SOCIETY OF ACTUARIES
Quantitative Finance and Investment Core

Exam QFICORE

MORNING SESSION

Date: Wednesday, November 2, 2016
Time: 8:30 a.m. – 11:45 a.m.

INSTRUCTIONS TO CANDIDATES

General Instructions

1. This examination has a total of 100 points. It consists of a morning session (worth 60 points) and an afternoon session (worth 40 points).

   a) The morning session consists of 10 questions numbered 1 through 10.

   b) The afternoon session consists of 5 questions numbered 11 through 15.

   The points for each question are indicated at the beginning of the question.

2. Failure to stop writing after time is called will result in the disqualification of your answers or further disciplinary action.

3. While every attempt is made to avoid defective questions, sometimes they do occur. If you believe a question is defective, the supervisor or proctor cannot give you any guidance beyond the instructions on the exam booklet.

Written-Answer Instructions

1. Write your candidate number at the top of each sheet. Your name must not appear.

2. Write on only one side of a sheet. Start each question on a fresh sheet. On each sheet, write the number of the question that you are answering. Do not answer more than one question on a single sheet.

3. The answer should be confined to the question as set.

4. When you are asked to calculate, show all your work including any applicable formulas. When you are asked to recommend, provide proper justification supporting your recommendation.

5. When you finish, insert all your written-answer sheets into the Essay Answer Envelope. Be sure to hand in all your answer sheets because they cannot be accepted later. Seal the envelope and write your candidate number in the space provided on the outside of the envelope. Check the appropriate box to indicate morning or afternoon session for Exam QFICORE.

6. Be sure your written-answer envelope is signed because if it is not, your examination will not be graded.

Tournez le cahier d'examen pour la version française.

1. (*4 points*) A risk-free asset B and two stocks, S and G, satisfy the following

$$dB_t = rB_t dt, \qquad B_0 = 1$$
$$dS_t = \mu S_t dt + \sigma S_t dW_t, \ S_0 = s_0$$
$$dG_t = v G_t dt + \tau G_t dW_t, G_0 = g_0$$

where W_t is a Wiener process under real world measure $\mathbb{P}$ and the coefficients r, μ, v, σ, and τ are constants.

Assume that the two stocks S and G pay continuous dividend D_t and E_t, respectively, where the dividend processes are given by:

$$dD_t = \sigma S_t dt$$
$$dE_t = \tau G_t dt.$$

You are given the following portfolio with weights θ_1, θ_2 and θ_3 which are not necessarily constant:

$$V_t = \theta_1 B_t + \theta_2 S_t + \theta_3 G_t$$

which is self-financing, that is,

$$dV_t = \theta_1 dB_t + \theta_2 \left(dS_t + dD_t \right) + \theta_3 \left(dG_t + dE_t \right).$$

Assume that the market is arbitrage-free. From Girsanov theorem there exists a Wiener process $\widetilde{W}_t$ under risk-neutral measure $\mathbb{Q}$ such that

$$dW_t = \gamma_t dt + d\widetilde{W}_t .$$

(a) (*2 points*) Prove that under the risk neutral measure $\mathbb{Q}$

$$dS_t = \left(r - \sigma \right) S_t dt + \sigma S_t d\widetilde{W}_t$$
$$dG_t = \left(r - \tau \right) G_t dt + \tau G_t d\widetilde{W}_t.$$

1.   Continued

Consider a contract with the following payout structure at maturity time *T:*

$$D_T - \lambda E_T$$

(b)   (*2 points*) Determine λ such that the price of the contract is 0 at time 0, assuming that $r \neq \sigma$ and $r \neq \tau$.

2. (*5 points*) Consider a stochastic process $S_t = W_t^2$ where W_t is a Wiener process with $W_0 = 0$.

Note that $E\left[\left(\dfrac{X-\mu}{\sigma}\right)^4\right] = 3$ if X is normally distributed with mean μ and standard deviation σ.

(a) (*2 points*) Determine the mean and variance of

$$\int_0^t W_s \, dW_s$$

(b) (*1 point*) Show that $e^{-rt} S_t$ is not a martingale if r is constant.

(c) (*2 points*)

   (i) Express $F_T = \int_0^T S_s \, ds$ in terms of S_T and $\int_0^T W_s^3 \, dW_s$.

   (ii) Determine the expectation of F_T.

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3. (*6 points*) The zero-coupon bond price $P(t,T)$ with maturity T at time t satisfies the following SDE:

$$\frac{dP}{P} = r(t)\,dt - \sigma B(t,T)\,dX_t,$$

where X_t is a Brownian motion under the risk neutral measure $\mathbb{Q}$, $r(t)$ is the short rate, $B(t,T) = \dfrac{1 - e^{-\kappa(T-t)}}{\kappa}$, and both σ and κ are positive constants.

(a)   (*1 point*) Derive $d(\log P)$ in terms of dt and dX_t using Ito's Lemma.

The price $F(t,T)$ of a derivative is known to satisfy the following SDE:

$$\frac{dF}{F} = r(t)\,dt - \sigma B(t,T)\,dX_t - \tau e^{-\alpha(T-t)}dY_t$$

where Y_t is a Brownian motion such that $[dY_t, dX_t] = \rho\,dt$, that is, $Y_t = \rho X_t + \sqrt{1-\rho^2}\,Z_t$ with Z_t being a Brownian motion independent of X_t under $\mathbb{Q}$, $-1 \le \rho \le 1$, α and $\tau > 0$ are constant parameters.

(b)   (*1.5 points*) Derive $d(\log F)$ in terms of $dt, dX_t,$ and dY_t.

Assume that $M(t,T) = \dfrac{F(t,T)}{P(t,T)}$ is also the price of a traded asset.

(c)   (*0.5 points*) Derive $d(\log M)$ using results of parts (a) and (b).

3. **Continued**

Using the Girsanov Theorem we can find an equivalent forward measure $\mathbb{Q}^T$ under which $M(t,T)$ is a martingale. That is,

$$\frac{dM}{M} = \xi(t,T)d\tilde{Y}_t^T$$

for some function $\xi(t,T)$ where $\tilde{Y}_t^T$ is a Brownian motion under measure $\mathbb{Q}^T$.

(d)   (*3 points*)

    (i)   Derive $d\tilde{Y}_t^T$ in terms of dY_t and dt.

    (ii)   Prove that

$$d(\log M) = -\frac{1}{2}\tau^2 e^{-2\alpha(T-t)}dt - \tau e^{-\alpha(T-t)}d\tilde{Y}_t^T.$$

4. (*9 points*) Assume that the continuously compounded annual rate at time t for the period $[t, t+1]$ follows the model:

$$r_t = r_{t-1} + \phi(t) + \sigma\epsilon_t, \ t = 1, 2, \ldots,$$

where $\phi(t)$ is a bounded deterministic function, ϵ_t, $t = 1, 2, \ldots$, are independent and identically distributed standard normal random variables under the risk neutral measure, and r_0 and σ are constants.

(a) (*1 point*) Show that for $T > t$

$$\sum_{k=t}^{T-1} r_k = r_{t-1}(T-t) + \sum_{k=t}^{T-1}(T-k)\left[\phi(k) + \sigma\epsilon_k\right]$$

(b) (*1 point*) Compute the price of a zero-coupon bond at time t with maturity $T > t$.

(c) (*0.5 point*) Identify the model as the period becomes infinitesimal.

Now consider the interest rate dynamics under the Vasicek model

$$dr(t) = a(b - r(t))dt + \sigma dW_t,$$

where W_t is a Wiener process, and a, b and $\sigma > 0$ are constants.

(d) (*2 points*)

(i) Solve the SDE above to give an explicit expression for $r(t)$ under the Vasicek model.

(ii) Determine the expectation and the variance of $r(t)$.

4. **Continued**

You are given the following results:

If $f(s)$ and $g(s,u)$ are bounded deterministic functions, then for fixed $t < T$

- Both $\int_t^T f(s)\,dW_s$ and $\int_{u=t}^T \int_{s=t}^u g(s,u)\,dW_s\,du$ are normally distributed random variables;

- $Var\left(\int_t^T f(s)\,dW_s\right) = \int_t^T f(s)^2\,ds$; and

- $Var\left(\int_{u=t}^T \int_{s=t}^u g(s,u)\,dW_s\,du\right) = \int_{s=t}^T \left[\int_{u=s}^T g(s,u)\,du\right]^2 ds$.

(e) (*2.5 points*) Determine the expectation and the variance of $\int_t^T r(u)\,du$ at time t.

(f) (*1 point*) Derive the price of a zero-coupon bond at time t with maturity $T \geq t$ under the Vasicek model using the results of part (e).

(g) (*1 point*) Compare and contrast the Vasicek Model and the model in part (c).

5. (*5 points*) The risk neutral dynamics of a stock S_t paying no dividend is given by the following stochastic differential equation (SDE):

$$\frac{dS_t}{S_t} = rdt + \sigma\, dW_t$$

where W_t is a standard Wiener process, and r and σ are positive constants.

Two derivatives on the stock, both of which mature at time T, have values $V(S_t, t)$ and $Q(S_t, t)$ at time t respectively. The following data is given at time t:

	Derivative V	Derivative Q
Delta	0.5	-0.5
Theta	-3.55	-1.40
Price	0.40	$Q(S_t, t)$

$$S_t^2\left(\frac{\partial^2 V(S_t,t)}{\partial S_t^2} + \frac{\partial^2 Q(S_t,t)}{\partial S_t^2}\right) = 1000$$

(a)   (*1.5 points*) Estimate $Q(S_t, t)$ assuming that $r = 6\%$ and $\sigma = 10\%$.

The value $U(S_t, t)$ of a third derivative on S_t with maturity at time $T > t$ has the general form $S_t^m e^{\left(r+\sigma^2\right)(T-t)}$ where m is a positive power.

(b)   (*2 points*)

    (i)   Solve for m and thus the explicit expression of $U(S_t, t)$.

    (ii)   Describe this derivative's payoff at maturity T.

5. **Continued**

Your intern, who was given only the SDE of S_t and derivative U's payoff at maturity, is trying to price $U(S_t,t)$ by simulation. The following are his steps:

    I.   Simulate 1000 stock prices at time T using the SDE.

    II.   Calculate the average stock price at time T based on the simulations in Step I.

    III.   Calculate U's payoff using the average stock price from Step II.

    IV.   Discount the payoff from Step III by risk-free rate from T to t.

(c)   (*1.5 points*)

    (i)   Assess whether your intern's method is expected to underestimate, match, or overestimate $U(S_t,t)$.

    (ii)   Recommend revisions for each of the above steps if any are needed to correctly price this derivative.

6. (*9 points*) The price $P(t,T)$ at time t of zero-coupon bonds maturing at time $T \geq t$, under the risk neutral measure $\mathbb{Q}$ with the filtration $\{\mathcal{F}_t\}_{t \geq 0}$, is governed by the following stochastic differential equation (SDE):

$$\frac{dP(t,T)}{P(t,T)} = r(t)\,dt - \sigma(t,T)\,dW_t$$

where $r(t)$ is the risk-free spot rate, $\sigma(t,T)$ is a non-negative and bounded function of t and T with $\sigma(T,T) = 0$ and W_t is a standard Wiener process.

(a) (*1.5 points*) Show that $e^{-\int_0^t r(u)\,du} P(t,T)$ is a martingale using Ito's Lemma.

(b) (*1 point*) Show that $P(t,T) = \mathbb{E}^{\mathbb{Q}}[e^{-\int_t^T r(u)\,du} \mid \mathcal{F}_t]$ using the result in part (a).

Consider the case when $\sigma(t,T) = \alpha(T-t)$ where α is a positive constant.

(c) (*2 points*) Show that the solution of the SDE for $P(t,T)$ is as given below:

$$P(t,T) = P(0,T)\exp\left(\int_0^t r(s)\,ds - \frac{\alpha^2}{6}\left(T^3 - (T-t)^3\right) - \alpha\int_0^t (T-s)\,dW_s \right)$$

(Hint: First obtain the SDE for $lnP(t,T)$ using Ito's Lemma).

Let $F(t,T) = -\dfrac{\partial lnP(t,T)}{\partial T}$ be the instantaneous forward rate at calendar time t for the forward period $[T, T+dt]$.

(d) (*1.5 points*) Show that

$$r(t) = F(0,t) + \frac{\alpha^2 t^2}{2} + \alpha W_t.$$

6. **Continued**

(e) (*2 points*) Evaluate the expectation in part (b) by using the result in part (d).

(Hint: For fixed $t < T$, $\alpha \int_t^T W_s ds \big| \mathcal{F}_t$ is normally distributed random variable with

mean $\alpha W_t (T - t)$ and variance $\dfrac{\alpha^2 (T - t)^3}{3}$.)

(f) (*1 point*) Show that evaluating the solution from part (c) leads to the same result as obtained in part (e).

7. (*5 points*) Andre writes deep out-of-the-money American put options on Mud and Puff (M&P) 500 stock index. When he writes them, he prices them using the implied volatility and then he delta-hedges the options using the Black-Scholes model. Subsequently, Andre dynamically rebalances his position at the opening of every day using the following approach:

- Model the M&P 500 return using IGARCH(1,1) model with i.i.d. normally distributed innovations.
- Predict the volatility using IGARCH(1,1) model.
- Use the predicted volatility in the Black-Scholes model to calculate delta and adjust the portfolio.

During the last year Andre made some profit from his strategy.

On October 24, 2016, based on past experience he calibrated the M&P 500 index return to be

$$r_t = 0.0067 + a_t, \quad a_t = \sigma_t \epsilon_t$$
$$\sigma_t^2 = 0.000119 + 0.8059 \sigma_{t-1}^2 + 0.1941 a_{t-1}^2$$

where $\{\epsilon_t\}$ is a sequence of i.i.d. random variables with mean 0 and variance 1.

And $\sigma_h^2(1) = 0.00012$ is the squared volatility forecasted for the 1st day from the calibrated day.

However, before rebalancing took place the M&P 500 index went down by 22%, causing all investors to exercise the options and causing Andre huge losses.

(a) (*1 point*) Describe mild randomness and wild randomness.

(b) (*1 point*) Compute the predicted volatility $\sigma_h(10)$ on the 10$^{\text{th}}$ day from the calibrated day.

(c) (*1 point*) Compute 0.1% percentile of the M&P 500 index return distribution on October 24, 2016 and interpret the result.

(d) (*1 point*) Provide reasons for Andre's initial success and his subsequent failure.

(e) (*1 point*) Suggest an alternative model that Andre could have used.

8. (*5 points*) The current value of the R&P 500 index is 1900. You obtain the following data relating to R&P 500 options:

Strike	1-Year Calls Implied Volatility
1855	24.5%
1880	24.0%
1905	23.5%
1930	23.0%
1955	22.5%

(a) (*1.5 points*)

    (i)   Identify the above volatility pattern and explain why it occurs.

    (ii)   Describe how you can use the concept of the deterministic volatility surface and stochastic volatility models to explain this pattern.

You find that the concept of the deterministic volatility surface is too complicated. You use a constant flat volatility σ of 23.0% across all strikes and you obtain the following values:

Strike	1-Year Calls Volatility σ	Black-Scholes Statistics d_1	d_2	Vega	$Vega\dfrac{d_1 d_2}{\sigma}$
1855	23.0%	0.244	0.081	520.3	44.7
1880	23.0%	0.162	-0.001	529.0	-0.4
1905	23.0%	0.081	-0.082	534.2	-15.4
1930	23.0%	0.000	-0.162	536.0	X
1955	23.0%	-0.079	-0.241	534.3	Y

(b) (*1.5 points*) Prove that $\dfrac{\partial Vega}{\partial \sigma} = Vega\dfrac{d_1 d_2}{\sigma}$.

(c) (*2 points*) Plot a graph of $\dfrac{\partial Vega}{\partial \sigma}$ versus the strike price and comment on the pattern observed.

9. (*7 points*) EFG Company offers a five-year equity-linked product that guarantees a return over the next five years. The guaranteed return is the maximum of the following two rates:

I.   1% per annum, compounded continuously

II. $p \cdot R$, where R is the return of the underlying Domestic Equity price index over the next five years and p is a percentage of participation (a participation rate) declared by the company at time 0.

You are given the following assumptions:

Domestic Equity Price Index	2,000
Risk-free interest rate (per year)	3%
Domestic Equity Index dividend	0%
Equity Index Volatility (per year)	16%
Total premium collected ($)	100 million
Costs & profits ($)	1% of premium, deducted upfront

EFG hedging strategy is to fully hedge the guarantee risk in the Equity-Linked product by using a portion of collected premium to purchase a zero-coupon Treasury bond which has the same maturity as the investment contract and then use the remaining amount, after taking operation & administration costs, to purchase equity options on the Domestic Equity Index. Assume that EFG uses Black-Scholes formula to calculate an option price.

(a)   (*2.5 points*) Determine the equity option; specify the maturity, strike, exercise and calculate the cost of the option if the participation rate p is 100%.

(b)   (*2 points*) Determine the participation rate p in order for EFG to break-even, given the hedge costs and profits target.

(c)   (*0.5 points*) Calculate Rho of the option in part (a).

Before EFG executes its hedging strategy, the risk-free rate dropped by 0.50%.

(d)   (*2 points*) Determine whether or not EFG should pursue the same hedging strategy, if the participation rate p determined in part (b) is maintained. Justify your answer.

10. (*5 points*) Clayton, an investment manager of a pension fund is considering hiring a market-oriented equity portfolio manager in order to enhance the return of the fund.

His research shows the following information about two portfolio managers:

Manager	A	B
Total return (TR)	12%	13%
Investor's Benchmark return (IR)	11%	11%
Manager's Normal benchmark return (MR)	10%	15%
Variance of (TR - IR)	.0017	.009
Variance of (MR - IR)	.0014	.0005

(a) (*2.5 points*) Evaluate and compare Manager A's and Manager B's efficiency of delivering active return by decomposing each manager's performance into true and misfit components.

(b) (*1 point*) Describe 4 reasons why investors are more risk averse when facing active risk vs. total risk.

Clayton has decided to use a questionnaire to screen potential managers.

(c) (*1.5 points*) Recommend topics that he should include in his questionnaire.

END OF EXAMINATION
Morning Session

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