ and $\sigma_D > 0$, W_X^P and W_D^P are $\mathbb{P}$-measure Weiner processes with instantaneous correlation ρ (i.e. $dW_X^P dW_D^P = \rho dt$).

Country A Savings Account $\left(M_t^A\right)$ available in country A (expressed in A's currency) follows:

$$\frac{dM_t^A}{M_t^A} = r_A dt, \text{ where } r_A \text{ is constant.}$$

Country B Savings Account $\left(M_t^B\right)$ available in country B (expressed in B's currency) follows:

$$\frac{dM_t^B}{M_t^B} = r_B dt, \text{ where } r_B \text{ is constant.}$$

Let N_t^B be the Country B Savings Account expressed in the Country A currency. Let $F_t = D_t X_t$ be the stock D price expressed in the Country A currency. Assume that the markets are complete.

(a)   (*1 point*) Derive the dynamics of N_t^B under the $\mathbb{P}$-measure.

2. **Continued**

(b)   (*1 point*) Derive the dynamics of X_t under Country A's martingale measure ($\mathbb{Q}^A$)

(c)   (*1 point*) Derive the dynamics of F_t under Country A's martingale measure ($\mathbb{Q}^A$)

(d)   (*1 point*) Derive the dynamics of D_t under Country A's martingale measure ($\mathbb{Q}^A$)

Let $G_t \left(0 \leq t \leq T \right)$ denote the value of the forward contract at time t, which at time $T \left(T > 0 \right)$ has the payoff: $G_T = X_0 \left(D_T - K \right)$ in the currency of country A, with K the strike price.

(e)   (*2 points*) Derive the fair strike price, K*, at time 0 of this forward contract.

3. (*9 points*) Suppose that the forward interest rate $F(t;T)$ and the equity $S(t)$ follow the stochastic differential equation (SDE) dynamics below:

$$dF(t;T) = m(t,T)dt + \upsilon(t,T)dX_t \quad \text{with } F(0;T) = F_0 > 0,$$
$$dS(t) = r(t)S(t)(dt) + \sigma_s S(t)dW_t \quad \text{with } S(0) = s_0 > 0$$

where $X_t = \rho W_t + \sqrt{1-\rho^2}Z_t$ with W_t and Z_t being independent Brownian motions under the risk neutral measure $\mathbb{Q}$; $-1 \le \rho \le 1, \sigma_s \ge 0$ are constant parameters; and

$m(t,T) = \upsilon(t,T)\int_t^T \upsilon(t,s)ds$.

Let $a_{t:n\rceil} = \sum_{i=1}^n P(t,T_i)$ where $P(t,T_i), 1 \le i \le n$, is the zero coupon bond price with maturity T_i at time t.

(a)   (*1 point*) Derive $d\left(a_{t:n\rceil}\right)$ in terms of dt and dX_t .

Given that

$$d\left[\log\left(\frac{a_{t:n\rceil}}{S(t)} \right) \right] = A\ dt + B\ dW_t + C\ dZ_t$$

(b)   (*3.5 points*) Derive expressions for A, B, and C.

(c)   (*2 points*) Derive $d\left(\dfrac{a_{t:n\rceil}}{S(t)} \right)$ based on your answer for part (b).

We would like to price an option with a payoff equal to max $\left(a_{T:n\rceil} - S(T), 0 \right)$ at option maturity time *T*.

(d)   (*2.5 points*)

    (i)   Derive the equivalent martingale measure for evaluating the expectation of the payoff of this option using Girsanov Theorem and all the previous results.

    (ii)   Describe how the expectation of this payoff can be evaluated using the results in part (d)(i).

4. (*4 points*) You are given the following two stochastic differential equations for X_t and Y_t, respectively:

$$dX_t = Y_t dt + t X_t dW_t$$
$$dY_t = X_t dt + t dV_t$$

where W_t and V_t are independent standard Wiener processes.

Let $Z_t = \dfrac{1}{\sqrt{2}}(W_t + V_t)$ and $F = t + X_t e^{Y_t}$.

(a) (*1.5 points*) Derive the stochastic differential equation for F in terms of Z_t.

(b) (*1 point*) Calculate the mean and variance of $(Z_{t+s} - Z_t)$ for $t \geq 0$ and $s > 0$.

(c) (*1.5 points*) Calculate the mean of $Z_t(Z_{t+s} - Z_t)$ for $t \geq 0$ and $s > 0$.

5. (*7 points*) Let W_t be a standard Weiner process, and T be a fixed time in the future.

Define a partition $(t_0, t_1, ..., t_n)$ of the interval $[0, T]$ such that $t_i = \frac{i}{n}T$ for $i = 0, 1, ..., n$.

You are given the following:

I. $\quad \Delta W_{t_i} = W_{t_{i+1}} - W_{t_i}$

II. $\quad E\left(\Delta W_{t_i} e^{\Delta W_{t_i}}\right) = h\, e^{\frac{h}{2}}$ for any $i = 0, 1, 2, ..., n-1$, where $h = \dfrac{T}{n}$

III. $\quad E\left[\left(e^{W_T - \frac{T}{2}} - 1\right)^2\right] = e^T - 1$

IV. Expected value of a lognormal distribution is $E(e^X) = e^{\mu + \frac{\sigma^2}{2}}$ where $X \sim N(\mu, \sigma^2)$

Note that

$$E\left[\left(\sum_{i=0}^{n-1} e^{W_{t_i} - \frac{t_i}{2}} \Delta W_{t_i}\right)^2\right]$$

$$= E\left(\sum_{i=0}^{n-1} e^{2W_{t_i} - t_i}\left(\Delta W_{t_i}\right)^2\right) + 2E\left(\sum_{i<j} e^{W_{t_i} - \frac{t_i}{2} + W_{t_j} - \frac{t_j}{2}} \Delta W_{t_i} \Delta W_{t_j}\right)$$

(a) (*1.5 points*) Calculate $E\left[\left(\sum_{i=0}^{n-1} e^{W_{t_i} - \frac{t_i}{2}} \Delta W_{t_i}\right)^2\right]$.

(b) (*2.5 points*) Calculate $E\left[\left(\sum_{i=0}^{n-1} e^{W_{t_i} - \frac{t_i}{2}} \Delta W_{t_i}\right)\left(e^{W_T - \frac{T}{2}} - 1\right)\right]$.

5. **Continued**

(c)　(*2 points*) Show that $\int_0^T e^{W_s - \frac{s}{2}} dW_S = e^{W_T - \frac{T}{2}} - 1$ by proving that $\sum_{i=0}^{n-1} e^{W_{t_i} - \frac{t_i}{2}} \Delta W_{t_i}$

converges to $e^{W_T - \frac{T}{2}} - 1$ in mean square convergence.

(d)　(*1 point*) Show that $\int_0^T e^{W_s - \frac{s}{2}} dW_S = e^{W_T - \frac{T}{2}} - 1$ by proving that $d\left(e^{W_t - \frac{t}{2}} \right) = e^{W_t - \frac{t}{2}} dW_t$

using Ito's Lemma.

6. (*5 points*) Company ABC has significant equity holdings in Company XYZ. ABC is considering hedging its position by trading equity options on XYZ's stock. Current market conditions are as below:

- Yields of non-coupon paying Treasury securities with different maturities:

(All rates are annual continuous.)

Term	1 month	3 month	6 month	1 year	2 year	3 year	5 year	7 year	10 year	20 year
Yield (%)	1.92	1.93	1.96	2.01	2.06	2.17	2.27	2.42	2.87	3.24

- Implied volatility of at-the-money (ATM) equity options on XYZ's stock with different maturity:

(All rates are annualized.)

	1-month	3-month	1-year	5-year	15-year
ATM volatility	20%	21%	22%	23%	24%

- The relationship between the implied volatility and the moneyness given term to maturity can be described using the following equation:

$$\sigma(k, T) = \sigma(1, T) + |k - 1|\, a(T)$$

where

$k = \dfrac{K}{S}$ is the ratio of fixed strike price K to the underlying stock price S

T is the term to maturity of the option

$a(T)$ is volatility adjustment, expressed as a function of term to maturity T and $a(T) > 0$

6. Continued

- ABC can borrow funds at 50 basis point spread over the Treasury rate with same term of maturity.
- ABC can loan out its fund at 30 basis point spread over the Treasury rate with same term of maturity.
- ABC operates its business in a country where income taxes are levied on investment incomes including dividend, interest and capital gains.
- The bid-ask spread in the market is 1%.
- By regulation, market participants are only allowed to short-sell the stocks when the most recent movement between traded prices was upward.
- A large corporation TDD is in a merger negotiation with company XYZ. If the merger is successful, all XYZ shares will be bought back and shares of the new entity will be issued.

(a) (*2.5 points*) Describe five differences in the actual market conditions above from the Black-Scholes assumptions.

Information on a European call option on XYZ' stock is given as below:

- The option now has 1 year left before expiration
- Underlying XYZ stock price at issue = 100
- Current XYZ stock price = 110
- The ratio k determined at issue = 1.21
- The volatility adjustment scheme is given as follow:

	1-month	3-month	1-year	5-year	15-year
Volatility adjustment $a(T)$	11%	12%	13%	14%	15%

There are two choices of volatility to use for calculating the option value:

    I.   The one-year volatility

    II.   Arithmetic average of the volatilities until expiration

(b) (*1.5 points*) Calculate the volatility under the above two assumptions, respectively.

(c) (*1 point*) Determine which volatility should be used to price the option and state why.

7. (*4 points*) Consider European options on a non-dividend paying stock S_t with strike price K, expiry time T, risk free interest rate r and volatility σ.

You are given the following identity

$$Ke^{-r(T-t)}\varnothing(d_2) = S_t\varnothing(d_1)$$

where

$$\varnothing(x) = \frac{1}{\sqrt{2\pi}}e^{-\frac{x^2}{2}}, d_1 \text{ and } d_2 \text{ are defined as in the Black Scholes formula.}$$

(a) (*2 points*) Show that, at time t, the theta θ_c of a call option is:

$$\theta_c = -S_t\frac{\varnothing(d_1)\sigma}{2\sqrt{T-t}} - rKe^{-r(T-t)}N(d_2)$$

(b) (*1 point*) Derive an expression for the theta θ_p of a put option.

(c) (*1 point*) Calculate $\lim_{S_t \to 0}\theta_c$ and $\lim_{S_t \to 0}\theta_p$

8. (*6 points*) The short rate $r(t)$ follows a one-factor Brownian motion process with constant drift a and constant volatility σ: $dr(t) = a\,dt + \sigma dW$. The arbitrage-free price $P(t,T)$ at time t of a zero coupon bond maturing at time T is given in the following formula:

$$P(t,T) = \exp\left(A(t,T) - (T-t)r(t)\right)$$

where

$$A(t,T) = -\frac{a(T-t)^2}{2} + \frac{\sigma^2}{6}(T-t)^3$$

Let $F(t,T)$ be the arbitrage-free forward rate at time t for the period from t to T.

(a)   (*2 points*) Derive the formula for $F(t,T)$

(b)   (*1 point*) Show that $dF(t,T) = \sigma^2(T-t)dt + \sigma dW$

(c)   (*1 point*) Show that $\dfrac{dP}{P} = r\,dt - \sigma(T-t)dW$

(d)   (*2 points*) Prove $F(t,T)$ satisfies the condition relating diffusion to drift in the HJM forward rate model.

9. (*4 points*) Consider a Vasicek Model:

$$dr_t = (\eta - \gamma r_t)\,dt + \beta^{\frac{1}{2}}dX_t$$

where X_t is a Wiener process, and η, γ and β are constants with $\gamma \neq 0$.

(a) (*2 points*) Show that for $t < T$,

$$r_T = \left(r_t - \frac{\eta}{\gamma}\right)e^{-\gamma(T-t)} + \frac{\eta}{\gamma} + \beta^{\frac{1}{2}}e^{-\gamma T}\int_t^T e^{\gamma s}dX_s$$

A forward contract is written on a zero-coupon bond which has principal 100 and matures in 20 years. The forward contract matures in 10 years and the contract price is 58.

(b) (*2 points*) Calculate the payoff of the forward contract at its maturity, assuming $\gamma = 0.1, \eta = 0.01, \beta = 0.004,$ and the prevailing short rate 10 years from now is 2%.

10. (*6 points*) You are the investment manager at a private University in the U.S., tasked with writing an Investment Policy for each of the following two funds:

- A fund supporting defined benefit (DB) pensions for the faculty and administration staff at the University, and
- An endowment to assist underprivileged students with tuition

(a) (*1 point*) Compare and contrast your fiduciary responsibilities for each fund.

(b) (*1 point*) Compare considerations in setting return objectives for DB plan assets and for endowments.

(c) (*1 point*) Compare considerations in setting risk objectives for DB plan assets and for endowments.

(d) (*2 points*) Identify the following investment policy constraints for each fund:

      I.   Liquidity requirements

      II.   Time horizon

      III.   Tax concerns

      IV.   Legal and regulatory requirements

      V.   Unique circumstances

The DB plan is underfunded, and the University has been using investment income from the endowment to keep the plan funding status from declining further.

(e) (*1 point*) Assess whether this is an acceptable strategy.

11. (*4 points*) You are responsible for managing the risks of a non-agency Residential Mortgage Back Securities (RMBS) portfolio and a Commercial Mortgage Back Securities (CMBS) portfolio of a large insurance company.

    (a)   (*1.5 points*) Compare the non-agency RMBS portfolio and the CMBS portfolio on the relative basis in terms of the following risks.

        (i)   Concentration risk

        (ii)   Interest rate risk

        (iii)   Prepayment risk

    (b)   (*1 point*) Describe how the following two measures can be used in assessing risk of CMBS.

        (i)   Debt Service Coverage Ratio (DSCR)

        (ii)   Loan-to-Value (LTV)

    (c)   (*1.5 points*) Describe a deterministic, rule-based default modeling framework for the CMBS portfolio, projecting DSCR and LTV to determine one of the three possible outcomes of the loan:

- Term default
- Timely pay-off
- Maturity default/loan extension

END OF EXAMINATION
Morning Session

USE THIS PAGE FOR YOUR SCRATCH WORK

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