QFI CORE Model Solutions
Fall 2016

1. **Learning Objectives:**

1. The candidate will understand the fundamentals of stochastic calculus as they apply to option pricing.

Learning Outcomes:

(1b) Understand the importance of the no-arbitrage condition in asset pricing.

(1g) Demonstrate understanding of the differences and implications of real-world versus risk-neutral probability measures.

(1j) Understand the Black Scholes Merton PDE (partial differential equation).

Sources:
An Introduction to the Mathematics of Financial Derivatives, Neftci, Salih, 3rd Edition. Ch. 2, 12, 14, 11.

Commentary on Question:
This question is to test candidates' understanding of the fundamentals of stochastic calculus and how they are applied to option pricing. Most candidates understood the difference between real-world measure and risk-neutral measure but could not apply it well.

Solution:

(a) Prove that under the risk neutral measure $\mathbb{Q}$

$$dS_t = (r-\sigma)S_t dt + \sigma S_t dW_t$$
$$dG_t = (r-\tau)G_t dt + \tau G_t dW_t.$$

Plugging all the applicable given equations we have

$$dV_t = (\theta_1 r B_t + \theta_2(\mu+\sigma)S_t + \theta_3(\nu+\tau)G_t)dt + (\theta_2\sigma S_t + \theta_3\tau G_t)dW_t$$

Since

$$dW_t = \gamma_t dt + d\widetilde{W}_t$$

we find

$$dV_t = (\theta_1 r B_t + \theta_2(\mu+\sigma+\gamma\sigma)S_t + \theta_3(\nu+\tau+\gamma\tau)G_t)dt + (\theta_2\sigma S_t + \theta_3\tau G_t)d\widetilde{W}_t$$

1. Continued

By no-arbitrage argument, V_t must have drift rV_t under $\mathbb{Q}$, which implies

$\mu + \sigma + \gamma\sigma = r$ — equation 1
$\nu + \tau + \gamma\tau = r$ — equation 2

Note that

$$dS_t = (\mu + \gamma\sigma)S_t dt + \sigma S_t d\widetilde{W}_t$$
$$dG_t = (\nu + \gamma\tau)G_t dt + \tau G_t d\widetilde{W}_t$$

Substituting the equations 1 and 2 into the last two equations gives

$$dS_t = (r - \sigma)S_t dt + \sigma S_t d\widetilde{W}_t$$
$$dG_t = (r - \tau)G_t dt + \tau G_t d\widetilde{W}_t$$

(b) Determine λ such that the price of the contract is 0 at time 0, assuming that $r \neq \sigma$ and $r \neq \tau$.

We need to compute λ such that

$$\exp(-rT)\, E^Q[D_T - \lambda E_t] = 0.$$

That is

$$\lambda = E^Q[D_T]/E^Q[E_T].$$

Calculate

$$E^Q[D_T] = E^Q[\int_0^T \sigma S_t dt] = \sigma E^Q[\int_0^T S_t dt]$$

Since

$$S_t = s_0 \exp(\left(r - \sigma - \frac{\sigma^2}{2}\right)t + \sigma\,\widetilde{W}_t)$$

and

$$\mathbb{E}^{\mathbb{Q}}\left[\exp\left(\sigma\widetilde{W}_t\right)\right] = \exp(\frac{\sigma^2 t}{2})$$

$$E^Q[D_T] = \sigma s_0[\int_0^T \exp(r - \sigma)\, t\, dt]$$

Therefore

$$E^Q[D_T] = \frac{\sigma s_0}{r - \sigma}[\exp((r - \sigma)T) - 1]$$

1. Continued

Similarly

$$E^Q[E_T] = \frac{\tau g_0}{r - \tau}[\exp((r - \tau)T) - 1]$$

And hence

$$\lambda = \frac{\sigma s_0(r - \tau\ )}{\tau g_0(\ r - \sigma)} \frac{[\exp((r - \sigma)T) - 1]}{[\exp((r - \tau)T) - 1]}$$

2. **Learning Objectives:**

1. The candidate will understand the fundamentals of stochastic calculus as they apply to option pricing.

Learning Outcomes:

(1c) Understand Ito integral and stochastic differential equations.

(1h) Define and apply the concepts of martingale, market price of risk and measures in single and multiple state variable contexts.

Sources:

Hirsa-Neftci 3rd edition Ch.6, 9, 10. (pp. 107-108, pp.153-155, pp.171-172)

Commentary on Question:

The question tests candidates' understanding of the relationships between Ito integral and Ito's lemma through the definition of Wiener process

Solution:

(a) Determine the mean and variance of

$$\int_0^t W_s dW_s$$

Commentary on Question:

Solving stochastic integral is a key to the distribution of at the certain time of processes

From Ito's lemma

$$dS_s = ds + 2W_s dW_s.$$

Rearranging it as

$$2W_s dW_s = dS_s - ds,$$

Integrating over $[0,t]$ and noting that $S_0 = 0$ we find

$$2\int_0^t W_s dW_s = \int_0^t dS_s - \int_0^t ds$$
$$= S_t - t,$$

Plugging $S_t = W_t^2$ we have

$$\int_0^t W_s \mathrm{d}W_s = \frac{1}{2}(W_t^2 - t) \qquad (1)$$

2. Continued

Hence the mean of the stochastic integral is

$$\mathrm{E}\left(\int_0^t W_s dW_s\right) = \frac{1}{2}E(W_t^2) - \frac{t}{2}.$$

But $E(W_t^2) = t$ by the property of Wiener process,

$$\mathrm{E}\left(\int_0^t W_s dW_s\right) = \frac{t}{2} - \frac{t}{2} = 0.$$

Alternative:
$E\left(\int_0^t W_s dW_s\right) = 0$ because $W_s dW_s$ is a martingale. See also (9.131) on page 160 of Neftci.

For the variance, *let* $A = \int_0^t W_s dW_s$. Then

$$Var(A) = E(A^2) - E(A)^2$$

From equation (1) above,

$$A^2 = \frac{1}{4}(W_t^4 - 2W_t^2 t + t^2),$$

Then

$$E(A^2) = \frac{1}{4}E(W_t^4) - \frac{t}{2}E(W_t^2) + \frac{t^2}{4}$$

Since

$$E\left(\frac{W_t - 0}{\sqrt{t}}\right)^4 = 3, \qquad => \quad E(W_t^4) = 3t^2,$$

Therefore, with $E(W_t^2) = t$,

$$E(A^2) = \frac{3t^2}{4} - \frac{t^2}{2} + \frac{t^2}{4} = \frac{t^2}{2}$$

Alternative:
$E\left[\left(\int_0^t W_s dW_s\right)^2\right] = \int_0^t E\left(W_s{}^2\right)ds = \int_0^t s ds = \frac{t^2}{2}$ where the first equality follows from (9.132) on page 160 of Neftci.

Finally, since $E(A) = 0$,

$$Var(A) = E(A^2) = \frac{t^2}{2}$$

2. Continued

(b) Show that $e^{-rt}S_t$ is not a martingale if r is constant.

Commentary on Question:
Some stochastic processes are not martingale even in the function of Wiener processes. Test candidates' understanding of risk neutral measure and martingales.

For the risk neutral measure, the stock price should be normalized with money market account risk free rate:

$$\widetilde{S_t} = \frac{S_t}{B_t}, dB_t = rB_t dt,$$

where $B_t = e^{rt}$

$$d\widetilde{S_t} = d\left(\frac{S_t}{B_t}\right) = \frac{dS_t}{B_t} - \widetilde{S_t}\frac{dB_t}{B_t} = \frac{dS_t}{S_t}\widetilde{S_t} - \widetilde{S_t} r dt,$$

$$\frac{d\widetilde{S_t}}{\widetilde{S_t}} = \frac{dS_t}{S_t} - rdt.$$

For the normalized process $\frac{d\widetilde{S_t}}{\widetilde{S_t}}$ to be martingale, the drift of $\frac{dS_t}{S_t}$ should be rdt, but

$$\frac{dS_t}{S_t} = \frac{dt}{S_t} + \frac{2W_t dW_t}{S_t} = \frac{dt}{W_t^2} + \frac{2dW_t}{W_t},$$

the drift is $\frac{1}{W_t^2}$. Therefore the normalized with risk free rate money market account is not martingale.

Alternative:
Drfit term obtained by Ito's lemma should be zero if the process were a martingale. Let

$$Z = e^{-rt}S_t = e^{-rt}W_t^2$$

$$dZ = \frac{\partial Z}{\partial t}dt + \frac{\partial Z}{\partial W_t}dW_t + \frac{1}{2}\frac{\partial^2 Z}{\partial {W_t}^2}(dW_t)^2$$

$$\frac{\partial Z}{\partial t} = -re^{-rt}W_t^2, \frac{\partial Z}{\partial W_t} = 2e^{-rt}W_t, \frac{\partial^2 Z}{\partial {W_t}^2} = 2e^{-rt}$$

Hence

$$dZ = -re^{-rt}W_t^2 dt + 2e^{-rt}W_t dW_t + e^{-rt}dt$$

$$dZ = e^{-rt}(1 - rW_t^2)dt + 2e^{-rt}W_t dW_t.$$

The drift term cannot be zero.

2. Continued

(c)

(i) Express $F_T = \int_0^T S_s ds$ in terms of S_T and $\int_0^T W_s^3 dW_s$.

(ii) Determine the expectation of F_T .

Commentary on Question:
It tests the capability of applying stochastic integral to find a solution of forward prices.

(i)
Since $S_t = W_t^2$,

$$\int_0^T S_s\, ds = \int_0^T W_s^2\, ds$$

From $G_t = S_t^2 = W_t^4$, then by Ito's lemma,

$$dG_t = dW_t^4 = 4W_t^3 dW_t + \frac{1}{2}12W_t^2(dW_t)^2 = 4W_t^3 dW_t + 6W_t^2 dt,$$

$$6W_t^2 dt = dG_t - 4W_t^3 dW_t,$$

$$6\int_0^T W_t^2 dt = \int_0^T dG_t - 4\int_0^T W_t^3 dW_t, \quad \text{(Equation 3)}$$

$$F_T = \int_0^T S_t dt = \frac{1}{6}\left[G_T - 4\int_0^T W_t^3 dW_t\right],$$

$$= \frac{G_T}{6} - \frac{2}{3}\int_0^T W_t^3 dW_t$$

(ii)
Taking expectations to above equation (3),

$$6E\left(\int_0^T W_t^2 dt\right) = E(W_T^4) - 4E\left(\int_0^T W_t^3 dW_t\right),$$

But from the solution for (a)

$$E(W_T^4) = 3T^2,$$

and

$$6E\left(\int_0^T W_t^2 dt\right) = 6\int_0^T E(W_t^2)dt = 6\int_0^T t dt = 3T^2,$$

So

$$\mathrm{E}\left(\int_0^T W_t^3 dW_t\right) = 0.$$

2. Continued

Alternative:
$E\left(\int_0^t W_s^{\,3} dW_s\right) = 0$ because $W_s^{\,3} dW_s$ is a martigale. See also (9.131) on page 160 of Neftci.

Therefore

$$E\left(\int_0^T W_t^2 dt\right) = E\left(\int_0^T S_t\, dt\right) = \frac{E(W_T^4)}{6} = \frac{T^2}{2}$$

3. **Learning Objectives:**

1. The candidate will understand the fundamentals of stochastic calculus as they apply to option pricing.

3. The candidate will understand the quantitative tools and techniques for modeling the term structure of interest rates and pricing interest rate derivatives.

Learning Outcomes:

(1c) Understand Ito integral and stochastic differential equations.

(1d) Understand and apply Ito's Lemma.

(1f) Demonstrate understanding of option pricing techniques and theory for equity and interest rate derivatives.

1h) Define and apply the concepts of martingale, market price of risk and measures in single and multiple state variable contexts.

(1i) Understand and apply Girsanov's theorem in changing measures.

(3c) Understand and apply popular one-factor interest rate models including Vasicek, Cox-Ross-Ingersoll, Hull-White, Ho-Lee, Black-Derman-Toy, Black-Karasinski.

Sources:

Neftci Ch. 6, 10, 14, 15, 17

Wilmott – Introduces Quantitative Finance Ch. 16, 17

Commentary on Question:

This question tests the practical use of Ito's lemma and the Girsanov theorem.

Most candidates attempted parts (a) to (c) and were able to apply Ito's lemma correctly, however made mistakes in deriving the final solution, which means they should brush up their calculus skill.
For part (d), some candidates did not attempt. For those who attempted, most people understood the concept of martingale and the Girsanov's theorem, but struggled with the practical use of it.

3. Continued

Solution:

(a) Derive $d(\log P)$ in terms of dt and dX_t using Ito's Lemma.

Commentary on Question:
Most candidates were able to complete question (a) correctly.

By applying Ito's lemma on $\log P$, we have

$$d(log\,P) = \frac{\partial\, log\,P}{\partial P}dP + \frac{(-\sigma BP)^2}{2}\frac{\partial^2 log\,P}{\partial P^2}dt$$
$$= \frac{1}{P}(rdt - \sigma BdX_t)P + \frac{(-\sigma BP)^2}{2}(-\frac{1}{P^2})dt = \left(r - \frac{\sigma^2 B^2}{2}\right)dt - \sigma BdX_t$$

(b) Derive $d(\log F)$ in terms of dt, dX_t, and dY_t.

Commentary on Question:
Most candidates attempted part (b), but some candidates struggled to derive the final solution and failed to handle the correlated terms dX and dY correctly.

By applying Ito's lemma on $log\,F$, we have

$$d(log\,F) = \frac{\partial\, log\,F}{\partial F}dF + \frac{1}{2}\frac{\partial^2 log\,F}{\partial F^2}(dF)^2$$
$$= \frac{1}{F}\left(rdt - \sigma BdX_t - \tau e^{-\alpha(T-t)}dY_t\right)F$$
$$- \frac{1}{2F^2}\left(rdt - \sigma BdX_t - \tau e^{-\alpha(T-t)}dY_t\right)^2 F^2$$
$$= rdt - \sigma BdX_t - \tau e^{-\alpha(T-t)}dY_t$$
$$- \frac{1}{2}\left((\sigma B)^2 + (\tau e^{-\alpha(T-t)})^2 + 2\rho\sigma\tau Be^{-\alpha(T-t)}\right)dt$$
$$= \left(r - \frac{1}{2}(\sigma B)^2 - \frac{1}{2}\tau^2 e^{-2\alpha(T-t)} - \rho\sigma\tau Be^{-\alpha(T-t)}\right)dt - \sigma BdX_t - \tau e^{-\alpha(T-t)}dY_t$$

(c) Derive $d(\log M)$ using results of parts (a) and (b).

Commentary on Question:
Most candidates were able to relate parts (a) and (b) to part (c). Partial marks are awarded for incorrect but consistent results.

Thus,

$$d(log\,M) = d(log\,F) - d(log\,P)$$
$$= -\tau e^{-\alpha(T-t)}dY_t - \left(\frac{1}{2}\tau^2 e^{-2\alpha(T-t)} + \rho\sigma\tau Be^{-\alpha(T-t)}\right)dt$$

3. **Continued**

(d)

(i) Derive $d\tilde{Y}_t^T$ in terms of dY_t and dt.

(ii) Prove that

$$d(\log M) = -\frac{1}{2}\tau^2 e^{-2\alpha(T-t)}dt - \tau e^{-\alpha(T-t)}d\tilde{Y}_t^T.$$

Commentary on Question:
Most candidates had basic understanding of Girsanov theorem and the property of martingale, but struggled to connect the dots and used Ito's lemma to solve this question.

Partial marks are awarded for candidates who had wrong answers in part (c) but applied Ito's lemma and Girsanov theorem correctly.

(i) Assume that $d\tilde{Y}_t^T = dY_t + \eta\, dt$. Then

$$\frac{dM}{M} = \xi(dY_t + \eta\, dt)$$

Using Ito's lemma we have

$$d(log\, M) = \left(\xi\eta - \frac{1}{2}\xi^2\right)dt + \xi dY_t.$$

Compare to the result in part (c) we find

$$\xi = -\tau e^{-\alpha(T-t)}$$
$$\eta = \rho\sigma B(t,T)$$

Thus

$$d\tilde{Y}_t^T = \rho\sigma B(t,T)\, dt + dY_t$$

(ii) From (c) and (d)(i)

$$d(log\, M) = -\tau e^{-\alpha(T-t)}d\tilde{Y}_t^T - \frac{1}{2}\tau^2 e^{-2\alpha(T-t)}dt$$

4. **Learning Objectives:**

3. The candidate will understand the quantitative tools and techniques for modeling the term structure of interest rates and pricing interest rate derivatives.

Learning Outcomes:

(3a) Understand and apply the concepts of risk-neutral measure, forward measure, normalization, and the market price of risk, in the pricing of interest rate derivatives.

(3b) Apply the models to price common interest sensitive instruments including: callable bonds, bond options, caps, floors, swaptions, caption, floortions.

Sources:

Wilmott 2nd ED: Ch16 one-factor interest rate modeling; Ch 17 Yield Curve Fitting

Commentary on Question:

Commentary listed underneath question component.

Solution:

(a) Show that for $T > t$

$$\sum_{k=t}^{T-1} r_k = r_{t-1}(T-t) + \sum_{k=t}^{T-1}(T-k)[\phi(k) + \sigma\epsilon_k]$$

Commentary on Question:

Candidates generally did well on this part. There are two ways to demonstrate an understanding here, either by writing out the terms and identifying the recursive formula or by proving by mathematical induction.

$$\sum_{k=t}^{T-1} r_k = r_t + r_{t+1} + \cdots + r_{T-1}$$

$$= [r_{t-1} + \phi(t) + \sigma\epsilon_t] + [r_t + \phi(t+1) + \sigma\epsilon_{t+1}] + \cdots + [r_{T-2} + \phi(T-1) + \sigma\epsilon_{T-1}]$$

$$= [r_{t-1} + \phi(t) + \sigma\epsilon_t] + [r_{t-1} + \phi(t) + \sigma\epsilon_t + \phi(t+1) + \sigma\epsilon_{t+1}] + \cdots + \big[r_{t-1} + [\phi(t) + \phi(t+1) + \cdots + \phi(T-1)] + [\sigma\epsilon_t + \sigma\epsilon_{t+1} + \cdots + \sigma\epsilon_{T-1}]\big]$$

$$= [(T-1) - t + 1]\, r_{t-1} + [(T-t)\phi(t) + (T-t-1)\phi(t+1) + \cdots + \phi(T-1)] + [(T-t)\sigma\epsilon_t + (T-t-1)\sigma\epsilon_{t+1} + \cdots + \sigma\epsilon_{T-1}]$$

$$= r_{t-1}(T-t) + \sum_{k=t}^{T-1}(T-k)[\phi(k) + \sigma\epsilon_k]$$

4. Continued

(b) Compute the price of a zero-coupon bond at time *t* with maturity $T > t$.

Commentary on Question:
Most candidates knew the generalized formula for the price of a zero coupon bond at time t with maturity T>t. The key here is to identify that $\sum_{k=t}^{T-1} r_k$ *is normally distributed random variable and therefore able to use the mean and variance to derive the price of the zero coupon bond.*

$$P(t,T) = E[exp(-\sum_{k=t}^{T-1} r_k)\ ]$$

From part (a) we can conclude that $\sum_{k=t}^{T-1} r_k$ is a normally distributed random variable with mean $r_{t-1}(T-t) + \sum_{k=t}^{T-1}(T-k)[\phi(k)]$ and variance $\sum_{k=t}^{T-1}(T-k)^2\sigma^2$.

Therefore the bond price is given by

$$P(t,T) = exp[-r_{t-1}(T-t) - \sum_{k=t}^{T-1}(T-k)\phi(k) + \frac{\sum_{k=t}^{T-1}(T-k)^2\sigma^2}{2}]$$

(c) Identify the model as the period becomes infinitesimal.

Commentary on Question:
This question is straightforward. Most candidates who did well on overall question did well on this part too.

$dr = \phi(t) + \sigma dW_t$
This is the Ho and Lee model.

(d)

(i) Solve the SDE above to give an explicit expression for $r(t)$ under the Vasicek model.

(ii) Determine the expectation and the variance of $r(t)$.

Commentary on Question:
Candidates did well on this question. If people understood the topic, they generally got full credit because it is straightforward. Some candidates messed up part (i) but got partial credit on (ii) because they properly understood the expectation characteristics.

4. Continued

(i)

Re-write $dr(t) = a\big(b - r(t)\big)dt + \sigma dW_t$ as

$$dr(t) + ar(t)dt = abdt + \sigma dW_t$$

Multiply by e^{at}

$$e^{at}dr(t) + ar(t)e^{at}dt = abe^{at}dt + \sigma e^{at}dW_t$$

That is

$$d\big(e^{at}r(t)\big) = abe^{at}dt + \sigma e^{at}dW_t$$

Integrating this from 0 to t we obtain

$$e^{at}r(t) - r(0) = b(e^{at} - 1) + \sigma \int_0^t e^{as}dW_s$$

$$r(t) = b(1 - e^{-at}) + \sigma e^{-at} \int_0^t e^{as}dW_s + e^{-at}r(0)$$

(ii)

r(t) is a normally distributed random variable with

$\mathbb{E}[r(t)] = b(1 - e^{-at}) + e^{-at}r(0)$

$Var[r(t)] = \sigma^2 e^{-2at} Var[\int_0^t e^{as}dW_s] = \sigma^2 e^{-2at} \int_0^t e^{2as}ds = \frac{\sigma^2}{2a}(1 - e^{-2at})$

(e) Determine the expectation and the variance of $\int_t^T r(u)du$ at time t.

Commentary on Question:

Candidates did relatively poorly on this part of the question. Most candidates did not realize that the question demands the expectation and variance conditional on time t and started the integral from 0 instead. Many candidates mistakenly used the expression for r(t) in part (d) (replacing t by u).

From part (d)(i) we have:

$$d\big(e^{at}r(t)\big) = abe^{at}dt + \sigma e^{at}dW_t$$

Integrating both sides of the above equation from t to u (0<t<u<T) we have

$$e^{au}r(u) - e^{at}r(t) = b(e^{au} - e^{at}) + \int_t^u \sigma e^{as}dW_s$$

Rearranging we get

$$r(u) = b - (b - r(t))e^{-a(u-t)} + \sigma e^{-au} \int_t^u e^{as}dW_s$$

4. Continued

Here for fixed t and u, $\int_t^u e^{as} dW_s$ is a normally distributed random variable with mean zero and variance $\int_t^u e^{(2as)} ds = \frac{1}{2a}(e^{2au} - e^{2at})$

Integrating the short-rate $r(u)$ from t to T we obtain

$$\int_t^T r(u)du = b(T-t) - (b - r(t)) \int_t^T e^{-a(u-t)} du + \sigma \int_{u=t}^T e^{-au} \int_{s=t}^u e^{as} dW_s \, du$$

$$\int_t^T r(u)du = b(T-t) - \frac{b - r(t)}{a}\left(1 - e^{-a(T-t)}\right) + \sigma \int_{u=t}^T \int_{s=t}^u e^{a(s-u)} dW_s \; du$$

We know the distribution of stochastic integral, so we know $\int_t^T r(u)du$ conditioned on r(t) is a normally distributed random variable with

$$Mean = b(T-t) - \frac{b - r(t)}{a}\left(1 - e^{-a(T-t)}\right)$$

$$Variance = \sigma^2 \int_{s=t}^T [\int_{u=s}^T e^{a(s-u)} du]^2 \; ds = \frac{\sigma^2}{a^2} \int_t^T \left(1 - e^{-a(T-s)}\right)^2 ds$$

$$= \frac{\sigma^2}{a^2} \int_t^T \left(1 - 2e^{-a(T-s)} + e^{-2a(T-s)}\right) ds$$

$$= \frac{\sigma^2}{a^2} [(T-t) - \frac{2}{a}\left(1 - e^{-a(T-t)}\right) + \frac{1}{2a}\left(1 - e^{-2a(T-t)}\right)]$$

(f) Derive the price of a zero-coupon bond at time t with maturity $T \geq t$ under the Vasicek model using the results of part (e).

Commentary on Question:
Since this part of question largely based on results of part (e), very few candidates received full credits of this part. Partial credit was given to candidates who demonstrated an understanding of the correct concept and logic. Most candidates did not realize that the solution should be conditional on time t. Many candidates did not realize that $\int_t^T r(u)du$ is Gaussian and hence that $exp(-\int_t^T r(u)du)|r(t)]$ is lognormal.

Price of the zero coupon bond is

$$\mathbb{E}[exp(-\int_t^T r(u)du)|r(t)]$$

Since $\int_t^T r(u)du$ is Gaussian and its mean and variances are as given in (e); then $exp(-\int_t^T r(u)du)|r(t)]$ is lognormal and its mean is given by

4. **Continued**

$$\mathbb{E}[\exp(-\int_t^T r(u)du)|r(t)] = \exp(-Mean + \frac{Variance}{2})$$

$$\frac{1}{2}Variance - Mean$$
$$= \left[\frac{1-e^{-a(T-t)}}{a} - (T-t)\right]\left(b - \frac{\sigma^2}{2a^2}\right) - \frac{\sigma^2}{4a}[\frac{1-e^{-a(T-t)}}{a}]^2$$
$$- r(t)\frac{1-e^{-a(T-t)}}{a}$$

Alternatively, the candidate may use the formula:
Price of the zero coupon bond is

$$\mathbb{E}[\exp(-\int_t^T r(u)du)|r(t)] = \exp(A(t,T) - r(t)B(t,T))$$

$$A(t,T) = [B(t,T) - (T-t)]\left(b - \frac{\sigma^2}{2a^2}\right) - \frac{\sigma^2}{4a}B(t,T)^2$$

$$B(t,T) = \frac{1-e^{-a(T-t)}}{a}$$

(g) Compare and contrast the Vasicek Model and the model in part (c).

Commentary on Question:
Many candidates only listed the differences between the two models but failed to mention the similarities of the two models. Candidates had to both compare and contrast (list both differences and similarities) to get full credit.

Differences:
Vasicek is mean-reverting while Ho-Lee model is not;
Ho-Lee model is arbitrage free while Vasicek is a one-factor model
Similarity:
Both allows for negative rates
Both assume constant volatility
Both are in the class of affine term structure models
Both models the short rate of interest, and both follows a normal process.

5. Learning Objectives:

1. The candidate will understand the fundamentals of stochastic calculus as they apply to option pricing.

Learning Outcomes:

(1c) Understand Ito integral and stochastic differential equations.

(1d) Understand and apply Ito's Lemma.

(1e) Understand and apply Jensen's Inequality.

(1f) Demonstrate understanding of option pricing techniques and theory for equity and interest rate derivatives.

Sources:

Nefci Ch. 12

Frequently Asked Questions, Wilmott Q23

Commentary on Question:

This question tests the candidates' understanding of option pricing techniques along with Ito's Lemma and Jensen's Inequality. Most people answered part (a) correctly but only received partial credits for parts (b) and (c).

Solution:

(a) Estimate $Q(S_t, t)$ assuming that $r = 6\%$ and $\sigma = 10\%$.

Commentary on Question:

Most candidates answered this part of the question correctly. Partial credits were given if the candidates were able to list the Black-Scholes PDE correctly.

Appling the Black-Scholes PDE on V and Q respectively:

$$-rV + rV_s S + V_t + \frac{1}{2}\sigma^2 S^2 V_{ss} = 0$$

$$-rQ + rQ_s S + Q_t + \frac{1}{2}\sigma^2 S^2 Q_{ss} = 0$$

$$-0.06 \times 0.4 + 0.06 \times 0.5 \times S - 3.55 + \frac{1}{2} \times 0.01 \times S^2 V_{ss} = 0$$

$$-0.06Q + 0.06 \times (-0.5) \times S - 1.4 + \frac{1}{2} \times 0.01 \times S^2 Q_{ss} = 0$$

Adding above 2 equations together:

$$-0.06 \times 0.4 - 0.06Q - 4.95 + \frac{1}{2} \times 0.01 \times 1000 = 0$$

Therefore Q = 0.43

5. Continued

(b)

(i) Solve for m and thus the explicit expression of $U(S_t, t)$.

(ii) Describe this derivative's payoff at maturity T.

Commentary on Question:
Few people received full marks on this part of the question. Partial credits were given to candidates who failed to derive the value of m but were able to express $U(S_t,t)$ *in terms of m and* S_t.

$$U(S_t,t) = S_t^m e^{(r+\sigma^2)(T-t)}$$
$$\frac{\partial U(S_t,t)}{\partial S_t} = mS_t^{m-1}e^{(r+\sigma^2)(T-t)}$$
$$\frac{\partial^2 U(S_t,t)}{\partial {S_t}^2} = m(m-1)S_t^{m-2}e^{(r+\sigma^2)(T-t)}$$
$$\frac{\partial U(S_t,t)}{\partial t} = S_t^m e^{(r+\sigma^2)(T-t)}(-r-\sigma^2)$$

Using the Black-Scholes PDE we find

$$S_t^m e^{(r+\sigma^2)(T-t)}(-r-\sigma^2) + \frac{1}{2}\sigma^2 S_t^2 m(m-1)S_t^{m-2}e^{(r+\sigma^2)(T-t)}$$
$$= rS_t^m e^{(r+\sigma^2)(T-t)} - rS_t m S_t^{m-1}e^{(r+\sigma^2)(T-t)}$$

Therefore

$$(-r-\sigma^2) + \frac{1}{2}\sigma^2 m(m-1) = r - rm$$

It follows that

$$(\frac{\sigma^2}{2}(m+1)+r)(m-2) = 0$$

Since $\frac{\sigma^2}{2}(m+1)+r > 0$, we find $m = 2$.
Thus $U(S_t,t) = S_t^2 e^{(r+\sigma^2)(T-t)}$.

At time T, $U(S_T,T) = S_T^2$. The derivative's payoff at time T is square of the underlying stock price at time T.

Alternative solution for $m = 2$:
Note that $e^{-r(T-t)}U(S_t,t)$ is a martingale and
$e^{-r(T-t)}U(S_t,t) = S^m e^{(r+\sigma^2)(T-t)-r(T-t)} = S^m e^{\sigma^2(T-t)}$.

5. **Continued**

Next applying Ito's lemma we have

$$d\left(S^m e^{\sigma^2(T-t)}\right)$$
$$= \left[(-2r-\sigma^2)S^m e^{(r+\sigma^2)(T-t)-rt} + \frac{1}{2}\sigma^2 m(m-1)S^m e^{(r+\sigma^2)(T-t)-rt}\right]dt$$
$$+ \left(mS^{m-1}e^{(r+\sigma^2)(T-t)-rt}\right)dS$$
$$= \left[(-2r-\sigma^2)S^m e^{(r+\sigma^2)(T-t)-rt} + \frac{1}{2}\sigma^2 m(m-1)S^m e^{(r+\sigma^2)(T-t)-rt}\right.$$
$$\left. + rmS^m e^{(r+\sigma^2)(T-t)-rt}\right]dt + \left(\sigma mS^m e^{(r+\sigma^2)(T-t)-rt}\right)dW$$

Since $e^{-r(T-t)}U(S_t,t)$ is a martingale, the drift term vanishes and thus

$$(-2r-\sigma^2)S^m e^{(r+\sigma^2)(T-t)-rt} + \frac{1}{2}\sigma^2 m(m-1)S^m e^{(r+\sigma^2)(T-t)-rt} + rmS^m e^{(r+\sigma^2)(T-t)-rt} = 0$$

That is

$$(-2r-\sigma^2) + \frac{1}{2}\sigma^2 m(m-1) + rm = 0, i.e., (m-2)\left(r + \frac{m+1}{2}\sigma^2\right) = 0$$

Thus m = 2, and $U(S_t,t) = S_t^2 e^{(r+\sigma^2)(T-t)}$.

(c)

(i) Assess whether your intern's method is expected to underestimate, match, or overestimate $U(S_t,t)$.

(ii) Recommend revisions for each of the above steps if any are needed to correctly price this derivative.

Commentary on Question:

Candidates have to determine if the function $U(S_t,t)$ is concave or convex in order to determine whether the intern underestimated/ overestimated the price. For part (i), full credits were given only when candidates had a correct assessment with a proper supporting explanation. Partial credits were given if the candidate weren't able to derive m from part (b) but gave the correct argument depending on the range of m.(when m>1 or 0<m<1 or m<0)

5. **Continued**

(i) The intern is underestimating the price of the derivative.
There are 2 ways to explain this:
$U(S_T, T) = S_T^2$ is a convex function, and therefore $U(E[S_T], T) < E[U(S_T, T)]$ by Jensen's inequality.

The intern's steps are estimating the discounted value of $U(\mathrm{E}[\mathrm{S}_T], T) = E[S_T]^2$, whereas the derivative's price is the discounted value of $E[U(\mathrm{S}_T, \mathrm{T})] = E\left[{S_T}^2\right] = E[S_T]^2 + VAR[S_T]$

(ii) The steps should be modified as follow:
Keep step 1 the same
Step 2: calculate the derivative's payoff at time T under each simulation in step 1
Step 3: calculate the expected payoff of the derivative based on the simulated payoffs in step 2
Step 4: discount the expected payoff by risk free rate

6. Learning Objectives:

1. The candidate will understand the fundamentals of stochastic calculus as they apply to option pricing.

3. The candidate will understand the quantitative tools and techniques for modeling the term structure of interest rates and pricing interest rate derivatives.

Learning Outcomes:

(1a) Understand and apply concepts of probability and statistics important in mathematical finance.

(1c) Understand Ito integral and stochastic differential equations.

(1d) Understand and apply Ito's Lemma.

(1h) Define and apply the concepts of martingale, market price of risk and measures in single and multiple state variable contexts.

(3a) Understand and apply the concepts of risk-neutral measure, forward measure, normalization, and the market price of risk, in the pricing of interest rate derivatives.

(3e) Understand and differentiate between the classical approach to interest rate modelling and the HJM modeling approach, including the basic philosophy, arbitrage conditions, assumptions, and practical implementations.

Sources:

Hirsa, Neftci, 3rd Edition, Chapters 10 and 11; Wilmott, Chapter 19

Commentary on Question:

Candidates struggled with several parts of this question. Because many of the parts required the candidates to "show" all intermediate steps to derive the result to receive full credits, candidates had to illustrate complete intermediate steps with justification to arrive at a given answer.

Solution:

(a) Show that $e^{-\int_0^t r(u)du} \; P(t,T)$ is a martingale using Ito's Lemma.

Commentary on Question:

To receive full credits on this part of the question, candidates had to demonstrate the ability of applying Ito's Lemma correctly and state that the result has zero drift with bounded diffusion. Partial credits were awarded for candidates who applied Ito's Lemma correctly but failed to mention that resulting zero drift implied it was a martingale.

6. Continued

Applying Ito's Lemma to $e^{-\int_0^t r(u)du}\ P(t,T)$ gives

$$d\left(e^{-\int_0^t r(u)du}P(t,T)\right) = d\left(e^{-\int_0^t r(u)du}\right)P + e^{-\int_0^t r(u)du}dP + d\left(e^{-\int_0^t r(u)du}\right)dP$$

$$= -r(t)e^{-\int_0^t r(u)du}P(t,T)dt$$
$$+ e^{-\int_0^t r(u)du}(r(t)P(t,T)dt - P(t,T)\sigma(t,T)dW_t)$$
$$= -e^{-\int_0^t r(u)du}P(t,T)\sigma(t,T)dW_t.$$

Since this has zero drift and the diffusion is bounded, it is a martingale.

(b) Show that $P(t,T) = \mathbb{E}^{\mathbb{Q}}[e^{-\int_t^T r(u)du}\ |\mathcal{F}_t]$ using the result in part (a).

Commentary on Question:
To receive full credits, candidates had to clearly state that P(T,T)=1 and the exponential can be moved into the expectation by properties of martingales.

The properties of martingales allow us to write
$e^{-\int_0^t r(u)du}P(t,T) = \mathbb{E}^{\mathbb{Q}}\left[\left(e^{-\int_0^T r(u)du}P(T,T)\right)\middle|\mathcal{F}_t\right]$. Since *P(T,T)=1*, after moving the exponential term from the left into the expectation and simplifying, the result is obtained.

(c) Show that the solution of the SDE for $P(t,T)$ is as given below:

$P(t,T) = P(0,T)\exp\left(\int_0^t r(s)ds - \frac{\alpha^2}{6}(T^3 - (T-t)^3) - \alpha\int_0^t (T-s)\,dW_s\right)$
(Hint: First obtain the SDE for $lnP(t,T)$ using Ito's Lemma).

Commentary on Question:
Partial credits were awarded for identifying correct partial derivatives and for integrating correctly. Full credits were awarded only for illustrating all intermediate steps from the SDE given in the problem statement to the final solution for P(t,T).

6. Continued

$$\frac{\partial lnP(t,T)}{\partial P} = \frac{1}{P(t,T)}$$
$$\frac{\partial^2 \ln P(t,T)}{\partial^2 P} = -\frac{1}{P(t,T)^2}$$

Now

$$d \ln P(t,T) = \frac{\partial \ln P(t,T)}{\partial t} dt + \frac{1}{2}\frac{\partial^2 \ln P(t,T)}{\partial^2 P}(dP)^2 + \frac{\partial \ln P(t,T)}{\partial t} dP$$
$$= -\frac{1}{2}\frac{1}{P(t,T)^2}(dP)^2 + \frac{1}{P(t,T)} dP$$
$$= -\frac{1}{2}\sigma(t,T)^2 dt + (r(t) - \sigma(t,T)dW_t)$$
$$= \left(r(t) - \frac{\alpha^2(T-t)^2}{2}\right) dt - \alpha(T-t)dW_t$$

Upon rearranging and substituting $\alpha(T-t)$ for *σ(t,T)*. Integrate this result from 0 to *t* and exponentiate both sides to obtain the desired result:

$$P(t,T) = P(0,T)\exp(\int_0^t r(s)ds - \frac{\alpha^2}{6}(T^3 - (T-t)^3) - \alpha\int_0^t (T-s)dW_s).$$

(d) Show that

$$r(t) = F(0,t) + \frac{\alpha^2 t^2}{2} + \alpha W_t.$$

Commentary on Question:
To receive full credits on this part of the questions, candidates had to present F(0,T), F(t,T), and r(t)=F(t,t) in their solutions.

Since $F(t,T) = -\frac{\partial \ln P(t,T)}{\partial T}$, we use the result in (c) to take the derivative with respect to *T*:

$$F(t,T) = -(\frac{\partial \ln P(0,T)}{\partial T} - \frac{\alpha^2}{6}(3T^2 - 3(T-t)^2) - \alpha\int_0^t dW_s)$$
$$= F(0,T) + \frac{\alpha^2}{2}(T^2 - (T-t)^2) + \alpha\int_0^t dW_s$$
$$= F(0,T) + \frac{\alpha^2}{2}(T^2 - (T-t)^2) + \alpha W_t.$$

As $r(t) = F(t,t)$, we obtain $r(t) = F(0,t) + \frac{\alpha^2 t^2}{2} + \alpha W_t$.

(e) Evaluate the expectation in part (b) by using the result in part (d).

6. Continued

(Hint: For fixedt $< \mathrm{T}, \alpha \int_t^T W_s ds \,|\mathcal{F}_t$ is normally distributed random variable with mean $\alpha W_t(T-t)$ and variance $\frac{\alpha^2(T-t)^3}{3}$.)

Commentary on Question:
Candidates received partial credits on this part of the question by showing Equation 1below and identifying the correct mean and variance of the normally distributed random variable $\int_t^T r(u)du\,|\mathcal{F}_t$.

Integrating the result in (d) from *t* to *T* gives

$$\int_t^T r(u)du = \int_t^T F(0,u)du + \frac{\alpha^2}{2}(T^3 - t^3) + \alpha \int_t^T W_u\, du \ (Equation\ 1).$$

Using the hint, $\int_t^T r(u)du|\mathcal{F}_t$ is a normally distributed random variable with mean $\mathbb{E}^{\mathbb{Q}}\left[\int_t^T r(u)du|\mathcal{F}_t\right] = \int_t^T F(0,u)\,du + \frac{\alpha^2}{6}(T^3 - t^3) + \alpha W_t(T-t)$ and variance $Var[\int_t^T r(u)du|\mathcal{F}_t] = \frac{\alpha^2}{3}(T-t)^3$.

Hence $e^{-\int_t^T r(u)du}\ |\mathcal{F}_t$ is a lognormal random variable so its expectation is given by

$$\begin{aligned}
&\mathbb{E}^{\mathbb{Q}}\left[\left(e^{-\int_t^T r(u)du}\right)\Big|\mathcal{F}_t\right] \\
&= \exp\left(-\int_t^T F(0,u)du - \frac{\alpha^2}{6}(T^3 - t^3) - \alpha W_t(T-t) + \frac{\alpha^2}{6}(T-t)^3\right) \\
&= \frac{P(0,T)}{P(0,t)}\exp\left(-\frac{\alpha^2}{6}(T^3 - t^3) - \alpha W_t(T-t) + \frac{\alpha^2}{6}(T-t)^3\right) \\
&= \frac{P(0,T)}{P(0,t)}\exp(\frac{\alpha^2}{6}((T-t)^3 - T^3 + t^3) - \alpha(T-t)W_t)
\end{aligned}$$

as $\exp(-\int_t^T F(0,u)du) = \frac{P(0,T)}{P(0,t)}$.

(f) Show that evaluating the solution from part (c) leads to the same result as obtained in part (e).

Commentary on Question:
Very few candidates attempted and received credits on this part of the question. More justification was required than providing the solution from (e) and stating it is the same as the solution from (c).

6. Continued

Examining the exponential terms from part (c) and substituting in the result from Equation 1 for $\int_0^t r(s)ds$ gives

$$\int_0^t r(s)ds - \frac{\alpha^2}{6}(T^3 - (T-t)^3) - \alpha \int_0^t (T-s)dW_s$$
$$= \int_0^t F(0,u)du + \frac{\alpha^2}{6}t^3 + \alpha \int_0^t W_u \; du - \frac{\alpha^2}{6}(T^3 - (T-t)^3) - \alpha \int_0^t (T-s)dW_s.$$

But integrating by parts gives $\alpha \int_0^t (T-s)dW_s = \alpha(T-t)W_t + \alpha \int_0^t W_u du.$
Thus we have

$$\int_0^t F(0,u)du + \frac{\alpha^2}{6}(t^3) - \frac{\alpha^2}{6}(T^3 - (T-t)^3) + \alpha \int_0^t W_u \; du - \alpha \int_0^t (T-s)dW_s$$
$$= \int_0^t F(0,u)du + \frac{\alpha^2}{6}t^3 - \frac{\alpha^2}{6}(T^3 - (T-t)^3) - \alpha(T-t)W_t.$$

Noting $\int_0^t F(0,u)du = -\ln P(0,t)$ and substituting this result into the exponential expression in (c) and simplifying gives

$$P(t,T) = P(0,T)\exp(\int_0^t F(0,u)du + \frac{\alpha^2}{6}t^3 - \frac{\alpha^2}{6}(T^3 - (T-t)^3) - \alpha(T-t)W_t)$$
$$= P(0,T)\exp(-\ln P(0,t) + \frac{\alpha^2}{6}t^3 - \frac{\alpha^2}{6}(T^3 - (T-t)^3) - \alpha(T-t)W_t)$$
$$= \frac{P(0,T)}{P(0,t)}\exp(\frac{\alpha^2}{6}((T-t)^3 - T^3 + t^3) - \alpha(T-t)W_t).$$

7. **Learning Objectives:**

2. The candidate will understand how to apply the fundamental theory underlying the standard models for pricing financial derivatives. The candidate will understand the implications for option pricing when markets do not satisfy the common assumptions used in option pricing theory such as market completeness, bounded variation, perfect liquidity, etc. The Candidate will understand how to evaluate situations associated with derivatives and hedging activities.

4. The candidate will understand the concept of volatility and some basic models of it.

Learning Outcomes:

(2a) Identify limitations of the Black-Scholes pricing formula

(2b) Compare and contrast the various kinds of volatility, (eg actual, realized, implied, forward, etc.).

(2c) Compare and contrast various approaches for setting volatility assumptions in hedging.

(2d) Understand the different approaches to hedging.

(2e) Understand how to delta hedge and the interplay between hedging assumptions and hedging outcomes.

(2f) Appreciate how hedge strategies may go awry.

(4b) Understand and apply various techniques for analyzing conditional heteroscedastic models including ARCH and GARCH.

Sources:

Tsay Chapter 3.8; QFIC-104-13_Diebold Financial Risk Chapter 3

Commentary on Question:

This question tests candidates' understanding of volatility concept, including the basic model used for volatility hedging and the associated underlying assumptions as well as limitation. Each part of the question can be viewed as an independent question but they are connected, thus to obtain full credits candidates need to show enough understanding of volatility.

Solution:

(a) Describe mild randomness and wild randomness.

7. Continued

Commentary on Question:
This part of the question tests candidates' fundamental knowledge of the volatility (mild vs. wild randomness). A small number of candidates fully understood the concept. Most candidates only received some partial credits.

Mild randomness:
The random event that can be modeled by a Gaussian curve (Normal distribution). There are many random variables that can be modeled by Gaussian: height, weight and intelligence of people.
If a population follows a mild type randomness, one single observation will disproportionately impact the total. These follow thin tails, smaller probability of extreme outcomes.

Wild randomness:
A single observation can impact the total in a disproportionately. These follow fat tails, meaning high probability of rare events.
Wild random events, examples: 1000 Americans are in a stadium and Bill Gates walk in; there total wealth increased disproportionally.

(b) Compute the predicted volatility $\sigma_h(10)$ on the 10th day from the calibrated day.

Commentary on Question:
This question is a simple calculation if candidates were able to identify the correct formula to use. However, a large number of candidates failed to identify the correct info provided in the question to derive predicted volatility and finished at the predicted variance.

Predicted variance $\sigma_h^2(l) = \sigma_h^2(1) + (l-1)\alpha_0$

$$\sigma_h^2(10) = 0.00012 + 9 * 0.000119 = 001191$$
$$\sigma_{h(10)} = 0.034551$$

(c) Compute 0.1% percentile of the M&P 500 index return distribution on October 24, 2016 and interpret the result.

Commentary on Question:
Most candidates received partial credits for identifying the normal distribution. Only a small number of candidates were able to correctly identify the percentile, normal distribution parameters to derive correct answers.

7. **Continued**

r_t is normally distributed random variable with mean 0.0067 and variance $\sigma_h^2(1) = 0.00012.$
Therefore 0.1% percentile = $0.0067 - 3.09 * .00012^{0.5} = -0.0277.$
We can be 99.9% certain that October 24, 2016 return is greater than -2.77%

(d) Provide reasons for Andre's initial success and his subsequent failure.

Commentary on Question:
This part of the question tests candidates' understanding of IGARCH model, its associated assumptions and limitation. To obtain full credits, candidates had to address both the initial success and subsequent failure.

Andre is using Delta hedging with relaxed constant volatility assumption.
Initial success is because the stock market does not show any wild randomness (i.e. shows mild randomness), and AR(1)-IGARCH(1,1) model can be used to describe its returns very well.
IGARCH(1,1) model is being widely used in practice (Risk Metric Var Calculation)
IGARCH(1,1) is an exponential smoothing model for a_t^2
The underlying assumption is that, ϵ_t,errors are normally distributed.
Also, using Black-Scholes model implies that continuously compounded index return is normal.

The subsequent failure is because the above conditions no longer hold when wild randomness volatility has occurred, and the IGARCH model does not predict rare events well.

(e) Suggest an alternative model that Andre could have used.

Commentary on Question:
Most candidates did poorly on this part of the question. Pareto or fat tail distribution should be used to capture the wild randomness.

The best model is fat tail model or Pareto type model where $P[X > x] = Kx^{-\alpha}$ This model provides a fatter tail than Normal.

8. Learning Objectives:

2. The candidate will understand how to apply the fundamental theory underlying the standard models for pricing financial derivatives. The candidate will understand the implications for option pricing when markets do not satisfy the common assumptions used in option pricing theory such as market completeness, bounded variation, perfect liquidity, etc. The Candidate will understand how to evaluate situations associated with derivatives and hedging activities.

Learning Outcomes:

(2a) Identify limitations of the Black-Scholes pricing formula

(2c) Compare and contrast various approaches for setting volatility assumptions in hedging.

(2d) Understand the different approaches to hedging.

(2g) Describe and explain some approaches for relaxing the assumptions used in the Black-Scholes formula.

Sources:

Quantitative Finance, Wilmott, Paul, 2nd Edition Ch. 8-10

Wilmott, Paul, *Frequently Asked Questions* in *Quantitative Finance*, 2nd Edition Ch. 2: Q38, Q39

Commentary on Question:

This question tests candidates' understanding of the volatility used in the Black-Scholes formula, and various concepts related to volatility.

Solution:

(a)

(i) Identify the above volatility pattern and explain why it occurs.

(ii) Describe how you can use the concept of the deterministic volatility surface and stochastic volatility models to explain this pattern.

Commentary on Question:

Most candidates answered part i) well, and identified the volatility smile/skew, as well as reasons why it occurs.

Candidates struggled with part ii). The ideal answer would be to define deterministic volatility and define stochastic volatility, and explain how these produce the volatility surface/smile.

8. **Continued**

Part i)

The volatility pattern is called volatility skew. This occurs because:

Supply/Demand:
People have demand for insurance and drive up the price of low strike puts. This increases IV at low strike.
People sell OTM calls to earn premium and this creates an oversupply. This will decrease IV at high strike.

IV reflects stock return kurtosis:
Stock returns are not normal and actually have fat tails relative to a normal distribution. Therefore the expected pay-off for very high/low strike options is higher than implied by a normal distribution, therefore a higher IV.

Correlation of volatility and stock returns:
When stock prices fall dramatically, there is often a temporary increase in volatility. To capture this effect, low strike options have higher IV.

Volatility gamma:
OTM options have much high volatility 'gamma' and this is incorporated into option price (driving up IV).

Part ii)

Deterministic Volatility Surface:

- Volatility is a known function of stock (S) and time (t) and this function outputs the volatility skew
- Is essentially an "inverse" problem to back-solve the actual volatility using market prices (implied volatility)
- Captures the volatility surface at a point in time but does not capture the dynamics and evolution of the surface over time

Stochastic Volatility:

- Stochastic volatility models have two sources of randomness: one for stock returns and one for volatility
- The correlation between stock return and volatility is an additional parameter that is typically negative. As a result, falls in stock price result in higher volatility, explaining the volatility skew
- Has greater potential for capturing dynamics and evolution of the surface over time
- Has problems when trying to determine which stochastic volatility mode to use and also when trying to determine appropriate parameters

8. Continued

(b) Prove that $\frac{\partial Vega}{\partial \sigma} = Vega \frac{d_1 d_2}{\sigma}$.

Commentary on Question:
This question is an application using basic calculus (chain rule and, depending on the solution approach, quotient rule).

From formula sheet: Vega = $\frac{S\sqrt{t}e^{-\mathrm{qt}-0.5(d_1^2)}}{\sqrt{2\pi}}$

$$\frac{\partial Vega}{\partial \sigma} = \frac{\partial}{\partial \sigma}(\frac{S\sqrt{t}e^{-qt-0.5(d_1^2)}}{\sqrt{2\pi}})$$
$$= (\frac{S\sqrt{t}e^{-qt-0.5(d_1^2)}}{\sqrt{2\pi}}) * \frac{\partial}{\partial \sigma}\ (-0.5(d_1^2))$$
$$= \text{Vega} * \ (-d_1) * \frac{\partial}{\partial \sigma}(\ d_1\ )$$
$$= \text{Vega} * \ (-d_1) * \frac{\partial}{\partial \sigma}\frac{ln\left(\frac{S}{K}\right)+(r-q+0.5*\sigma^2)*t}{\sigma\sqrt{t}}$$
$$= \text{Vega} * \ (-d_1) * \ \frac{\sigma t*\sigma\sqrt{t} - \ \sqrt{t}*\left(\ln\left(\frac{S}{K}\right)+(r-q+0.5*\sigma^2)*t\right)}{\sigma^2 t}$$
$$= \text{Vega} * \ (-d_1) * \ \frac{\sigma^2 t - \ \left(ln\left(\frac{S}{K}\right)+(r-q+0.5*\sigma^2)*t\right)}{\sigma^2\sqrt{t}}$$
$$= \text{Vega} * \ (-d_1) * \ \frac{-\left(ln\left(\frac{S}{K}\right)+(r-q-0.5*\sigma^2)*t\right)}{\sigma^2\sqrt{t}}$$
$$= \text{Vega} \ * \ (-d_1) * \ \ (-d_2) \ \ * \frac{1}{\sigma}$$
$$= \text{Vega} \ * \ (\frac{d_1 d_2}{\sigma})$$

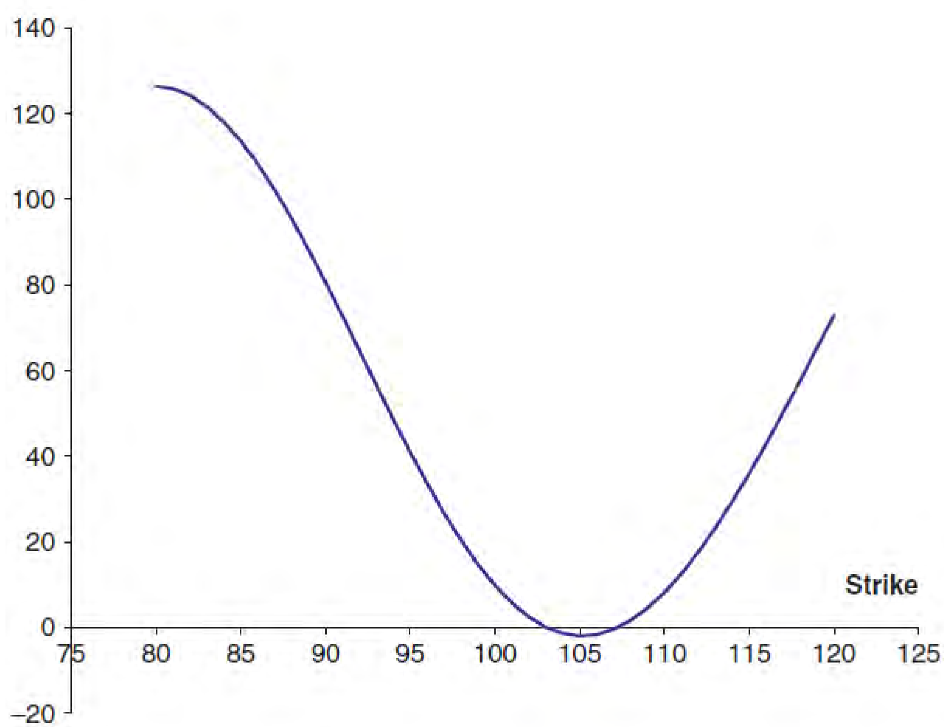
(c) Plot a graph of $\frac{\partial Vega}{\partial \sigma}$ versus the strike price and comment on the pattern observed.

Commentary on Question:
Most candidates were able to produce the correct values for X and Y and draw an appropriate graph, but few candidates were able to discuss the results.

8. **Continued**

		Black-Scholes Statistics			*Answer*
Strike	**Implied Volatility** σ	$\boldsymbol{d_1}$	$\boldsymbol{d_2}$	**Vega**	$\frac{\partial Vega}{\partial \sigma}$
1855	23.0%	0.244	0.081	520.3	44.7
1880	23.0%	0.162	-0.001	529.0	-0.4
1905	23.0%	0.081	-0.082	534.2	-15.4
1930	23.0%	0.000	-0.162	536.0	0 = X
1955	23.0%	-0.079	-0.241	534.3	44.2 = Y

As noted in the Wilmott reading, the volatility gamma (vomma) pattern should look something like:



$\partial^2 V/\partial\sigma^2$ versus strike.

Note that the strikes and $\frac{\partial Vega}{\partial \sigma}$ should be the ones used in the table. Also, the tapering at lower strikes will not be seen, since only 5 data points are used.

8. Continued

Some comments about the pattern observed:

- Parabola shape (and/or tapering expected at lower strikes)
- Vomma is positive OTM/ITM (so vega has positive sensitivity to volatility change when OTM/ITM)
- Vomma is negative close to ATM (so vega has negative sensitivity to volatility change when near ATM)
- Vomma increases moving away from ATM (so vega sensitivity to volatility is increasing as one moves deeper OTM/ITM)
- Vomma is positive when both d_1 and d_2 are positive, or both d_1 and d_2 are negative
- Vomma is negative when d_1 and d_2 have different signs
- Vomma is zero when d_1 or $d_2 = 0$

9. **Learning Objectives:**

2. The candidate will understand how to apply the fundamental theory underlying the standard models for pricing financial derivatives. The candidate will understand the implications for option pricing when markets do not satisfy the common assumptions used in option pricing theory such as market completeness, bounded variation, perfect liquidity, etc. The Candidate will understand how to evaluate situations associated with derivatives and hedging activities.

Learning Outcomes:

(2c) Compare and contrast various approaches for setting volatility assumptions in hedging.

(2d) Understand the different approaches to hedging.

(2e) Understand how to delta hedge and the interplay between hedging assumptions and hedging outcomes.

Sources:

Paul Wilmott Introduces Quantitative Finance, Wilmott, Paul, 2nd Edition: Ch. 8, Ch. 10

Commentary on Question:

Candidates generally did poorly of this question. For those who made the attempt, most of them received some credits only on parts (a) and (c) the question.
The question is relatively demanding in application and knowledge utilization because it tests the exam material in a more real-life-like situation. The question tries to present one of the situations where the option pricing formula and hedging can be used in life insurance practice (with simplification for exam purpose). The calculation behind the practice setting is a basic Black-Scholes option value calculation. Unfortunately, most candidates struggled to put together the information and translate it into a calculation task.

Solution:

(a) Determine the equity option; specify the maturity, strike, exercise and calculate the cost of the option if the participation rate p is 100%.

Commentary on Question:

Most candidates made the attempt to answer part (a) of the question and received credits for the Black-Scholes option pricing calculation.
Common mistakes made by candidates who did not received full marks on this part include:

1. *Failure to calculate the correct strike K. Some candidates used 1% as annual effective rate; some candidates treated it as an At-the-money option.*
2. *Failure to identify the type of the option: considerable number of candidates calculated it as a put option. In that case they were given partial credits for correct calculation using the appropriate parameters.*

9. **Continued**

EFG is allowing policyholder to participate in the upside potential of the underlying equity index, which from policyholder's perspective they long a call option. Since the guaranteed amount (defined by the 1% growth) is backed by a zero-coupon bond, any excess return above 1% from the equity index growth will be EFG's obligation.

To hedge the upside equity risk, EFG needs to purchase call option.

- The option maturity is 5 years
- The exercise is European, the option can only be exercised at the end of its life (5 years)
- The strike K = \$2000 * $e^{(0.01)*5}$= \$2102.54
  (It is also correct if candidate used the total account value \$100 Million. In that case, the strike K = \$100M * $e^{(0.01)*5}$ = 2102.54 * 50000 = \$105,127,110 or \$105.127 Million.)

Using Black-Scholes formula to calculate the option price:

$$C = N(d_1) \cdot S - N(d_2) \cdot Ke^{-r(T-t)}$$

$$P = C + Ke^{-r(T-t)} - S$$

$$d_1 = \frac{\ln\left(\frac{S}{K}\right) + \left(r + \frac{\sigma^2}{2}\right)(T-t)}{\sigma\sqrt{T-t}}$$

$$d_2 = d_1 - \sigma\sqrt{T-t}$$

Key factors of Black-Scholes formula

- S = \$2000 (or \$100 Million)
- K = \$2102.54 (or \$105.127 Million)
- r = 3%
- q = 0%
- T = 5
- σ =16%

$$d_1 = \frac{log\left(\frac{2000}{2102.54}\right)+\left(0.03+\frac{1}{2}*0.16^2\right)*5}{0.16*\sqrt{5}} = 0.45$$

Given the standard normal table N (0.45) = 0.6736

$$d_2 = d_1 - \sigma\sqrt{T} = 0.45 - 0.16 * \sqrt{5} = 0.10$$

9. Continued

$N(d_2) = N(0.10) = 0.5398$

Call option price ($2000 as base)
= $2000 * N(0.45) – $2102.54*$e^{-0.03*5}$*N(0.10)
=$375.97

OR

Call option price ($100 million as base)
$100 Million * N(0.45) – $105.127 Million *$e^{-0.03*5}$*N(0.10)
=$18,798,500 or $18.8 Million

(b) Determine the participation rate p in order for EFG to break-even, given the hedge costs and profits target.

Commentary on Question:
Candidates did rather poorly on part (b). Most candidates who made an attempt on this part of the question received credits for calculating the correct cost & profits. Only a few candidates demonstrated their understanding of the relationships among bond price, cost & profits, option price and the participation rate.

Total Premium Collected = ZCB price + Cost & Profits + Option Price * p

Approach 1: using the total account value ($100 Million)

To meet the minimum 1% per year crediting for the 5-year investment contract, the face value of zero-coupon Treasury bond should be

B = $100\ million * e^{(1\%-3\%)*5}$ = $90.48 million

Costs & profits deducted upfront = $100million * 1% = $1 million

Remaining budget for purchasing call option
= $100 million - $90.48 million - $1 million
= $8.52 million

The amount of money needed to purchase call option for $100 million face value
= $100 million/ $2000 * $375.97 = $18.80 million

This requires the participation rate that EFG can afford to provide to policyholders be $\frac{\$8.52M}{\$18.80M} = 45\%$.

9. Continued

Approach 2: using the $2000 as base

It's acceptable if a candidate calculated the bond face value using the $2000 base as follows

B = $\$2000 * e^{(1\%-3\%)*5}$= $1809.67

Costs & profits deducted upfront = 2000 * 1% = $20

Remaining budget for buying call options
= $2000 - $1809.67 - $20
=$170.33

Because the remaining budget for buying call options is $170.33, the participation rate that EFG can afford to provide to policyholders is $\frac{\$170.33}{\$375.97} = 45\%$.

(c) Calculate Rho of the option in part (a).

Commentary on Question:
Candidates had to provide both formula and numeric answer to receive full credits. For candidates who assumed a put option as the answer to part (a) of the question, partial credits were given if they calculated the correct Rho for put option.

$\text{Rho} = \text{K} * (\text{T} - \text{t}) * \text{e}^{-\text{r}*(\text{T}-\text{t})} * \text{N}(\text{d}_2)$
$= 2102.54 * 5 * \text{e}^{-0.03*5} * 0.5398$
$= 4884.31$

If candidate used $100 million as the calculation base, the Rho calculated will be 4884.31 * 50000 = $244,215,500 or $244.2155 million.

(d) Determine whether or not EFG should pursue the same hedging strategy, if the participation rate p determined in part (b) is maintained. Justify your answer.

Commentary on Question:
Candidates generally did poorly on part (d) of the question. Most likely because they struggled to derive the solution for part (b). No credit was given if candidates came up with the right conclusion by guessing but without any justification. Candidates were given partial credits by providing correct argument and explanation without calculation.

9. **Continued**

After the risk-free rate drops 50 basis points, the bond that needs to purchase to guarantee the minimum crediting rate (1%) has increased in value. A larger portion out of the premium collected needs to be spent on purchasing the zero-coupon bond, leaving less for purchasing call options if the same costs and profits target are to be maintained.

Approach 1: using the total account value ($100 million)

New bond price: $B_{new} = 100\ million * e^{(1\%-2.5\%)*5}$=$92.77 million

Costs & profits deducted upfront remains same = $100million * 1% = $1 million

New remaining budget for purchasing call option
= $100 million - $92.77 million - $1 million
= $6.23 million

Because of the drop in the interest rate, the call option price drops as well. The new call option price can be approximated using the Rho calculated:

New call option= $18.8 million – 244.2155 million * 0.0050 = $17.58 million

(Candidates were given full credit for full recalculation of the new call option price using Black-Scholes formula with r = 2.5%)

The amount of money needed to purchase new call option for $100 million face value with same participation rate of 45% determined in part (b)

= $17.58 million * 45% = $7.91 million

This is more than the new remaining budget of $6.23M.

Therefore, EFG cannot pursue the same hedge strategy under the new interest rate environment, otherwise it would lock in a loss of $7.91 million - $6.23 million = $1.68M

Approach 2: using the $2000 as base

$B_{new} = \$2000 * e^{(1\%-2.5\%)*5}$ = $1855.49

If the same costs & profits are to be maintained, $20 based on the $2000 premium collected, is deducted upfront.

9. Continued

The remaining option budget
= $2000 - $20 - $1855.49
= $124.51

Because of the drop in the interest rate, the call option price drops as well. The new call option price can be approximated using the Rho calculated:

New call option= $375.97 – 4884 * 0.0050 = $351.55

(Candidates were given full credit for full recalculation of the new call option price using a Black-Scholes formula with r = 2.5%)

The amount of money needed to purchase new call option for $2000 face value with same participation rate of 45% determined in part (b)

$351.55 * 45% = $158.20

This is more than the remaining option budget $124.51.

Therefore, EFG cannot pursue the same hedge strategy under the new interest rate environment, otherwise it would lock in a loss of $158.20 - $124.51 = $33.69

10. Learning Objectives:

6. The candidate will understand the variety of equity investments and strategies available for portfolio management.

Learning Outcomes:

(6g) Recommend and justify, in a risk-return framework, the optimal portfolio allocations to a group of investment managers.

(6j) Describe the process of identifying, selecting, and contracting with equity managers.

Sources:

Managing Investment Portfolios: A Dynamic Process, Maginn & Tuttle, 3rd Edition
Ch. 7 Equity Portfolio Management p. 432-433, 460-473

Commentary on Question:

Overall, this question tests the process of identifying, selecting and contrasting the equity managers. To receive full credits, the candidates had to demonstrate a relatively complete knowledge on the topic.

Solution:

(a) Evaluate and compare Manager A’s and Manager B’s efficiency of delivering active return by decomposing each manager’s performance into true and misfit components.

Commentary on Question:

This question tests the process of selecting and contrasting the equity managers. To receive full credits, the comparison had to include on the total and active components, based on return and information ratio.

Candidates generally did well on the returns part, but had trouble with the Information Ratio components. Some candidates did all the correct comparison of return and information ratio but failed to explicitly state which manager was better. Some of the candidates stated which manager was better without explaining why.

The model solution is based on active and total components, but calculation and comparison are based on misfit components also qualify for partial credits.

10. Continued

Manager A:
Outperformed benchmark by 12% - 11% = 1% (total active return)
Outperformed normal portfolio by 12% - 10% = 2% (true active return)
Total active risk = (0.0017)^0.5 = 4.1%
True active risk = (.0017 - .0014)^0.5 = 1.7%
Total information ratio = 1%/4.1% = 0.24
True information ratio = 2%/1.7% = 1.18

Manager B:
Outperformed benchmark by 13% - 11% = 2%
Underperformed normal portfolio by 13% - 15% = -2%
Total active risk = (.009)^0.5 = 9.5%
True active risk = (.009 - .0005)^0.5 = 9.2%
Total information ratio = 2%/9.5% = 0.21
True information ratio = -2%/9.2% = -0.22

Even though manager B has a higher total active return compared to manager A, manager B has a lower, and in fact, negative true active return. Credit was also awarded for describing performance relative to the benchmarks rather than the phrases "total active return" and "true active return".
After taking into account active risk, manager A has a higher total information ratio (0.24 vs 0.21).
Manager A also has a higher true information ratio.

Overall, manager A is a better manager.

(b) Describe 4 reasons why investors are more risk averse when facing active risk vs. total risk.

Commentary on Question:
Many candidates did not answer this part of the question, or only gave definitions for active and total risk. Statement on the difficulty for active management to beat benchmark consistently qualifies for partial credit.

Investors must believe that successful active management is possible;
Investors must believe that they have to skills to select active managers who will outperform;
Supervisors will judge investors on how well the overall portfolio performs relative to benchmark, but since active management is difficult, many active managers underperform;
As one moves up on the efficient frontier assuming more active risk, less manager diversification exists;

10. **Continued**

(c) Recommend topics that he should include in his questionnaire.

Commentary on Question:
A brief description was required to earn full marks for each topic. Many candidates were able to identify at least investment philosophy, performance, and fee schedule.

Staff/personnel and organization structure
E.g.: Describe the firm's organization and who will be managing the portfolio, vision of the firm, its competitive advantages, how it defines success, the organization of the group and the role of portfolio managers, traders, and analysts; the delegation of responsibility for decisions, the structure of the compensation program; the background of the professionals directly involved in managing the assets, the length of time the team has been together and the reasons for any turnover.

Investment philosophy and procedures
E.g.: Asks questions about how the equity portfolio will be managed, research process, risk management function, how the firm monitors the portfolio's adherence to its stated investment style, philosophy, and process; stock selection process, portfolio construction process.

Resources and how research is conducted and used
E.g.: Look at the allocation of resources within the organization, how and by whom research is conducted, the outputs of this research, how the research outputs are communicated, and how the research is incorporated into the portfolio construction process.

Performance
E.g.: What the equity manager considers to be an appropriate benchmark (and why), what level of excess return is appropriate, how performance is evaluated within the firm.

Fee schedule
E.g.: What is included in the fee, the type of fee (ad valorem or performance based), and any specific terms and conditions relating to the fees quoted

11. **Learning Objectives:**

5. The candidate will understand and identify the variety of fixed instruments available for portfolio management. This section deals with fixed income securities. As the name implies the cash flow is often predictable, however there are various risks that affect cash flows of these instruments. In general the candidates should be able to identify the cash flow pattern and the factors affecting cash flow for commonly available fixed income securities. Candidates should also be comfortable using various interest rate risk quantification measures in the valuation and managing of investment portfolios. Candidates should also understand various strategies of managing the portfolio against given benchmark.

8. The candidate will understand the theory and techniques of portfolio asset allocation.

Learning Outcomes:

(5i) Construct and manage portfolios of fixed income securities using the following broad categories.
 (i) Managing funds against a target return
 (ii) Managing funds against liabilities.

(8a) Explain the impact of asset allocation, relative to various investor goals and constraints.

(8b) Propose and critique asset allocation strategies.

(8c) Evaluate the significance of liabilities in the allocation of assets.

(8d) Incorporate risk management principles in investment policy and strategy, including asset allocation.

(8e) Understand and apply the concept of risk factors in the context of asset allocation.

Sources:

Managing Investment Portfolios: A Dynamic Process, Maginn & Tuttle, 3rd Edition

- Ch. 5 *Managing Investment Portfolios: A Dynamic Process*, Maginn & Tuttle, 3rd Edition
- Ch. 6, Fixed Income Portfolio Management

Commentary on Question:

Commentary listed underneath question component.

11. Continued

Solution:

(a) Assess whether ABC Insurance has a high ability to take risk or low ability to take risk. Justify your answer.

Commentary on Question:

Candidates needed to justify their assessment by providing justification.

Reasons supporting high risk tolerance:

- net insurance inflow in the next 8 years,
- long liability duration indicate long time horizon,
- Book value surplus way above required capital,
- no market value capital requirement.

Reasons supporting low risk tolerance:

- negative market value of surplus,
- regulation may change,
- insurance liability maybe under reserved.

(b)

(i) List four types of risks the benchmark portfolio faces.

(ii) Compare the benchmark portfolio with Efficient Portfolio H regarding the risks described in part (b)(i).

Commentary on Question:

Many of the candidates were able to name the key risks of the portfolio.

(i)

- Interest risk
- Yield curve risk
- Spread risk
- Credit risk,
- Sector risk (can't compare)
- Prepayment risk
- Convexity risk

(ii) Compared to the benchmark, H has

- lower interest risk because of lower duration
- higher yield curve risk because of maturity bucket concentration
- higher spread and credit risk because of lower average quality
- higher prepayment and convexity risk because of higher MBS and ABS
- higher equity risk (benchmark has non)

11. Continued

(c)

(i) Determine the most appropriate allocation on the efficient frontier for ABC.

(ii) Describe how you can achieve a portfolio with better return to risk trade-off than the solution in part (c)(i) by utilizing the capital allocation line.

(iii) Explain why your portfolio in part (c)(ii) might not be feasible to implement.

Commentary on Question:
Many candidates were able to calculate the utility for the portfolio. Some candidates mis-interpreted the risk/reward chart (i.e. flip the inputs backwards) and thus arrived at an incorrect conclusion. Some of candidates also computed Sharpe ratio for each of the portfolio to determine the optimal portfolio. Such approach is not ideal as it does not account the risk aversion factor.

(i) In order to maintain long term MV surplus, the portfolio need to earn on average of at least 2.5% (the discount rate of FV liability). Therefore A-F is not
UG=2.51-0.005*7*5.38^2=1.49

	A	B	C	D	E	F	G	H	I	J
Risk	1.697447	1.71784	1.899902	2.422795	3.551098	4.960235	5.375111	6.730765	7.52659	12.54356
Reward	0.777365	1.071836	1.352083	1.627966	1.986546	2.370851	2.509144	2.875372	3.081019	4.253128
Utility	0.676519	0.968552	1.225746	1.422518	1.545185	1.509713	1.497931	1.28976	1.098284	-1.2538

Not necessary to calculate all utilities. Enough to notice that it starts declining after H
G has highest utility with reward above 2.5% (2 points for final answer)

(ii) Joining the risk free asset and the tangent portfolio with the highest Sharpe ratio, one can produce a capital market line that is more efficient than the efficient frontier. Assuming that the Tangent portfolio is to the left of G portfolio (tangent portfolio is less risky than G), you will have to short the risk free asset to match G portfolio risk.

By shorting the risk free rate and investing the proceeds into the tangent portfolio, one can produce a portfolio with same risk as G by higher return.

11. Continued

(iii) Your company may not be able to borrow at risk free rate, depend on the borrowing rate, investing the market portfolio at margin may not be more efficient than simply investing in G.

(d)

(i) List the advantages of investing in agency MBS.

(ii) Identify challenges encountered modeling MBS.

Commentary on Question:
This is a relative straight forward part of the question. Most of the candidates were able to answer the question without challenges.

(i) High credit quality
Yield pick-up
Liquidity
Diversification

(ii) Prepayments
Path dependence of mortgage pools
nonlinearity of prepayment models

(e) Identify the efficient frontier model used by the Investment Department and assess the appropriateness of the investment portfolios determined above with respect to characteristics of ABC's liabilities.

Commentary on Question:
Most of the candidates were able to recognize the efficient frontier model was based on an asset only approach and subsequently proposed ALM is better. Candidates did quite well in this part of the question overall.

The efficient frontier starts with a low fixed income duration at low risk level for a company that has a liability duration of 12.

This suggests it is an asset-only efficient frontier. The asset-only efficient frontier does not correctly capture interest risks faced by an insurance company. In this case, ABC will face tremendous reinvestment risk if asset allocation A is being used, even though it is considered as a low risk portfolio in the efficient frontier.

11. **Continued**

(f) Recommend alternative approaches that may be more appropriate for determining ABC's long-term allocation. Justify your answer.

Commentary on Question:
This is an open end question. Any method that included liability values in the risk reward metrics would be acceptable, partial credit only for answering ALM without a method suggested .

For example, Bob should have used an ALM efficient frontier approach; one way to do that is by optimizing the risk and return of the market value surplus.

12. **Learning Objectives:**

5. The candidate will understand and identify the variety of fixed instruments available for portfolio management. This section deals with fixed income securities. As the name implies the cash flow is often predictable, however there are various risks that affect cash flows of these instruments. In general the candidates should be able to identify the cash flow pattern and the factors affecting cash flow for commonly available fixed income securities. Candidates should also be comfortable using various interest rate risk quantification measures in the valuation and managing of investment portfolios. Candidates should also understand various strategies of managing the portfolio against given benchmark.

Learning Outcomes:

(5h) Demonstrate an understanding of the characteristics and mechanics of fixed income ETFs.

(5i) Construct and manage portfolios of fixed income securities using the following broad categories.
- (i) Managing funds against a target return
- (ii) Managing funds against liabilities.

Sources:

Managing Investment Portfolios (MIP): A Dynamic Process, Maginn & Tuttle, 3rd edition, Chapter 6

The Handbook of Fixed Income Securities (HFIS), Fabozzi, 8th edition, Chapter 21

Commentary on Question:

This question tests candidates' understanding of ALM for relatively straightforward liabilities, the use of different instruments / derivatives to achieve the ALM objectives.

Most candidates scored high in identifying the advantages of using Futures derivative as an ALM practice, and were able to correctly identify the impact on Market Liquidity and ETF's creation cost when market condition shifts; however, very few candidates were able to answer fully key advantages and disadvantages of using ETF instead of Futures, and most struggled to accurately calculate the rebalancing activities needed or the number of Futures contracts required for the rebalancing.

Solution:

(a) Describe the limitations of using single period immunization technique in asset/liability matching (ALM) application to rebalance duration of the portfolio.

Commentary on Question:

Most candidates focused their answers on why Single-Period Immunization technique not working in Multi-period situation, however, the question was looking for a more general "description" of the limitations of this method.

12. Continued

There are four key limitations of using a Single-Period Immunization technique:

1. The immunization portfolio needs to use only high quality, very liquid instruments
2. The method only immunize for parallel yield curve movements
3. The technique assumes no interim cash flows before the fixed horizon date
4. The target value of the portfolio value does not change if no interest rate structure change

(b) Determine the new positions in securities TB-1 and TB-3, if action 1 were taken and the security TB-2 were sold entirely.

Commentary on Question:
Many candidates failed to correctly identify what the target liability duration the portfolio should target at, some candidates did not fully understand what the rebalancing activity was and still included TB-2 allocation. (Full credits were given for answers only differ by rounding.)

The targeted liability duration = 4 / 1.03 = 3.88 years
Hence the dollar duration = 3.88 x (140 + 105 + 135) = 1,475.728 K
Now suppose we put X in TB-1 and (380 – X) in TB-3
6.4 x X + 2.6 x (380 – X) = 1,475.728
X = 128.35 K
Therefore we shall allocation $128,350 into TB-1 and $251,650 into TB-3.

(c) Assess advantages and disadvantages of using futures contracts rather than actual securities repositions to adjust duration of a portfolio.

Commentary on Question:
Most candidates were able to identify the advantages, but not as many were doing well on the disadvantages side. Candidates were expected to elaborate (hence the verb "Assess") on how Basis Risk exists rather than simply list the "Basis Risk" in their answers.

Advantages of using Futures rather than actual securities:

1. Futures are more liquid
2. Futures generally are more cost-effective
3. Futures are very effective instruments for duration management

12. **Continued**

Disadvantages of using Futures rather than actual securities:

1. Basis Risk – the difference between the cash price and future price will change in an unpredictable way
2. Cross Hedging – securities to be hedged is not identical to the securities underlying the future contract
3. Cheapest-to-deliver (CTD) and Delivery Options (sum of quality option, timing option and wild card option) inherited in Futures contract means the long positioned securities and timing of delivery are unknown in advance.

(d) Determine the number of futures contracts you would short or long, if action 2 were taken.

Commentary on Question:
Many candidates failed to identify the correct liability duration target (many assumed 0 and some simply took the current asset duration).

Again, difference due to rounding in using Portfolio Duration rather than Dollar Duration received full credits.

The Dollar Duration of liability from part b) = $1,475,728.155
The Dollar Duration of current assets =
(140,000 x 6.4 + 105,000 x 1.4 + 135,000 x 2.6) = 1,394,000
This is less than the Dollar Duration of liabilities hence we need to long Futures.
# of Futures to long = (1,475,728.155 – 1,394,000) / (92 x 7.5) x 1.2 = 142
Hence Long 142 contracts

(e) Describe sources of liquidity for fixed income ETFs.

Commentary on Question:
Most candidates correctly identified that ETFs are traded on the Exchange; fewer went into details in describing other sources of liquidity from the underlying security liquidity or contingent Exchange liquidity through the use of limit orders.

Sources of liquidity for fixed income ETFs are:

1. The liquidity provided by two complementary markets – the Primary market, in which fund shares are created and redeemed, and the Secondary market, in which existing shares are traded throughout the day on a Stock Exchange
2. The liquidity in the underlying bond market which can be accessed through the creation / redemption process
3. The contingent liquidity in the Exchange which may be accessed through the use of limit orders
4. Authorized participants generally work with both investors and ETF providers to maintain liquidity in the primary market

12. Continued

(f) State the formula for the fixed income ETFs' premium/discount and describe each of its three components.

Commentary on Question:
Most candidates demonstrated a better understanding of the Flow Factor, some were able to correctly describe the concept of Creation Cost, but not many were describing Execution Risk Adjustment sensibly. (Stating "Execution Risk Adjustment is an adjustment for execution risk" will receive no credit, for example.)

A few candidates forgot to state the formula.

Fixed Income ETFs' Premium / Discount =
Creation Cost x Flow Factor + Execution Risk Adjustment

Creation Cost reflects the transaction cost such as bid / ask spread when originating the new FI ETF shares
Flow Factor is a scalar between 0 to 1, reflecting the net balance of ETF flows in the market (0 = all Sell orders; 1 = all Buy orders)
Execution Risk Adjustment is the compensation for execution and liquidity risk broker- dealer bear when executing trades and aggregating bond portfolios to facilitate bond creation and redemption.

(g) Assess advantages and disadvantages of using fixed income ETFs rather than futures contracts to adjust duration of a portfolio.

Commentary on Question:
Similar to part c), candidates scored better on the Advantages than Disadvantages

Advantages of using FI ETFs:
1. ETFs has a broader investment breadth than Futures contracts
2. ETFs has greater tax efficiency comparing to Future
3. ETFs has minimal counterparty risk

Disadvantages of using FI ETFs:
1. The level of execution risk adjustment may be significant in highly stressed market
2. Many underlying securities in ETF basket have discontinuous liquidity, make it virtually impossible to fully replicate a broad market benchmark.
3. Active FI ETF providers publish their holding on a daily basis, which limits managers to invest more liquid securities classes or hold securities till maturity. This minimizes the chances of taking advantages by opportunistic investors.

12. Continued

(h) Assess the impact on liquidity and creation cost of a fixed income ETF if bid/offer spread of underlying securities widen. Justify your answer.

Commentary on Question:
Most candidates were able to correctly identify the impact on liquidity and creation cost as a result of widened bid / offer spread of the underlying. Some did not justify their answers hence did not get full credits.

Bid / offer spread widening leads to a higher creation cost, as the transaction cost for pulling the ETF become higher.
Bid / offer spread widening reduces liquidity since the normal Exchange liquidity drops due to higher transaction costs.

13. Learning Objectives:

5. The candidate will understand and identify the variety of fixed instruments available for portfolio management. This section deals with fixed income securities. As the name implies the cash flow is often predictable, however there are various risks that affect cash flows of these instruments. In general the candidates should be able to identify the cash flow pattern and the factors affecting cash flow for commonly available fixed income securities. Candidates should also be comfortable using various interest rate risk quantification measures in the valuation and managing of investment portfolios. Candidates should also understand various strategies of managing the portfolio against given benchmark.

Learning Outcomes:

(5b) Demonstrate an understanding of par yield curves, sport curves, and forward curves and their relationship to traded security prices; and understanding of bootstrapping and interpolation.

Sources:

Paul Wilmott Introduces Quantitative Finance, Wilmott, Paul, 2nd Edition, Ch.14

Commentary on Question:

For this question, the candidates were expected to demonstrate an understanding of par yield curves, spot curves, and forward curves; their relationship to traded security prices, and an understanding of bootstrapping and interpolation.
Another part of the question was about the concept of Break-even inflation rate and its relation to nominal rates and TIPS.

In the first part related to the yield curve, the candidates did not perform as well as expected. They often did not use the correct formula in calculating the 2-year spot rate. They also often did not select the appropriate yield in calculating the 1-year forward 1-year rate and the repurchase price of a 6-month repo of a 10-year zero-coupon bond.

Solution:

(a) Define zero-coupon spot curve and implied forward curve.

Zero-coupon spot curve is the internal rate of return of zero-coupon bond of a certain maturity.

Implied Forward curve is the par or spot curve in a future time implied from current yields.

13. **Continued**

(b) Calculate 2-year spot rate, using the information given above.

$101.5(1+r)^{-\frac{1}{2}} = 100$; r = 0.030225

$1.4(1.030225)^{-\frac{1}{2}} + 101.4(1+r)^{-1} = 100$; r = 0.02818

$1.3(1.030225)^{-\frac{1}{2}} + 1.3(1.02818)^{-1} + 101.3(1+r)^{-\frac{3}{2}} = 100$; r = 0.026134

$$1.2(1.030225)^{-\frac{1}{2}} + 1.2(1.02818)^{-1} + 1.2(1.026134)^{-\frac{3}{2}} + 101.2(1+r)^{-2} = 100$$

r= 0.024083

(c) Calculate 1-year forward 1-year rate.

$$1+f = (1.024083)^2 / (1.02818) = 1.02000$$

(d) Calculate the repurchase price at which you will buy back.

$$10mm\,(1.030225)^{\frac{1}{2}} = 10.15mm$$

(e) Describe your interest rate exposure to the 10-year rate change under this repo agreement.

In repo agreement, since the seller will buy the security back at the specified price, the seller still have long-term interest rate exposure. [Make money when the rate decrease just like owning a bond].

(f)

(i) Calculate break-even annualized inflation rates for 1-year and 2-year periods.

(ii) Assess the market expected inflation rate over the next 2 years based on the calculation in part (f)(i). Justify your answer.

Commentary on Question:
The results were reasonable for the first part of the question. However, the second part was missed by most of the candidates. They usually try to estimate the figure itself of the 1-year or 2-year periods annualized inflation rate, while it was expected that they give an analytical relationship between the two based on the level of the risk premium in a nominal bond.

13. Continued

(i) Break-even infl. rate = (1+nom. yield)/(1+TIPS real yield) – 1
1-year BEIR = 1.02818/1.0095 – 1 = 1.85%
2-year BEIR = 1.02408/1.0075 – 1 = 1.64%

(ii) BEIR likely overstates market expected inflation rate since nominal bonds include inflation risk premium.
Researchers estimate inflation risk premium at 0.5% to 1%.
Market expected inflation rate likely 0.5% lower than calculated in (i).

14. Learning Objectives:

6. The candidate will understand the variety of equity investments and strategies available for portfolio management.

Learning Outcomes:

(6c) explain the basic active equity selection strategies including value, growth and combination approaches.

(6f) compare techniques for characterizing investment style of an investor.

(6h) Describe the core-satellite approach to portfolio construction with a completeness fund to control overall risk exposures.

Sources:

Maginn & Tuttle Chapter 7; QFIC-110-15 Liquidity as an Investment Style

Commentary on Question:

The main objective of the question was to describe the core-satellite approach to portfolio construction, to compare the styles of an active equity investment, and to characterize the investment style of an investor. The question did well in general.

Solution:

(a) Explain the merits of a core-satellite approach.

Core-satellite approach

The objective is to anchor with an index portfolio or enhanced index portfolio, and use active managers around the anchor to achieve acceptable level of active return.

It provides opportunity for upside to the benchmark with limited downside relative to the benchmark. It may add diversification.

(b) Describe the primary focuses, advantages, and major risks of the following styles of active equity investment.

(i) Growth

(ii) Market Oriented (Core or Blend)

(iii) Market Capitalization

(iv) Liquidity

14. Continued

Commentary on Question:
The candidates would have received more credits if they had explained more the advantages and risks of each style, and, in some cases, if they had a better description of the liquidity style.

(i) Growth

Having high sale growth relative to the overall market, trading at high price to earning (P/E) , price to book (P/B) , and price to sale ratios;

The risk is that the forecast EPS growth does not materialize as expected; P/E may contract at the same time as EPS amplifying investors' losses.

(ii) Market Oriented (Core or Blend)

Price of equity bellow its perceived intrinsic values; no matter of value / growth spectrum;

Risk: The portfolio achieves only market level of return;

Passive indexing or enhanced indexing equity investment based on broad equity market index will likely be lower cost and more effective alternatives.

(iii) Market Capitalization

Small Capitalization: lowest market capitalization of equities in the countries in which they invest; Mid Capitalization : middle capitalization of equities; Large Capitalization: large capitalization of equities;

Risk: Small Capitalization: assuming that better growth prospect or chance of earning high rate of return does not materialize;

Mid Capitalization: may be less well researched than the largest capitalization companies

Large Capitalization: can add value to investors with superior analysis and insight.

14. Continued

(iv) Liquidity

Turnover of equities; bid-Offer spread of equities;

Less liquid portfolio may involve tail risk;

The risk of needing to quickly liquidate positions in a crisis.

(c) Compare returns-based, holdings-based, and the style box approach to determining style.

Returns based

Developed by Sharpe, regresses returns on a set of indices, such as large-cap value, large-cap growth, and small stocks

Holdings based
categorizes securities by their characteristics to reach a determination.

Style box

Draw the box, 3x3, Large, mid and small on one axis, value, core, growth on other
Morningstar classifies stocks by box, and boxes add up to 100%

(d) Assess the Manager X's style.

Commentary on Question:
The candidates were able to identify some important characteristics of the investor's style in comparing the Manager's sector choices relative to the benchmark. However, many failed to assess the global style as it was more a combination of styles. They failed also to identify liquidity as a style.

Number of stocks and market cap says average market cap per stock in the index = $20B; mkt cap per stock in active portfolio is $40B = large cap style
High P/E, high P/B, low dividend yield = growth style
Growth portfolios tend to hold higher EPS share growth rates, but not this one, so this doesn't help
Higher turnover = high liquidity style = lower returns
Sector weightings (Finance, Utilities, Health Care, IT) all indicate value style, but this approach is not always correct.
Overall, large-cap growth, with high liquidity

14. Continued

(e) Manager X has recently started practicing socially responsible investing (SRI).

(i) Explain how SRI may have an impact on Active Manager X's style.

(ii) Explain the benefits to monitoring potential style bias arising from SRI screens.

Commentary on Question:
Generally speaking, most candidates did well on this part.

(i) Some screens may exclude basic industry and energy stocks, which are typically value stocks, making the portfolio more tilted towards growth;

SRI portfolios have found to be biased towards small cap stocks.

(ii) Portfolio manager can take steps to minimize the bias if it's inconsistent to the investor's objectives;

Portfolio manager can determine an appropriate benchmark for the SRI portfolio.

15. **Learning Objectives:**

7. The candidate will understand how to develop an investment policy including governance for institutional investors and financial intermediaries.

8. The candidate will understand the theory and techniques of portfolio asset allocation.

Learning Outcomes:

(7a) Explain how investment policies and strategies can manage risk and create value.

(7c) Determine how a client's objectives, needs and constraints affect investment strategy and portfolio construction. Include capital, funding objectives, risk appetite and risk-return trade-off, tax, accounting considerations and constraints such as regulators, rating agencies, and liquidity.

(7d) Incorporate financial and non-financial risks into an investment policy, including currency, credit, spread, liquidity, interest rate, equity, insurance product, operational, legal and political risks.

(8e) Understand and apply the concept of risk factors in the context of asset allocation.

Sources:

Maginn & Tuttle Chapter 3

Podkaminer, Risk Factors as Building Blocks

Commentary on Question:

Commentary listed underneath question component.

Solution:

(a) Propose risk objectives for the DB Plan.

Commentary on Question:

Most candidates were able to identify a single risk objective but full credit required listing multiple risk objectives. Key words were "maintain" or "minimize".

- Minimize surplus volatility
- Minimize shortfall risk
- Maintain funded status above certain level (fund liabilities)
- Minimize amount of future pension contributions
- Maintain or increase pension income impact of bottom line (maintain surplus; avoid bankruptcy of EFG)

15. Continued

(b) Assess the DB Plan's risk tolerance.

Commentary on Question:
Most candidates received full credits on this part of the question. To receive full credits candidates were required to state the overall low risk tolerance as well as the specific reasons for reaching that conclusion.

EFG has a low risk tolerance for the following reasons:

- Declining industry and falling earnings = low risk tolerance
- High debt to equity ratio = low risk tolerance
- Small surplus = higher risk tolerance (relative to no surplus)
- Assets 5 times larger than the market capitalization = very low risk tolerance
- Average age of EFG's workforce is 50 = low risk tolerance (older age)
- Liabilities for active lives represents 30% of total liabilities = low risk tolerance (more certain liability)
- Plan provides generous subsidies on early retirement = low risk tolerance

(c) Propose a return objective for the DB Plan.

Commentary on Question:
Most candidates were able to identify a single return objective. Common answers included offsetting liabilities or maintaining surplus.

- Match or exceed discount rate used to calculate PV of liabilities
- Return needed to make future contributions equal zero
- Return needed to maintain or increase pension accounting income
- Since the plan has a low risk tolerance, less aggressive return objective than for a plan with a high risk tolerance would be appropriate

(d) Identify the following IPS elements for the DB Plan:

(i) Liquidity requirements

(ii) Time Horizon

(iii) Tax Concerns

(iv) Legal and Regulatory Requirements

(v) Unique Circumstances

15. Continued

Commentary on Question:
Candidates did very well on this part of the question. The most common mistake was defining (v) Unique Circumstances

(i) Higher liquidity is needed due to greater number of retired lives, smaller corporate contributions relative to benefit disbursements, early retirement and lump sum payout options.

(ii) Shorter time horizon due to a mature plan, low proportion of active lives and relatively high age of the workforce. Partial credit was also given for calculating the number of years as long as the assumed age of retirement was specified.

(iii) Typically exempt from taxation

(iv) All retirement plans are governed by laws and regulation. In US, it is ERISA. In Canada, it is provincial regulations. If unions, Taft-Hartley applies.

(v) Human and financial resources for smaller plans, complex due diligence, self-imposed constraints against investing in certain industries.

(e) Explain the benefits of your approach.

Commentary on Question:
Candidates did not do well on this part of the question. Multiple benefits were required to receive full credits.

- Merely using risk, return, and correlation forecasts is insufficient to create robust portfolios. Credit was also given for answers related to the efficiency and diversification of using risk factors as benefits.
- Better to use an expanded collection of risk factors and construct the return-seeking portfolio with factors to prevent overlap with the liability hedge.
- Performing a surplus optimization using factors rather than asset classes leads to a greater consistency in portfolio construction.
- Overlaps and gaps become more readily apparent through the single lens of risk factors.

(f) Propose how a portfolio for the DB Plan could look like, expressed using the risk factors.

15. Continued

Commentary on Question:
Candidates did poorly on this part of the question.
To earn full credits, candidates had to:

- *Explain both the liability hedge and return-seeking parts of the portfolio;*
- *Stating the risks;*

Most candidates were able to state some risks, but failed to describe the liability hedge and return-seeking parts of the portfolio.

- Liability factor exposures expressed through physical bonds and derivatives benchmarked to a granular liability benchmark.
- Portfolio of risk factor exposures constructed to minimally overlap with the liability hedge.
- Portfolio with exposures to duration, inflation, credit quality, and other curve characteristics, also default, leverage, liquidity, GDP growth, manager skill, real interest rates, currency.

(g) Identify specific elements of the DC Plan IPS that distinguish it from the DB Plan IPS.

Commentary on Question:
Identifying the general differences of DB and DC plans without attaching to the IPS does not earn any credit. For candidates that did describe the IPS, the most common answers were related to #1 and #3 below.

1. Need to document adequate process for selecting and evaluating investment options. In other words, educating participants.
2. Includes procedures to ensure myriad of individual investors objectives and constraints are properly addressed.
3. Does not establish overall objectives and constraints. Risk/return objectives are with the participants.
4. Serves as governing document describing investment strategies available to a group of participants.
5. Lists choices to be offered and criteria for the funds.