

QFI Core Model Solutions

Spring 2015

1. Learning Objectives:

1. The candidate will understand the fundamentals of stochastic calculus as they apply to option pricing.

Learning Outcomes:

- (1a) Understand and apply concepts of probability and statistics important in mathematical finance.
- (1b) Understand the importance of the no-arbitrage condition in asset pricing.
- (1i) Define and apply the concepts of martingale, market price of risk and measures in single and multiple state variable contexts.

Sources:

Neftci, Chapter 2

Commentary on Question:

This question tests candidates' understanding of stock option pricing, condition of arbitrage-free in stock option pricing formula, and impact of stock dividend on option pricing. A majority of candidates did very well on this question.

Solution:

- (a) Derive the price of the option assuming $d(1+\delta) < 1+r < u(1+\delta)$.

We can set up a replicating portfolio consisting of φ_1 units of the underlying stock and φ_2 units of risk-free bond. The simultaneous equations are:

$$\begin{aligned}\varphi_1 S u(1+\delta) + \varphi_2(1+r) &= C^u \\ \varphi_1 S d(1+\delta) + \varphi_2(1+r) &= C^d\end{aligned}$$

Solving the two equations, we get

$$\varphi_1 = \frac{C^u - C^d}{S(1+\delta)(u-d)} \quad \text{and} \quad \varphi_2 = \frac{u C^d - d C^u}{(1+r)(u-d)}$$

The value of the option at time 0 is

1. Continued

$$\begin{aligned} C &= \varphi_1 S + \varphi_2 \\ &= \frac{C^u - C^d}{S(1+\delta)(u-d)} S + \frac{uC^d - dC^u}{(1+r)(u-d)} \\ &= \frac{1}{1+r} \left[\frac{(1+r)/(1+\delta) - d}{u-d} C^u + \frac{u - (1+r)/(1+\delta)}{u-d} C^d \right] \end{aligned}$$

let $q = \frac{(1+r)/(1+\delta) - d}{u-d}$, we can write C as the expected discounted value of the option

$$C = \frac{1}{1+r} [qC^u + (1-q)C^d]$$

- (b) Explain why it must be assumed that $d(1+\delta) < 1+r < u(1+\delta)$.

The condition is required for arbitrage-free.

Otherwise, there are arbitrage opportunities by taking either long or short positions on the stock that could generate guaranteed profits.

Also, if the inequalities did not hold, the quantity q would lie outside of range of $(0, 1)$ and hence could not represent a probability.

- (c) Analyze the impact on the price of the option if the stock dividend is increased to $\delta^* > \delta$.

When the dividend δ is increased, $q = \frac{(1+r)/(1+\delta) - d}{u-d}$ is decreased if other parameters remain the same.

The impact on the option price $C = \frac{1}{1+r} [qC^u + (1-q)C^d]$ depends upon the payoffs C^u and C^d :

The value C of the option will be lower if $C^u > C^d$.

The value C of the option will be higher if $C^u < C^d$.

No impact if the payoffs are the same on both paths.

2. Learning Objectives:

1. The candidate will understand the fundamentals of stochastic calculus as they apply to option pricing.

Learning Outcomes:

- (1b) Understand the importance of the no-arbitrage condition in asset pricing.
- (1c) Understand Ito integral and stochastic differential equations.
- (1d) Understand and apply Ito's Lemma.
- (1f) Demonstrate understanding of option pricing techniques and theory for equity and interest rate derivatives.
- (1k) Understand the Black Scholes Merton PDE (partial differential equation).
- (1l) Identify limitations of the Black-Scholes pricing formula.

Sources:

Neftci Ch. 10.4, 10.7.1, 12.2, 13.2

Wilmott – Introduces Quantitative Finance Ch. 16.3

Commentary on Question:

Commentary listed underneath question component.

Solution:

- (a) Derive $d\Pi(S_t, v_t, t)$, the change in the portfolio value, in terms of dt , dS , and dv_t .

Commentary on Question:

While a number of candidates did not attempt the questions, those candidates who attempted this question did fairly well. The key is to apply Ito's Lemma to get the change of $f(S_t, v_t, t)$ the value at time t of a European derivative with expiration T and terminal payoff $g(S_T)$ and $U(S_t, v_t, t)$ the value at time t of a European derivative with expiration T and terminal payoff $h(v_T)$. Some candidates missed a few terms in the calculations and scored partial credits.

2. Continued

For ease of notations, we suppress the arguments for $f(S_t, v_t, t)$ and denote it as $f = f(S_t, v_t, t)$.

By Ito's Lemma,

$$df = \frac{\partial f}{\partial t} dt + \frac{\partial f}{\partial S} dS + \frac{\partial f}{\partial v} dv + \frac{1}{2} v S^2 \frac{\partial^2 f}{\partial S^2} dt + \frac{1}{2} v \sigma^2 \frac{\partial^2 f}{\partial v^2} dt + v \sigma S \rho \frac{\partial^2 f}{\partial S \partial v} dt$$

(a-1)

Similar result for dU , namely:

$$dU = \frac{\partial U}{\partial t} dt + \frac{\partial U}{\partial S} dS + \frac{\partial U}{\partial v} dv + \frac{1}{2} v S^2 \frac{\partial^2 U}{\partial S^2} dt + \frac{1}{2} v \sigma^2 \frac{\partial^2 U}{\partial v^2} dt + v \sigma S \rho \frac{\partial^2 U}{\partial S \partial v} dt$$

(a-2)

Thus, (a-1) and (a-2) lead to the result

$$d\Pi = df + \delta dS + \gamma dU = \left(\frac{\partial f}{\partial t} + \frac{1}{2} v S^2 \frac{\partial^2 f}{\partial S^2} dt + \frac{1}{2} v \sigma^2 \frac{\partial^2 f}{\partial v^2} dt + v \sigma S \rho \frac{\partial^2 f}{\partial S \partial v} \right) dt + \gamma \left(\frac{\partial U}{\partial t} + \frac{1}{2} v S^2 \frac{\partial^2 U}{\partial S^2} + \frac{1}{2} v \sigma^2 \frac{\partial^2 U}{\partial v^2} + v \sigma S \rho \frac{\partial^2 U}{\partial S \partial v} \right) dt + \left(\frac{\partial f}{\partial S} + \gamma \frac{\partial U}{\partial S} + \delta \right) dS + \left(\frac{\partial f}{\partial v} + \gamma \frac{\partial U}{\partial v} \right) dv$$

(a-3)

- (b) Solve for the weights δ and γ so as to make the portfolio Π riskless. Express the resulting $d\Pi(S_t, v_t, t)$ with the parameters δ and γ obtained.

Commentary on Question:

Candidates did not do well in this question. The key is that the diffusion terms have to vanish in order to have no randomness (thus riskless). A number of candidates recognized this however not a lot of them were able to perform the calculation.

$$\begin{aligned} \frac{\partial f}{\partial S} + \gamma \frac{\partial U}{\partial S} + \delta &= 0 \\ \frac{\partial f}{\partial v} + \gamma \frac{\partial U}{\partial v} &= 0 \end{aligned}$$

(b-1)

Using simple algebra, we have the following results

$$\gamma = - \frac{\frac{\partial f}{\partial v}}{\frac{\partial U}{\partial v}}$$

2. Continued

$$\delta = -\frac{\partial f}{\partial S} + \left(\frac{\frac{\partial f}{\partial v}}{\frac{\partial U}{\partial v}} \right) \frac{\partial U}{\partial S}$$

(b-2)

$$d\Pi = \left(\frac{\partial f}{\partial t} + \frac{1}{2} \nu S^2 \frac{\partial^2 f}{\partial S^2} dt + \frac{1}{2} \nu \sigma^2 \frac{\partial^2 f}{\partial v^2} dt + \nu \sigma S \rho \frac{\partial^2 f}{\partial S \partial v} \right) dt - \frac{\frac{\partial f}{\partial v}}{\frac{\partial U}{\partial v}} \left(\frac{\partial U}{\partial t} + \frac{1}{2} \nu S^2 \frac{\partial^2 U}{\partial S^2} + \right.$$

$$\left. \frac{1}{2} \nu \sigma^2 \frac{\partial^2 U}{\partial v^2} + \nu \sigma S \rho \frac{\partial^2 U}{\partial S \partial v} \right) dt$$

(b-3)

3. Learning Objectives:

1. The candidate will understand the fundamentals of stochastic calculus as they apply to option pricing.

Learning Outcomes:

- (1a) Understand and apply concepts of probability and statistics important in mathematical finance.
- (1b) Understand the importance of the no-arbitrage condition in asset pricing.
- (1c) Understand Ito integral and stochastic differential equations.
 - (1i) Define and apply the concepts of martingale, market price of risk and measures in single and multiple state variable contexts.
 - (1j) Understand and apply Girsanov's theorem in changing measures.

Sources:

Wilmott, Paul, Frequently Asked Questions in Quantitative Finance, 2nd Edition Ch. 2: Q26

An Introduction to the Mathematics of Financial Derivatives, Neftci, Salih, 3rd Edition. Ch. 2, 6, 8, 11, 12, 14, 15

Commentary on Question:

Candidates did generally well on Part A.

Many candidates attempted a more complex answer to Part B (or skipped it), although the solution was straightforward.

For Part C, there were several components necessary to prove that $W_2(t)$ is a Brownian motion under the probability measure \mathbb{P}_2 . Candidates received full credit for identifying three properties, including the independence of increments, the mean and variance.

Part D called for the application of Ito's lemma and Girsanov's theorem to a vector setting. Many candidates received partial credit for identifying this most but had trouble with its proper formulation.

Part E required the construction of a hedging strategy that showed zero volatility. Candidates received credit for identifying the components of the hedging strategy (three assets: S_1 , S_2 , and savings account) and the correct derivation of the weights in each asset class.

3. Continued

Solution:

- (a) Derive an expression for k in terms of μ_2 and σ_2 .

$$\begin{aligned}d[\ln(e^{-kt} U_2(t))] &= -kdt + d(\ln U_2(t)) \\ &= -kdt + (\mu_2 dt + \sigma_2 dW(t) - \sigma_2^2/2 dt) \\ &= (\mu_2 - \sigma_2^2/2 - k)dt + \sigma_2 dW(t)\end{aligned}$$

You need the drift term to equal zero for $\ln(e^{-kt} U_2(t))$ to be a martingale
 $\rightarrow k = \mu_2 - \sigma_2^2/2$

- (b) Write an expression for the relationship among μ_1 , σ_1 , and μ_2 , σ_2 , and r necessary for the no-arbitrage condition to hold.

Commentary on Question:

The answer was very straightforward, although some candidates developed the solution mathematically. No points were awarded or subtracted for this. Some candidates showed an alternative form of the same expression and were awarded full credit.

Note that $\frac{\mu_2 - r}{\sigma_2}$ is the market price of risk

Under a no-arbitrage condition, then:

$$\frac{\mu_1 - r}{\sigma_1} = \frac{\mu_2 - r}{\sigma_2}$$

- (c) Prove that $W_2(t)$ is a Brownian motion under the probability measure \mathbb{P}_2 .

Commentary on Question:

There were several components necessary to prove that $W_2(t)$ is a Brownian motion under the probability measure \mathbb{P}_2 . Candidates received full credit for identifying three properties, including the independence of increments, the mean and variance.

$$W_2(t) - W_2(s) = \rho(W_1(t) - W_1(s)) + \sqrt{(1 - \rho^2)} (W_0(t) - W_0(s))$$

1. Since $|\rho| \leq 1$, W_2 is a linear combination of two continuous Brownian motions, thus W_2 is also continuous
2. Since W_2 is a linear combination of two normal distributions, W_2 is also normally distributed. Similarly, W_2 has independent increments
3. $E[W_2(t) - W_2(s)] = 0$ (straightforward)

3. Continued

$$4. \quad V[W_2(t) - W_2(s)] = V[\rho(W_1(t) - W_1(s)) + \sqrt{(1 - \rho^2)}(W_0(t) - W_0(s))] = \rho^2 V[W_1(t) - W_1(s)] + (1 - \rho^2)V[W_0(t) - W_0(s)] = \rho^2(t - s) + (1 - \rho^2)(t - s) = t - s$$

For asset 1, applying Ito's lemma to $e^{-rt}S_1(t)$, we have that

$$d(e^{-rt}S_1(t)) = S_1(t)d(e^{-rt}) + e^{-rt}dS_1(t)$$

And given the definition of $dS_1(t)$, under \mathbb{P}_2

$$d(e^{-rt}S_1(t)) = e^{-rt}(m_1 - r)S_1(t)dt + e^{-rt}v_1S_1(t)dW_1(t)$$

Under \mathbb{P}_2 , this is not a martingale since $(m_1 - r)$ could be $\neq 0$

Girsanov's theorem states if W_1 is a standard Wiener process under \mathbb{P}_2 , there exists another standard Wiener process, \tilde{W}_1 , where Q is a martingale measure, defined as

$$d\tilde{W}_1(t) = dW_1(t) + dX_1(t)$$

Moving from \mathbb{P}_2 to Q ,

$$d(e^{-rt}S_1(t)) = e^{-rt}(m_1 - r)S_1(t)dt + e^{-rt}v_1S_1(t)(d\tilde{W}_1(t) - dX_1(t))$$

To be a martingale the drift term should be equal to zero, thus

$$e^{-rt}(m_1 - r)S_1(t) - e^{-rt}v_1S_1(t)dX_1(t) = 0$$

Thus

$$dX_1(t) = \frac{(m_1 - r)}{v_1}dt$$

For asset 2, also applying Ito's lemma

$$d(e^{-rt}S_2(t)) = S_2(t)d(e^{-rt}) + e^{-rt}dS_2(t)$$

And given the definition of $dS_2(t)$, under \mathbb{P}_2

$$d(e^{-rt}S_2(t)) = e^{-rt}(m_2 - r)S_2(t)dt + e^{-rt}v_2S_2(t)dW_2(t)$$

$$\text{Since } W_2(t) = \rho W_1(t) + \sqrt{1 - \rho^2} W_0(t),$$

3. Continued

$$dW_2(t) = \rho dW_1(t) + \sqrt{1 - \rho^2} dW_0(t)$$

We have

$$\begin{aligned} d(e^{-rt}S_2(t)) &= e^{-rt}(m_2 - r)S_2(t)dt \\ &\quad + e^{-rt}v_2S_2(t) \left[\rho dW_1(t) + \sqrt{1 - \rho^2} dW_0(t) \right] \end{aligned}$$

Girsanov's theorem states if W_0 is a standard Wiener process under \mathbb{P}_2 , there exists another standard Wiener process, \tilde{W}_0 , where Q is a martingale measure, defined as

$$d\tilde{W}_0(t) = dW_0(t) + dX_0(t)$$

Moving from \mathbb{P}_2 to Q , and knowing the definition of $dX_1(t)$ under Q , we have

$$\begin{aligned} d(e^{-rt}S_2(t)) &= e^{-rt}(m_2 - r)S_2(t)dt \\ &\quad + e^{-rt}v_2S_2(t) \left[\rho (d\tilde{W}_1(t) - dX_1(t)) \right. \\ &\quad \left. + \sqrt{1 - \rho^2} (d\tilde{W}_0(t) - dX_0(t)) \right] \end{aligned}$$

To be a martingale the drift term should be equal to zero, thus

$$e^{-rt}(m_2 - r)S_2(t)dt = e^{-rt}v_2S_2(t) \left[\rho(dX_1(t)) + \sqrt{1 - \rho^2} (dX_0(t)) \right]$$

Solving for $dX_0(t)$ we get

$$dX_0(t) = \frac{\frac{(m_2 - r)}{v_2} dt - \rho dX_1(t)}{\sqrt{1 - \rho^2}}$$

Which can only exist when $|\rho| < 1$. Q is the only measure under which $e^{-rt}S_1(t)$ and $e^{-rt}S_2(t)$ are martingales, otherwise there would be multiple arbitrage free prices.

- (e) Construct a self-financing trading strategy with non-zero weights using the two assets in Portfolio 2 and the risk-free savings account such that the resulting portfolio has zero volatility.

3. Continued

Commentary on Question: None

Candidates received full credit for part E if they stated the hedging strategy components:

- Setting up a portfolio using the two assets in portfolio II and the risk free savings account without volatility and showing the differential equation of the portfolio under $W_1(t)$ terms
- Identifying that the volatility coefficient needs to be zero in order to have a risk-free portfolio
- Stating (and showing) the strategy: Long X units of S_1 , short $v_1 S_1 X / v_2 S_2$ and lend/borrow the remaining funds in the risk free savings account.
- Identifying that since the market price of risk is the same for S_1 and S_2 , the resulting portfolio has zero volatility

4. Learning Objectives:

1. The candidate will understand the fundamentals of stochastic calculus as they apply to option pricing.
2. The candidate will understand how to apply the fundamental theory underlying the standard models for pricing financial derivatives. The candidate will understand the implications for option pricing when markets do not satisfy the common assumptions used in option pricing theory such as market completeness, bounded variation, perfect liquidity, etc. The Candidate will understand how to evaluate situations associated with derivatives and hedging activities.

Learning Outcomes:

- (1h) Demonstrate understanding of the differences and implications of real-world versus risk-neutral probability measures.
- (1l) Identify limitations of the Black-Scholes pricing formula.
- (2a) Identify limitations of the Black-Scholes pricing formula
- (2d) Understand the different approaches to hedging.
- (2e) Understand how to delta hedge and the interplay between hedging assumptions and hedging outcomes.

Sources:

Paul Wilmott, Intro to Quant Finance, 2nd ed., Ch. 6, Ch. 8

Commentary on Question:

The purpose of this question is to test candidates on the pricing a non-standard option under Black Scholes framework. We further test candidates on deriving the Greeks associated with the non-standard option and interpreting the meaning on the Greeks.

Solution:

- (a) Sketch the payoff diagram of this option at maturity.

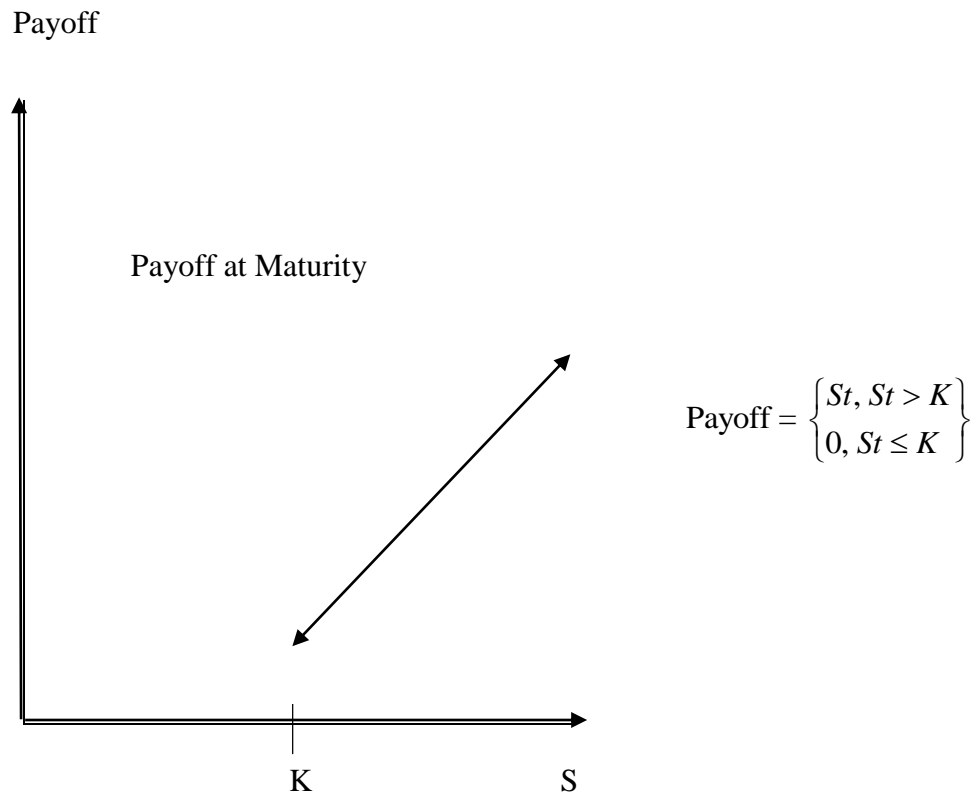
Commentary on Question:

This question is designed to help candidate picture the problem solving at hand and structure the rest of the solution to the problem.

Majority of the candidates were able to get the correct answer.

It is key for the candidate to recognize the payoff diagram has a jump at $S = K$.

4. Continued



- (b) Show that V_T can be restated as $C_T + K I(S_T \geq K)$ where $I(A)$ denotes the indicator function for an event A .

Commentary on Question:

This question is designed to help candidate further understand the problem. That is, the option at hand can be simplified into two basic options.

One can easily visualize from the diagram in (a) that the only difference between this option payoff and that of a call option with exercise price K is the value of the exercise price.

Using indicator functions, one can also show that

$$\begin{aligned} V_T &= S_T I(S_T \geq K) - K I(S_T \geq K) + K I(S_T \geq K) \\ &= (S_T - K) I(S_T \geq K) + K I(S_T \geq K) \\ &= C_T + K I(S_T \geq K) \end{aligned}$$

- (c) Derive a formula for the value of this option at time 0.
(Hint: $\Pr(S_T \geq K) = N(d_2)$)

4. Continued

Commentary on Question:

This question is designed to test the candidate on pricing a non-standard call/put option under the Black Scholes framework.

Calculating expectations based on risk-neutral measure, using part (b), and noting that the value is the expectation of the discounted payoff, we have

$$\begin{aligned}V_0 &= E(e^{-rT}C_T) + Ke^{-rT}E[I(S_T \geq K)] \\&= C_0 + Ke^{-rT}\Pr(S_T \geq K) \\&= S_0N(d_1) - Ke^{-rT}N(d_2) + Ke^{-rT}\Pr(S_T \geq K)\end{aligned}$$

where $C_0 = S_0N(d_1) - Ke^{-rT}N(d_2)$ is the Black-Scholes formula for a call option on a non-dividend paying stock. Now by noting that

$$\Pr(S_T \geq K) = N(d_2)$$

we find the value of the option at time 0 to be

$$V_0 = S_0N(d_1).$$

Additional Commentary on Question:

Another (although quite lengthy) approach was to derive the expectation under first principles. This is not the recommended approach, but partial points were given to candidates attempting it.

A handful of candidates made the following mistake:

$$\begin{aligned}V_0 &= E(S_T | S_T \geq K)\Pr(S_T \geq K) + E(S_T | S_T \leq K)\Pr(S_T \leq K) \\&= S_0N(d_2) + 0 \\&= S_0N(d_2)\end{aligned}$$

Above statements are not correct because $E(S_T | S_T \geq K)$ is not equal to S_T . To see this more clearly, one can take a closer look at the lengthy solution presented above.

A handful of candidates also invoke “ μ ” when pricing the option. This is not correct. Recall: When we are pricing a market consistent (i.e. no arbitrage) option/derivative, we should always do so in a risk neutral framework. In a risk neutral framework, all assets grow at risk free rate. As a result, the expected return “ μ ” under Physical measure should not come into play.

(d) Show that

$$\frac{\partial d_1}{\partial S} = \frac{1}{S\sigma\sqrt{T}}$$

4. Continued

Commentary on Question:

This question is designed to help candidate solve part e.

Recall that

$$d_1 = \frac{\log(S/K) + (r + \frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}}$$

Straightforward differentiation leads us to

$$\frac{\partial d_1}{\partial S} = \frac{1}{S\sigma\sqrt{T}}$$

- (e) Show that the Gamma for this option can be expressed as

$$\Gamma = \frac{N'(d_1)}{S\sigma\sqrt{T}} \left(1 - \frac{d_1}{\sigma\sqrt{T}}\right)$$

using the result in part (d) where $N'(\cdot)$ is the density of a standard normal random variable.

Commentary on Question:

This question is designed to test candidates on the definition behind Gamma as well as meaning behind Gamma (part f).

There are two approaches to solve this question:

Approach One:

Applying chain rule of differentiation, we find that

$$\begin{aligned} \frac{\partial V_0}{\partial S} &= N(d_1) + SN'(d_1) \frac{\partial d_1}{\partial S} \\ &= N(d_1) + SN'(d_1) \frac{1}{S\sigma\sqrt{T}} \\ &= N(d_1) + N'(d_1) \frac{1}{\sigma\sqrt{T}} \end{aligned}$$

Taking the derivative of both sides once more to get the Gamma, we have

4. Continued

$$\begin{aligned}\Gamma &= \frac{\partial^2 V_0}{\partial S^2} = N'(d_1) \frac{\partial d_1}{\partial S} - \frac{1}{\sigma\sqrt{T}} N'(d_1) d_1 \frac{\partial d_1}{\partial S} \\ &= N'(d_1) \frac{\partial d_1}{\partial S} \left[1 - \frac{d_1}{\sigma\sqrt{T}} \right] \\ &= \frac{N'(d_1)}{S\sigma\sqrt{T}} \left[1 - \frac{d_1}{\sigma\sqrt{T}} \right]\end{aligned}$$

where we note that

$$N'(d_1) = \frac{1}{\sqrt{2\pi}} e^{-\frac{d_1^2}{2}}$$

so that

$$\frac{\partial}{\partial S} = -\frac{1}{\sqrt{2\pi}} e^{-\frac{d_1^2}{2}} d_1 \frac{\partial d_1}{\partial S} = -N'(d_1) d_1 \frac{\partial d_1}{\partial S}$$

Approach Two:

Leveraging on the results from part b, we know this option V can be broken down into a two simpler options. We can derive the Gamma of each of the options separately.

Gamma of the Call option is straightforward. It is given in the formula sheet.

Gamma of the "KI($S_T \geq K$)" piece can be obtained by taking the second derivative of $Ke^{-rT}N(d_2)$ with respect to S. Note: The "KI($S_T \geq K$)" piece is also known as the Cash-or-nothing-call. Therefore, the Gamma of V is given by the sum of Gamma of Call and Gamma of the Cash-or-nothing-call.

- (f) Your actuarial colleague made the following observation: "Because the payoff for this option is always larger than the payoff of a call option with the same exercise price, this requires less frequent rebalancing in order to maintain a delta-neutral position."

Critique your colleague's statement.

Commentary on Question:

This question is designed to test candidates on the meaning behind Gamma

First of all, your colleague's reasoning is incorrect.

It is not the value of the payoff that determines how frequently delta hedge has to be rebalanced. However, sometimes, his conclusion may appear correct due to the following reasoning:

4. Continued

We note that the ‘Greek’ gamma gives a measure of how frequent you need to rebalance your position to be considered delta-hedged. Recall that the ‘Greek’ gamma of a call option with the same exercise price is given by

$$\Gamma_C = \frac{\phi(d_1)}{S\sigma\sqrt{T}}$$

which is the first term of the ‘Greek’ gamma of this option. A smaller gamma would lead to less frequent rebalancing, and your colleague is correct to the extent that the second term must be positive. This is the case if d_1 is positive. From the formula for d_1

$$d_1 = \frac{\log(S/K) + (r + \frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}}$$

we note that $d_1 > 0$ when you are ‘in the money’. However, when you are ‘out of the money’, there is a possibility that $d_1 < 0$ in which case your colleague could be wrong.

5. Learning Objectives:

1. The candidate will understand the fundamentals of stochastic calculus as they apply to option pricing.
2. The candidate will understand how to apply the fundamental theory underlying the standard models for pricing financial derivatives. The candidate will understand the implications for option pricing when markets do not satisfy the common assumptions used in option pricing theory such as market completeness, bounded variation, perfect liquidity, etc. The Candidate will understand how to evaluate situations associated with derivatives and hedging activities.

Learning Outcomes:

- (1f) Demonstrate understanding of option pricing techniques and theory for equity and interest rate derivatives.
- (2b) Compare and contrast the various kinds of volatility, (eg actual, realized, implied, forward, etc.).
- (2c) Compare and contrast various approaches for setting volatility assumptions in hedging.
- (2d) Understand the different approaches to hedging.
- (2e) Understand how to delta hedge and the interplay between hedging assumptions and hedging outcomes.

Sources:

Quantitative Finance, Wilmott, Paul, 2nd Edition Ch. 6, 8, 10

QFIC-103-13: How to Use the Holes in Black-Scholes

QFIC-105-13: Section IX of Carr, Peter, *FAQ's in Option Pricing Theory* – “Which volatility should one hedge at - historical or implied?” (pp. 26-28)

Wilmott, Paul, *Frequently Asked Questions in Quantitative Finance*, 2nd Edition Ch. 2: Q38, Q39

Commentary on Question:

This question tested a candidate's understanding of the Black-Scholes equation and required a working familiarity with calculus. Most candidates were able to answer parts (a) and (b), but were not as successful for latter parts.

5. Continued

Solution:

- (a) Describe the disadvantages if ABC uses Assumption 1 to price the option.

Commentary on Question:

The best candidates listed more than one disadvantage.

- 1) ABC will initially suffer arbitrage losses
 - 2) Lacks forward-looking view
 - 3) Inconsistent with market prices
 - 4) Ignore the possibility of market crashes
 - 5) Slow to adapt to changing market conditions
 - 6) Historic requires a model (parameter calibration issues, data cleaning, etc.)
- (b) Calculate the number of futures needed at $t = 0$ to hedge the position using Assumption 2, stating whether a long or short position should be created.

The delta of a long position in one contract of the 1-year future is $\exp((r-q)*t) = \exp((0.04-0.02)*1) = 1.0202$

The delta of the (sold) 5-year call option = $-10 N(d1) \exp(-qT)$

$$d1 = \left(\ln\left(\frac{S}{K}\right) + \left(r - q + \sigma^2 \frac{1}{2}\right) T \right) / \sigma \sqrt{T}$$

$$= [\ln(1400/1900) + (0.04 - 0.02 + \frac{1}{2}(0.2)^2)*5] / (0.2(5)^{.5}) = -0.23564$$

$$\text{So delta of option} = -10 * N(-0.23564) * \exp(-0.02*5) = -3.6814$$

The bank should buy futures to hedge short position

$$\text{Futures to buy} = 3.6814 / 1.0202 = 3.6085 \text{ (or 4)}$$

- (c) Prove using the above equation that the Vega of a European call option under Assumption 2 is

$$\frac{\partial C^{\text{Assumption 2}}}{\partial \sigma} = S e^{-qt} N'(d_1) \sqrt{\tau}$$

where $\tau =$ time to maturity.

Commentary on Question:

This question can also be answered without using the relationship between $d1$ and $d2$ (a shortcut used in this solution). However, many candidates failed to properly differentiate $d1$ and $d2$ with respect to sigma (note that the quotient rule would be applied).

5. Continued

$$Vega = \frac{\partial C}{\partial \sigma} = Se^{-q\tau} \phi(d_1) \left(\frac{\partial d_1}{\partial \sigma}\right) - Ke^{-r\tau} \phi(d_2) \left(\frac{\partial d_2}{\partial \sigma}\right)$$

$$\frac{\partial d_1}{\partial \sigma} - \frac{\partial d_2}{\partial \sigma} = \sqrt{\tau}$$

$$= Se^{-q\tau} \phi(d_1) \left(\frac{\partial d_2}{\partial \sigma} + \sqrt{\tau}\right) - Ke^{-r\tau} \phi(d_2) \left(\frac{\partial d_2}{\partial \sigma}\right)$$

$$= Se^{-q\tau} \phi(d_1) \sqrt{\tau} + \left(\frac{\partial d_2}{\partial \sigma}\right) [Se^{-q\tau} \phi(d_1) - Ke^{-r\tau} \phi(d_2)]$$

Using the equation given above the question

$$= Se^{-q\tau} \phi(d_1) \sqrt{\tau} + \left(\frac{\partial d_2}{\partial \sigma}\right)(0) = Se^{-q\tau} \phi(d_1) \sqrt{\tau}$$

- (d) Show that the Delta of the call option on the index under Assumption 3 is equal to

$$\frac{\partial C}{\partial S} \stackrel{\text{Assumption 3}}{=} \frac{\partial C}{\partial S} \stackrel{\text{Assumption 2}}{=} -\frac{1}{10,000} \frac{\partial C}{\partial \sigma}$$

Commentary on Question:

The simplest solution is to use the multivariable chain rule, as outlined below. However, it is also possible to derive the solution from first principles.

Let $C_3 =$ Call Price under assumption 3.

So $C_3 = f(S, K, r, T, q, \sigma_{adj})$ where f is the standard Black-Scholes formula

Note that σ_{adj} is a function of S

Then:

$$\frac{\partial C_3}{\partial S} = \frac{\partial f}{\partial S} + \frac{\partial f}{\partial \sigma_{adj}} \frac{\partial \sigma_{adj}}{\partial S}$$

And:

$$\sigma_{adj}(S) = \sigma_{fixed} - \frac{(S - 1,400)}{10,000}$$

So:

$$\frac{\partial \sigma_{adj}}{\partial S} = -\frac{1}{10,000}$$

5. Continued

Therefore:

$$\frac{\partial C_3}{\partial S} = \frac{\partial f}{\partial S} - \frac{1}{10,000} \frac{\partial f}{\partial \sigma_{adj}}$$

Finally, $\frac{\partial f}{\partial S}$ is the standard delta formula, albeit with σ_{adj} in d1.

Likewise, $\frac{\partial f}{\partial \sigma_{adj}}$ is the standard vega formula, albeit with σ_{adj} in d1.

In both cases, the formulas are identical to the corresponding derivatives using assumption 2 when $S = 1400$ (so σ_{adj} equals σ).

- (e) Calculate the number of futures needed to hedge the delta at $t = 0$ under Assumption 3.

Commentary on Question:

While most candidates did not arrive at the correct final answer, significant credit was awarded for attempting to use previous results.

Using part (c) and (d),

Delta of the adjusted call option = Unadjusted delta – notional amount * $-1/10000$
* $S e^{-q\tau} N'(d_1) \sqrt{\tau}$

Where $S = 1400$ and $\tau = 5$

$$\begin{aligned} &= 3.6814 - 10 (1/10000) 1400 \exp(-0.02*5) * N'(d1) \text{sqrt}(5) \\ &= 3.6814 - 10 (1/10000) 1400 \exp(-0.02*5) * \exp(-(-0.23564^2)/2)/\text{sqrt}(2*\text{pi}) * \\ &\text{sqrt}(5) \\ &= 2.5823 \end{aligned}$$

Call is sold, so buy futures.

Revised futures to buy = $2.5823 / .98019 = 2.6344$ (or 3) contracts

- (f) Explain the rationale and the impact of using Assumption 3.

Rationale:

- This is an adjustment for volatility skew
- This is an adjustment for "sticky delta"

Impact:

- As stock price goes below 1400, the implied volatility increases and option price increases (and vice versa)

6. Learning Objectives:

1. The candidate will understand the fundamentals of stochastic calculus as they apply to option pricing.
3. The candidate will understand the basic concepts underlying interest rate option pricing models

Learning Outcomes:

- (1b) Understand the importance of the no-arbitrage condition in asset pricing.
- (1c) Understand Ito integral and stochastic differential equations.
- (1d) Understand and apply Ito's Lemma.
- (1f) Demonstrate understanding of option pricing techniques and theory for equity and interest rate derivatives.
- (1h) Demonstrate understanding of the differences and implications of real-world versus risk-neutral probability measures.
- (1i) Define and apply the concepts of martingale, market price of risk and measures in single and multiple state variable contexts.
- (1j) Understand and apply Girsanov's theorem in changing measures.
- (3a) Demonstrate understanding of interest rate models.

Sources:

Neftci Ch. 6, 9, 10, 14, 15

Wilmott Frequent Asked Questions pg. 113-115

Wilmott – Introduces Quantitative Finance Ch. 16, 17, 18

Commentary on Question:

Candidate did poorly in this question. While many candidates were able to put down some advantages and disadvantages, some of the candidates got it opposite by thinking the model is tractable and mean-reverting. Part (b) was done relatively well: different candidates have different ways of doing the proof, and each scored as long as the answer makes sense. Not a lot of candidates were able to get Part (c) and a lot of candidates left Part (d) and Part (e) unanswered.

Solution:

- (a) Describe briefly the advantages and disadvantages of using the above model.

6. Continued

Advantages

- Given that the volatility is a function of the interest rate level, the interest rate can't easily go negative at low levels of interest rate.
- Simpler model and fewer parameters to maintain.

Disadvantages

- The model is not tractable. No simple closed-form solutions. Thus, more computation-intensive and time-consuming calculations for complicated derivatives.
- No mean-reverting feature, which is a popular feature of an interest rate model.
- Since the parameters are not time-dependent, the model cannot fit the interest rate curve exactly. The model's inability to fit the curve exactly also leads to the model prices being not adequately matching the observed market prices. Thus, it is not an arbitrage-free model. (p. 374 of Wilmott – Introduces Quantitative Finance Ch. 17.1)
- Model only has one-factor, which is inadequate for pricing derivatives with payoffs that are sensitive to the tilting of yield curve. (p. 394 of Wilmott – Introduces Quantitative Finance Ch. 18.7)
- If $\mu > 0$, the interest rate will “blow up” over time because $E(X_t) = X_0 + \mu t$ as shown in Part (d) below.

(b)

- (i) Derive the SDE for G_t and show that G_t is a martingale;

Applying Ito's Lemma to G_t as a function of t and W_t

$$dG_t = \left(\frac{\partial G_t}{\partial t} + \frac{1}{2} \frac{\partial^2 G_t}{\partial W_t^2} \right) dt + \frac{\partial G_t}{\partial W_t} dW_t.$$

From Calculus

$$\frac{\partial G_t}{\partial t} = -\frac{1}{2} \alpha^2 G_t,$$

$$\frac{\partial G_t}{\partial W_t} = \alpha G_t,$$

$$\frac{\partial^2 G_t}{\partial W_t^2} = \alpha^2 G_t.$$

6. Continued

It follows that

$$dG_t = \left(-\frac{1}{2}\alpha^2 G_t + \frac{1}{2}\alpha^2 G_t\right) dt + \alpha G_t dW_t = \alpha G_t dW_t.$$

And then for $t_2 > t_1$

$$E[G_{t_2}] = G_{t_1} + E\left[\int_{t_1}^{t_2} \alpha G_s dW_s\right] = G_{t_1}$$

This shows that G_t is a martingale.

(ii) Prove that $d(X_t F_t) = \mu F_t dt$.

Since

$$\frac{\partial F_t}{\partial W_t} = -\alpha F_t, \quad \frac{\partial^2 F_t}{\partial W_t^2} = \alpha^2 F_t, \quad \frac{\partial F_t}{\partial t} = \frac{1}{2}\alpha^2 F_t,$$

by Ito's formula we have

$$dF_t = \left(\frac{1}{2}\alpha^2 + \frac{1}{2}\alpha^2\right) F_t dt - \alpha F_t dW_t = \alpha^2 F_t dt - \alpha F_t dW_t$$

By the multivariate form of the Ito's Lemma (p. 173 of Neftci Ch. 10.7.1)

$$\begin{aligned} d(X_t F_t) &= F_t dX_t + X_t dF_t + dX_t dF_t \\ &= F_t(\mu dt + \alpha X_t dW_t) + X_t(\alpha^2 F_t dt - \alpha F_t dW_t) \\ &\quad + (\mu dt + \alpha X_t dW_t)(\alpha^2 F_t dt - \alpha F_t dW_t) \\ &= F_t(\mu dt + \alpha X_t dW_t) + X_t(\alpha^2 F_t dt - \alpha F_t dW_t) + (\alpha X_t dW_t)(-\alpha F_t dW_t) \\ &= F_t(\mu dt + \alpha X_t dW_t) + X_t(\alpha^2 F_t dt - \alpha F_t dW_t) - \alpha^2 F_t X_t dt \\ &= F_t \mu dt \end{aligned}$$

Alternative solution:

Applying Ito's Lemma to X_t as a function of t and W_t

$$dX_t = \left(\frac{\partial X_t}{\partial t} + \frac{1}{2}\frac{\partial^2 X_t}{\partial W_t^2}\right) dt + \frac{\partial X_t}{\partial W_t} dW_t.$$

Comparing it with the given SDE we find

$$\frac{\partial X_t}{\partial W_t} = \alpha X_t, \quad \text{and} \quad \frac{\partial X_t}{\partial t} + \frac{1}{2}\frac{\partial^2 X_t}{\partial W_t^2} = \mu.$$

Then

$$\frac{\partial X_t}{\partial t} = \mu - \frac{1}{2}\frac{\partial^2 X_t}{\partial W_t^2} = \mu - \frac{1}{2}\alpha^2 X_t.$$

6. Continued

It follows from Calculus that

$$\frac{\partial(X_t F_t)}{\partial t} = \frac{\partial X_t}{\partial t} F_t + \frac{\partial F_t}{\partial t} X_t = (\mu - \frac{1}{2} \alpha^2 X_t) F_t + (\frac{1}{2} \alpha^2 F_t) X_t = \mu F_t,$$

$$\frac{\partial(X_t F_t)}{\partial W_t} = \frac{\partial X_t}{\partial W_t} F_t + \frac{\partial F_t}{\partial W_t} X_t = (\alpha X_t) F_t + (-\alpha F_t) X_t = 0,$$

$$\frac{\partial^2(X_t F_t)}{\partial W_t^2} = 0.$$

Applying Ito's Lemma to $(X_t F_t)$ as a function of t and W_t

$$d(X_t F_t) = \left(\frac{\partial(X_t F_t)}{\partial t} + \frac{1}{2} \frac{\partial^2(X_t F_t)}{\partial W_t^2} \right) dt + \frac{\partial(X_t F_t)}{\partial W_t} dW_t = \mu F_t dt.$$

- (c) Solve for X_t in terms of X_0 , m , a , and W_s for all $s \leq t$.

Integrating $d(X_t F_t)$ gives us

$$X_t F_t = X_0 F_0 + \mu \int_0^t F_s ds$$

Thus,

$$\begin{aligned} X_t &= X_0 F_t^{-1} + \mu \int_0^t F_s F_t^{-1} ds \\ &= X_0 e^{\alpha W_t - \frac{1}{2} \alpha^2 t} + \mu \int_0^t e^{\alpha(W_t - W_s) - \frac{1}{2} \alpha^2 (t-s)} ds \end{aligned}$$

- (d) Derive the expected value of X_t .

From (b.i) $dG_t = \alpha G_t dW_t$, where $G_t = F_t^{-1}$, and $E(G_t) = G_0 = 1$.

Also, note that $\alpha(W_t - W_s) \sim \text{Normal}(0, \alpha^2(t-s))$. Thus,

$$\begin{aligned} E(X_t) &= X_0 + \mu \int_0^t E\left(\exp\left(\alpha(W_t - W_s) - \frac{1}{2} \alpha^2 (t-s)\right)\right) ds \\ &= X_0 + \mu \int_0^t 1 ds \\ &= X_0 + \mu t \end{aligned}$$

6. Continued

Alternative solution:

Integrate the given SDE $dX_t = \mu dt + \alpha X_t dW_t$ to get

$$X_t = X_0 + \mu t + \alpha \int_0^t X_s dW_s.$$

Since $E\left(\int_0^t X_s dW_s\right) = 0$, we have

$$E(X_t) = X_0 + \mu t.$$

(e) Prove that
$$\mathbb{E}[X_t^2] = X_0^2 e^{\alpha^2 t} + 2\mu X_0 \left(\frac{e^{\alpha^2 t} - 1}{\alpha^2}\right) + \mu^2 \mathbb{E}\left[F_t^{-2} \left(\int_0^t F_s ds\right)^2\right].$$

$$\begin{aligned} X_t^2 &= X_0^2 e^{2\alpha W_t - \alpha^2 t} + 2\mu X_0 e^{\alpha W_t - \frac{1}{2}\alpha^2 t} \int_0^t e^{\alpha(W_t - W_s) - \frac{1}{2}\alpha^2(t-s)} ds \\ &\quad + \mu^2 \left(\int_0^t e^{\alpha(W_t - W_s) - \frac{1}{2}\alpha^2(t-s)} ds\right)^2 \\ &= X_0^2 e^{2\alpha W_t - \alpha^2 t} + 2\mu X_0 \int_0^t e^{\alpha(2W_t - W_s) - \frac{1}{2}\alpha^2(2t-s)} ds \\ &\quad + \mu^2 \left(\int_0^t e^{\alpha(W_t - W_s) - \frac{1}{2}\alpha^2(t-s)} ds\right)^2 \end{aligned}$$

Note that $\alpha(2W_t - W_s) = \alpha(W_t - W_s + W_s - W_0 + W_t - W_s) = \alpha(2(W_t - W_s) + (W_s - W_0))$, where $W_0 = 0$.

Thus, $\alpha E(2W_t - W_s) = 0$ and

$$\begin{aligned} &\alpha^2 E[(2W_t - W_s)^2] \\ &= \alpha^2 E[4(W_t - W_s)^2 + 4(W_t - W_s)(W_s - W_0) + (W_s - W_0)^2] \\ &= \alpha^2(4(t-s) + s) \end{aligned}$$

So, $\alpha(2W_t - W_s) \sim \text{Normal}(0, \alpha^2(4(t-s) + s))$.

$$\begin{aligned} E(X_t^2) &= X_0^2 e^{\alpha^2 t} + 2\mu X_0 \int_0^t e^{\alpha^2(t-s)} ds + \mu^2 E\left\{\left(\int_0^t e^{\alpha(W_t - W_s) - \frac{1}{2}\alpha^2(t-s)} ds\right)^2\right\} \\ &= X_0^2 e^{\alpha^2 t} + 2\mu X_0 \left(\frac{\exp(\alpha^2 t) - 1}{\alpha^2}\right) + \mu^2 E\left\{F_t^{-2} \left(\int_0^t F_s ds\right)^2\right\} \end{aligned}$$

7. Learning Objectives:

3. The candidate will understand the basic concepts underlying interest rate option pricing models

Learning Outcomes:

- (3a) Demonstrate understanding of interest rate models.
- (3c) Understand the HJM model and the HJM no-arbitrage condition.

Sources:

An Introduction to the Mathematics of Financial Derivatives, Neftci, Salih, 3rd Edition, Ch 19

Commentary on Question:

Many candidates did not do well on this question. This question is a simple application of HJM model whereby: (a) there is a relationship between the drift and volatility for the forward rate; (b) the short rate is a stochastic integral of the forward rate which leads to a simple calculation of the expected value and variance for the short rate; (c) the drawbacks of the model are as explained in the syllabus. Partial credit is given for each step completed correctly.

Solution:

- (a) Derive formulae for $m(t, T, B(t, T))$ and $v(t, T, B(t, T))$ in terms of α , β , t , and T .

The forward rate volatility is driven by the bond price volatility $\sigma(t, T, B_t)$ which is given in (19.15) of Hirs/Neftci

$$\sigma(t, T, B(t, T)) = \alpha(T - t)^\beta$$

Therefore the forward rate volatility from (19.21) of Hirs/Neftci is given by

$$v(t, T, B(t, T)) = \frac{\partial \sigma}{\partial T} = \alpha\beta(T - t)^{\beta-1}$$

The forward rate drift $m(t, T, B(t, T)) = \sigma(t, T, B(t, T)) v(t, T, B(t, T)) = \alpha^2\beta(T - t)^{2\beta-1}$

- (b) Derive the risk-neutral expected value and variance of $r(t)$, predicted at time 0 for $t > 0$, in terms of α , β , t , and $F(0, t)$.

First let us obtain $F(t, T)$:

$$F(t, T) = F(0, t) + \int_0^t m(s, T, B(s, T)) ds + \int_0^t v(s, T, B(s, T)) dW_s$$
$$F(t, T) = F(0, t) + \int_0^t \alpha^2\beta(T - s)^{2\beta-1} ds + \int_0^t \alpha\beta(T - s)^{\beta-1} dW_s$$

7. Continued

$$F(t, T) = F(0, t) + \frac{\alpha^2}{2} [T^{2\beta} - (T - t)^{2\beta}] + \int_0^t \alpha\beta(T - s)^{\beta-1} dW_s$$

Now we can obtain the short rate $r(t)$:

$$r(t) = F(t, t) = F(0, t) + \frac{\alpha^2}{2} t^{2\beta} + \int_0^t \alpha\beta(t - s)^{\beta-1} dW_s$$

Note that the stochastic integral on RHS is a Gaussian process. It has zero mean and its variance can be calculated using Ito-Isometry.

$$\mathbb{E}[r(t)] = F(0, t) + \frac{\alpha^2}{2} t^{2\beta} + \mathbb{E} \int_0^t \alpha\beta(t - s)^{\beta-1} dW_s$$

Since $\mathbb{E} \int_0^t \alpha\beta(t - s)^{\beta-1} dW_s = 0$

$$\mathbb{E}[r(t)] = F(0, t) + \frac{\alpha^2}{2} t^{2\beta}$$

$$\text{Var}[r(t)] = \text{Var} \left[\int_0^t \alpha\beta(t - s)^{\beta-1} dW_s \right]$$

$$\text{Var}[r(t)] = \mathbb{E} \left[\int_0^t \alpha\beta(t - s)^{\beta-1} dW_s \right]^2$$

From Ito-isometry:

$$\text{Var}[r(t)] = \mathbb{E} \left[\int_0^t (\alpha\beta(t - s)^{\beta-1})^2 ds \right]$$

$$\text{Var}[r(t)] = \int_0^t \alpha^2 \beta^2 (t - s)^{2\beta-2} ds$$

$$\text{Var}[r(t)] = \frac{\alpha^2 \beta^2}{2\beta - 1} t^{2\beta-1}$$

- (c) Describe drawbacks of this model.

Interest rate can become negative due to normal distribution assumption in the model. This suggests an arbitrage opportunity in the model.

Interest rate is expected to explode with passage of time:

$$E[r(t)] = F(0, t) + \frac{\alpha^2}{2} t^{2\beta} \rightarrow \infty \text{ as } t \rightarrow \infty$$

This is not a desired property of the model as it will introduce major instabilities in the pricing effect.

8. Learning Objectives:

1. The candidate will understand the fundamentals of stochastic calculus as they apply to option pricing.
5. The candidate will understand and identify the variety of fixed instruments available for portfolio management. This section deals with fixed income securities. As the name implies the cash flow is often predictable, however there are various risks that affect cash flows of these instruments. In general the candidates should be able to identify the cash flow pattern and the factors affecting cash flow for commonly available fixed income securities. Candidates should also be comfortable using various interest rate risk quantification measures in the valuation and managing of investment portfolios.

Learning Outcomes:

- (1a) Understand and apply concepts of probability and statistics important in mathematical finance.
- (1b) Understand the importance of the no-arbitrage condition in asset pricing.
- (1f) Demonstrate understanding of option pricing techniques and theory for equity and interest rate derivatives.
- (1h) Demonstrate understanding of the differences and implications of real-world versus risk-neutral probability measures.
- (1i) Define and apply the concepts of martingale, market price of risk and measures in single and multiple state variable contexts.

Sources:

Neftci Ch 17 page 379-399

Commentary on Question:

In general, candidates seemed to avoid conceptual question as opposed to mechanical computational problem. The question 8 is an important pre-requisite concept to study advanced interest rate derivatives, but many candidates were not ready for this type of questions.

8. Continued

Solution:

(a)

- (i) Calculate the current spot rate r known at the present time t .
- (ii) Calculate the risk neutral probabilities for the economy at time $t+2$.
- (iii) Calculate the state price of (down, down) for the economy at time $t+2$.

- (i) One year bond price = sum of state prices at $t+1$
 $= 0.3960 + 0.5941 = 0.99$.

$$\text{Spot rate} = 1/0.99 - 1 = 0.01$$

Or

$$1 = (1+r)*0.3960 + (1+r)*0.5941, r = 0.01$$

(ii)

$$\tilde{P}^{ij} = (1 + r_{t_1})(1 + r_{t_2}^i)\psi^{ij}$$

Then

$$rn_prob(up, up) = (1+0.01)*(1+0.02)*0.2330 = 0.24$$

$$rn_prob(dn, up) = (1+0.01)*(1+0.015)*0.2536 = 0.26$$

$$rn_prob(up, dn) = (1+0.01)*(1+0.02)*0.2621 = 0.27$$

$$rn_prob(dn, dn) = 1 - (0.24 + 0.26 + 0.27) = 0.23$$

(iii)

$$\frac{0.23}{(1 + 0.01)(1 + 0.015)} = 0.2244$$

- (b) Calculate the arbitrage-free price of a 2-year zero-coupon bond issued currently at time t .

Bond price is the sum of the state prices

$$0.2330 + 0.2536 + 0.2621 + 0.2244 = 0.973$$

- (c) Show that for a bond with the maturity longer than 2 years, the price after normalization by the 2-year bond is a martingale under the forward measure.

For the normalized bond price to be martingale under the forward measure π , it

needs to show $\frac{B_t}{B_t^s} = E_t^\pi \left(\frac{B_{t+2\Delta}}{B_{t+2\Delta}^s} \right)$

8. Continued

Commentary on Question:

Many candidates didn't give correct answers. Some of them progressed well but didn't finalize. It needs understanding of concepts of state-price-forward measure under no arbitrage. It also requires basic understanding of bond markets.

From the matrix (17.66) in P.282, it can be written for four states

$$B_t = B_{t+2\Delta}(s1) * \psi^1 + B_{t+2\Delta}(s2) * \psi^2 + B_{t+2\Delta}(s3) * \psi^3 + B_{t+2\Delta}(s4) * \psi^4$$

Normalizing the longer term bond by short bond by dividing by

$$B_t^s,$$

$$\frac{B_t}{B_t^s} = B_{t+2\Delta}(s1) * \frac{\psi^1}{B_t^s} + B_{t+2\Delta}(s2) * \frac{\psi^2}{B_t^s} + B_{t+2\Delta}(s3) * \frac{\psi^3}{B_t^s} + B_{t+2\Delta}(s4) * \frac{\psi^4}{B_t^s}$$

From

$$\pi^i = \psi^i / B_{t_1}^s$$

$$\frac{B_t}{B_t^s} = B_{t+2\Delta}(s1) * \pi^1 + B_{t+2\Delta}(s2) * \pi^2 + B_{t+2\Delta}(s3) * \pi^3 + B_{t+2\Delta}(s4) * \pi^4$$

$$= E^\pi [B_{t+2\Delta}] = E^\pi \left[\frac{B_{t+2\Delta}}{B_{t+2\Delta}^s} \right], \text{ because } B_{t+2\Delta}^s = 1.$$

Hence any arbitrage free bond price is martingale under the forward measure based on numeraire B_t^s

- (d) Calculate the arbitrage-free forward LIBOR rate using:
- (i) The forward measure;
 - (ii) The risk-neutral measure.

Commentary on Question:

The question asks candidates to compare two methods and let them understand the convenience in the forward measure pricing of interest rate derivatives.

8. Continued

(i)

$$\pi^i = \psi^i / B_{t_1}^S$$

$$\pi^1 = 0.2330 / 0.973 = 0.2394$$

$$\pi^2 = 0.2536 / 0.973 = 0.2606$$

$$\pi^3 = 0.2621 / 0.973 = 0.2693$$

$$\pi^4 = 0.2244 / 0.973 = 0.2306$$

$$F_{t_1} = \left[L_{t_2}^u \pi^{uu} + L_{t_2}^u \pi^{ud} + L_{t_2}^d \pi^{dd} + L_{t_2}^d \pi^{du} \right]$$

Using forward measure

$$F_{t_1} = 0.03 * 0.2394 + 0.03 * 0.2693 + 0.02 * 0.2606$$

$$+ 0.02 * 0.2306 = 0.025$$

(ii) Using risk neutral measure

$$F_{t_1} = \frac{1}{E^{\tilde{P}} \left[\frac{1}{(1+r_{t_1})(1+r_{t_2})} \right]} E^{\tilde{P}} \left[\frac{1}{(1+r_{t_1})(1+r_{t_2})} L_{t_2} \right]$$

$$E^{\tilde{P}} \left[\frac{1}{(1+r_{t_1})(1+r_{t_2})} \right] = 0.973$$

$$E^{\tilde{P}} \left[\frac{1}{(1+r_{t_1})(1+r_{t_2})} L_{t_2} \right] = 0.03 * 0.9707 * 0.24 + 0.02 * 0.9755 * 0.26$$

$$+ 0.03 * 0.9707 * 0.27 + 0.02 * 0.9755 * 0.23 = 0.0244$$

$$F_{t_1} = \frac{0.0244}{0.973} = 0.025$$

9. Learning Objectives:

6. The candidate will understand the variety of equity investments and strategies available for portfolio management.
7. The candidate will understand how to develop an investment policy including governance for institutional investors and financial intermediaries.

Learning Outcomes:

- (6d) Demonstrate an understanding of equity indices and their construction, including distinguishing among the weighting schemes and their biases.
- (7a) Explain how investment policies and strategies can manage risk and create value.

Sources:

Managing Investment Portfolios: A Dynamic Process, Maginn & Tuttle, 3rd Edition Ch. 1, 3, 6, 7

Commentary on Question:

Commentary listed underneath question component.

Solution:

- (a) Outline the steps that you would follow.

Commentary on Question:

The overall candidates' performance was satisfactory. For the full credit three steps must be listed: planning, execution, and feedback with adequate descriptions. Partial credit is given for providing list/description only on some of the steps.

The steps of the portfolio management process are planning, execution, and feedback.

1. Planning
 - Identify and specify the investor's objectives and constraints
 - Prepare the investment policy statement (IPS)
 - Form capital market expectations
 - Determine strategic asset allocation
2. Execution
 - Select the specific assets for the portfolio
 - Implement these decisions

9. Continued

3. Feedback

- Ensure the client's current objectives and constraints are satisfied
- Monitor investor-related factors and economic and market input factors
- Rebalance the portfolio
- Evaluate performance to assess progress of the investor toward achievement of investment objectives as well as to assess portfolio management skills

(b) Assess the merits of the following for your client:

- (i) Fixed income
- (ii) Equity investment

Commentary on Question:

The overall candidates' performance was satisfactory.

(i) Advantages of fixed income:

- Predetermined income
Dedication strategies can be used to accommodate specific funding needs of the investor, allowing investors to plan for known liabilities
- Risk control
A variety of quantitative tools allow fixed income portfolio managers to control risk and explain small variations in desired performances
- However, may not offer enough return for this client

(ii) Advantages of equity investment

- Inflation hedge
Common equities should offer superior protection against unanticipated inflation because companies earnings tend to increase with inflation whereas payments on conventional bonds are fixed in nominal terms. Tuition is likely to rise with inflation.
- Equities have high historical long-term real rates of return compared with bonds. About a 6% return for the endowment fund to last.
- Investing across multiple markets offers diversification benefits.

(c) Identify additional information you need to make a recommendation of an investment strategy for your client.

Commentary on Question:

Not all of the points listed below needed to be mentioned to receive the full credit for this question.

9. Continued

Additional information needed to recommend an investment strategy for the client are:

- Time horizon
- Liquidity needs
- Return objectives
- Risk tolerance
- Regulatory and tax environment
- Constraints and limitations regarding specific investments
- Performance measures and benchmarks
- Preferred investment strategies and styles
- Responsibilities of parties involved (governance).

(d) Recommend whether or not to use MSA40 index as a benchmark.

Commentary on Question:

The overall candidates' performance was satisfactory.

Recommendation: MSA40 should not be used as a benchmark index.

- The stock index only contains 40 constituents out of the 1000 potential companies. The greater the index's number of stocks and the more diversified by industry and size the better the index will measure broad market performance. MSA40 does not accurately represent the investment universe.
- The index used price return but dividends are reinvested in the portfolio. Total return should be used.
- Committee-determined indices tend to have lower turnover and therefore transaction cost and tax advantages, but more easily drift away from the market segment they are intended to cover.
- Price-weighted indices are simple to calculate and historical data is easy to obtain. However, the index is not adjusted to maintain continuity in the series, take account of stock splits, or additions/deletions of components.

(e) Recommend which of these four styles should be selected and justify your answer.

Commentary on Question:

Full credit is given if a candidate recommended any style and provided a proper justification.

9. Continued

- College savings fund has an initial investment of \$5,000,000, future contributions are uncertain, and goal is to fund tuition of \$300,000 per year.
- Size – investing in companies based on market capitalization (large, medium, and small) may produce high returns to support the college savings fund.
- Value/Growth – investing in a mixture of value stocks and growth stocks may produce high returns to support the college savings each year. Value style looks for stocks that are cheap compared to their earnings or assets. These companies earning tend to revert to mean Growth style companies will continue to grow earnings per share when their prices go up.
- Momentum – Purchase securities that had large increases in price over the past twelve months with the expectation additional gains or higher returns will follow. The returns can be used to support the college tuitions.
- Liquidity – Purchase low liquidity stocks which tend to have higher returns. The returns can be used to support college fund tuitions.

10. Learning Objectives:

2. The candidate will understand how to apply the fundamental theory underlying the standard models for pricing financial derivatives. The candidate will understand the implications for option pricing when markets do not satisfy the common assumptions used in option pricing theory such as market completeness, bounded variation, perfect liquidity, etc. The Candidate will understand how to evaluate situations associated with derivatives and hedging activities.
4. The candidate will understand the concept of volatility and some basic models of it.

Learning Outcomes:

- (2c) Compare and contrast various approaches for setting volatility assumptions in hedging.
- (4a) Compare and contrast the various kinds of volatility, (eg actual, realized, implied, forward, etc.).
- (4b) Understand and apply various techniques for analyzing conditional heteroscedastic models including ARCH and GARCH.

Sources:

Wilmott, Paul: Paul Wilmott Introduce Quantitative Finance Ch. 8 and 9

Wilmott, Paul: Frequently Asked Questions in Quantitative Finance Q38

Tsay, Ruey: Analysis of Financial Time Series, Ch. 3

Commentary on Question:

For Part A, candidates were asked to identify, describe and compare strengths and weaknesses of Model I and II. For each model, candidates received full credit if they completed the following actions:

- *Correctly identified the model*
- *Described one key feature of the model*
- *Compared three characteristics, providing at least one strength and one weakness*

Note that the sample solution provided below includes more information than a candidate would need to get full credit.

Candidates did well on Parts B and C(i), but struggled on C(ii) if they did not recognize that the model was an IGARCH model ($\alpha + \beta = 1$)

10. Continued

Solution:

- (a) Identify, describe, and compare strengths and weaknesses of Model I and II.

Model I: Auto-regressive conditional heteroscedastic model - ARCH(1)

The key characteristics of the ARCH(1) model are:

1. The innovation terms for the asset returns (rt) are serially uncorrelated, but dependent
2. This dependence can be described by a simple quadratic function of its lagged value

Strengths:

1. There is volatility clustering. Large shocks tend to be followed by another large shock. Observed in other financial time series (e.g., the Deutsche mark/US dollar exchange rate in 10 minute intervals)
2. The excess kurtosis of the innovation term is positive and the tail distribution is heavier than that of a normal distribution. More likely to produce outliers, which agrees with empirical evidence.

Weaknesses:

1. The model assumes that positive and negative shocks have the same effects on volatility because it depends on the square of the previous shocks
2. The model is restrictive. α_2 must be between the interval $[0, 1/3]$ if the series has a finite fourth moment, limiting the model's ability to capture excess kurtosis
3. The model does not provide any new insight for understanding the source of variations. Provides a mechanical way to describe the behavior of the conditional variance, but gives no indication about its causes
4. The model is likely to overpredict the volatility since it responds slowly to large isolated shocks to the return series.

Model II: Exponentially weighted moving average (EWMA)

The key characteristics of the EWMA model are:

1. EWMA is a special case of ARCH(n) model where the weights decrease exponentially as it goes back through time

Strengths:

1. Requires a relatively small amount of data to be stored. Instead of n historical observations, the model only requires the current estimate of the variance rate and the most recent observation of the market variable
2. The model tracks changes in the volatility estimate and updates it through time

10. Continued

Weaknesses:

1. Lacks a mean reversion component

(b) Determine $\text{Var}(r_n)$ of Model I (assuming r_n and σ_n are stationary processes).

$$\text{Var}(r_n) = \frac{0.0388}{(1 - 0.0615)} = 0.04134$$

(c) Using, $\sigma_n^2(1) = 0.0016\%$,

(i) Calculate $\sigma_n^2(20)$, the expected variance of daily returns for Model II.

(ii) Calculate $\sigma_n^2(20)$, the expected variance of daily returns for Model III.

(i) For the EWMA model, the best estimate of volatility k periods into the future is today's value = 0.0016%

(ii) For an IGARCH(1,1) model, the best estimate of volatility increases by the constant coefficient term every time period:

$$\begin{aligned}\sigma_n^2(20) &= (20 - 1) \times 0.00024\% + \sigma_n^2(1) \\ \sigma_n^2(1) &= 0.0016\% \\ \sigma_n^2(20) &= 0.00616\%\end{aligned}$$

11. Learning Objectives:

5. The candidate will understand and identify the variety of fixed instruments available for portfolio management. This section deals with fixed income securities. As the name implies the cash flow is often predictable, however there are various risks that affect cash flows of these instruments. In general the candidates should be able to identify the cash flow pattern and the factors affecting cash flow for commonly available fixed income securities. Candidates should also be comfortable using various interest rate risk quantification measures in the valuation and managing of investment portfolios.

Learning Outcomes:

- (5a) Explain the cash flow characteristics and pricing of Treasury securities.
- (5f) Evaluate different private money market instruments.

Sources:

Fabozzi Ch9, Ch 16

Commentary on Question:

This is an easy question but many candidates either missed part c) completely or did not address it appropriately.

Solution:

- (a) List the four types of securities issued by the U.S. Treasury, describe their cash flow characteristics and discuss appropriateness of each of them for a money market fund.

1) Treasury Bills: Short term in nature (<1yr maturity); discount-value (no coupons paid).

It is appropriate for money market fund (MM).

2) Treasury Notes: Medium term in nature (1-10 year maturity); interest bearing and coupons paid with face value return at maturity.

11. Continued

It is not appropriate for MM.

3) Treasury Bonds: Long term in nature (>10 year maturity); interest bearing and coupons paid with face value return at maturity.

It is not appropriate for MM.

4) Treasury Inflation Protected Securities (TIPS): 5/10/30 year maturity; includes interest payments which adjusts based on the change of inflation.

It is not appropriate for MM.

- (b) Describe the characteristics of the three assets listed above and explain why they offer higher return possibilities than U.S. treasuries.

Considerations:

1) Commercial Paper

- Short-term unsecured notes to open market, typically <270days maturity;
- Yields are higher than treasury due to Credit Risk and lower liquidity;
- Different taxation than treasuries.

2) Certificate of Deposit

- Certificate issued by a bank that a specific sum is deposited at the issuing depository institution. They are negotiable or non-negotiable. If negotiable it can be sold on open market.
- Yields are higher due to credit risk and low liquidity.

3) Repurchase agreement

- Sale of security with commitment to buy it back in the future at a specified price. Thus is effectively a secured loan which is secured by the underlying asset.
- Higher yield due to exposure to non-performance of underlying asset.

- (c) Identify and describe the characteristics of these assets that reduce market risk and liquidity risk.

Commentary on Question:

Many candidates skipped this part and those who tried to answer it generally repeated that the money market instruments were very short term, that they were very tradable so that there is no problem of liquidity risk or of market value risk.

11. Continued

However, the question was about the characteristics that can reduce those risks (lines of credit; mark collateral to market; keep the amount loaned less than the value of the collateral, etc...).

Commercial paper

- Have liquidity enhancements (LOC Papers) against the Rollover risk (risk of not being able to issue new paper at Maturity);
- Secure backup lines of credit;
- Get credit support by a firm (credit supported commercial paper);
- Letter of credit;
- Collateralize the issue with high quality assets (asset-backed commercial paper).

Certificate of deposit

- Have negotiable CD's that can be sold to open market prior to maturity (liquidity);
- CDs insured by the FDIC.

Repurchase agreement

- Keep amount loaned less than the value of the collateral;
- Mark collateral to market;
- Ensure a liquid collateral is provided;
- Ensure a collateral with a stable market value is provided;
- Adequate security around holding of collateral (ie transfer to lenders account rather than just hold in segregated account);
- Avoid substitution clauses for the collateral.

(d) Describe the following risk-quantification metrics for the fund and recommend one:

- (i) Roy's Safety First Criterion
- (ii) Risk adjusted expected return

Commentary on Question:

Although the RSF criterion is in our view more appropriate in the case of a money market fund where safety is paramount, points were awarded for the RAER recommendation based on the fact that the management was willing to increase the risk tolerance factor.

11. Continued

Roy's safety first criterion

-Safety First Ratio (SF) = $(\text{Exp}(R_p) - R_L) / \sigma_p$, where $\text{Exp}(R_p)$ is expected return of portfolio, R_L is minimum threshold return, and σ_p is standard deviation of return.

The goal is to minimize the probability that portfolio return falls below the threshold level (i.e. maximize the SF ratio).

-Risk adjusted expected return

Investor's Utility $U_m = \text{Exp}(R_m) - 0.005 R_A * \sigma^2$, where $E(R_m)$ is expected return, σ is variance of returns, R_A is investor's risk aversion

Expected return is adjusted for the risk.

Investor select higher Utility investment (after risk is adjusted)

Recommendation: As the risk objective is to minimize risk to capital, the Safety First Ratio is recommended.

12. Learning Objectives:

8. The candidate will understand the theory and techniques of portfolio asset allocation

Learning Outcomes:

- (8a) Explain the impact of asset allocation, relative to various investor goals and constraints.
- (8b) Propose and critique asset allocation strategies.
- (8c) Evaluate the significance of liabilities in the allocation of assets.

Sources:

QFIC-107-13: Revisiting the Role of Insurance Company ALM within a Risk Management Framework

Commentary on Question:

Commentary is listed underneath question component.

Solution:

- (a) Describe a typical ALM approach from a “bottom-up” perspective.

Commentary on Question:

Most candidates tried to describe the ALM approach in a general sense instead of focusing on the approach in a “bottom-up” perspective. Partial credits are awarded if a candidate mentions a typical ALM approach strategy, such as cash flow matching and duration matching.

A typical ALM approach from the “bottom-up” perspective would be such that:

- Focus will be on assets backing reserves independent of surplus
- For insurers writing many different liability product types, there is a separate investment portfolio backing the reserves for each major liability type
- The reserve backing portfolio has an objective of closely matching the cash flows, or interest rate duration, of the liabilities
- The insurer’s surplus portfolio is managed consistently with the goal of capital preservation

- b) Propose an action plan to accomplish the third step if the investment portfolio is required to support liability cash flows as well as adequate surplus.

Commentary on Question

Many candidates spent too much effort on describing an ideal asset composition of the portfolio (invest in bonds versus corporate stocks or treasuries or surplus notes etc.) instead of on how to match the characteristics of the liability to mitigate risks.

12. Continued

1. Understand the liability profile: using best estimate liability cash flows, analyze the liability profile and capture features such as the maturity profile and minimum guarantees
 2. Develop the duration profile: Construct the overall duration as well as the key rate duration (KRD) profile of the liabilities to capture the sensitivity of the liability value to movements and changes in the term structure of interest rates.
 3. Design a risk minimizing profile: Replicate the liability cash flow using investable or hedge assets to match key characteristics of the liabilities including KRD and guarantees.
- (c) List four risk metrics that can be used to quantify the risk characteristics of an investment strategy in the context of ALM and SAA.

Commentary on Question:

Partial credit was given to other answers such as duration gap, duration, RACS. Candidates are not required to come up with the full list to receive full credit.

- Asset-only volatility
 - Surplus volatility
 - Economic capital
 - Required capital
 - Value at Risk (VaR)
 - Conditional Value at Risk (CVaR)/ CTE (Contingent Tail Expected Value)
 - Surplus drawdown risk
- (d) Identify which approach is the most efficient in an asset-only framework based on the case study result shown above. Explain why.

Commentary on Question:

Candidates did well on this part. Most candidates were able to identify the most efficient approach and provide a reasonable explanation for their choice.

Approach 1 is the most efficient approach.

Its efficient frontier is constructed to minimize asset portfolio volatility, while those of Approach 2 and Approach 3 are constructed to minimize the economic surplus volatility, which has incorporated the liability cash flows.

- (e) Identify which approach is the most efficient if both the asset and the liability are taken into consideration. Explain why.

12. Continued

Commentary on Question:

Candidates did well on this part. Most candidates were able to identify the most efficient approach and provide a reasonable explanation for their choice. The critical part is to differentiate Approach 3 and Approach 2 (i.e., identify benefits of Approach 3 over Approach 2 under given consideration).

Approach 3 is the most efficient approach.

This is because approach 3 seeks to minimize surplus volatility, while imposing different duration constraints. Approach 2 has a tighter duration constraint than Approach 3. Approach 3 allows for more duration mismatches and therefore gets more diversification benefits.

- (f) Identify any potential risk for the approach you have identified in (e). Explain if you should be concerned.

Commentary on Question:

Most candidates were able to recognize that there will be a larger duration gap for Approach 3. Some candidates only got partial credit if they did not explain how the duration gap will create risk for the portfolio.

The additional risk that Approach 3 introduces is more interest rate risk. The duration mismatch for this approach is much higher than Approach 2 and Approach 1.

- (g) Rank the three approaches in terms of the following key measures:
- Asset portfolio risk
 - Surplus risk
 - Economic capital requirement
 - Diversification across market risk

Commentary on Question:

Most candidates look at each of the key measures and rank the three approaches from best to worst, instead of analyzing the approaches according to their chances to meet the goals. This is acceptable as long as candidates appropriately recognize the risk levels for each of the approaches. While the table is directly from the reading material, it requires candidates to fully understand the pros and cons of different approaches under different risk measures.

12. Continued

Approach	Allocation of Market Risk			Diversification Across Market Risk
	Portfolio Risk	Surplus Risk	Economic Capital Requirements	
1. Minimize portfolio volatility with KRD constraints	✓✓✓	x	x	✓
2. Minimize surplus volatility with KRD constraints	✓	✓	✓	✓✓
3. Minimize surplus volatility relaxing KRD constraints	x	✓✓✓	✓✓✓	✓✓✓

- | | |
|-----|-----------------------------|
| ✓✓✓ | Most likely
to meet goal |
| ✓✓ | More likely
to meet goal |
| ✓ | Less likely to
meet goal |
| x | Not likely to
meet goal |

13. Learning Objectives:

5. The candidate will understand and identify the variety of fixed instruments available for portfolio management. This section deals with fixed income securities. As the name implies the cash flow is often predictable, however there are various risks that affect cash flows of these instruments. In general the candidates should be able to identify the cash flow pattern and the factors affecting cash flow for commonly available fixed income securities. Candidates should also be comfortable using various interest rate risk quantification measures in the valuation and managing of investment portfolios.

Learning Outcomes:

- (5h) Construct and manage portfolios of fixed income securities using the following broad categories.
- (i) Managing funds against a target return
 - (ii) Managing funds against liabilities.

Sources:

Managing Investment Portfolios: A Dynamic Process, Maginn & Tuttle, 3rd Edition Ch. 6, Fixed Income Portfolio Management

Commentary on Question:

Commentary listed underneath question component.

Solution:

- (a) List advantages and disadvantages of using forwards rather than futures to lengthen the duration of the bond portfolio to match liabilities.

Commentary on Question:

Most candidates did well on this question. Many candidates received full credit and some received partial credit.

Advantages of futures:

- The futures market is more liquid than the forward market, easier to exit contract if necessary
- Lower transaction costs for futures contract
- Futures contracts have lower credit risk due to being traded on exchange

Advantages of forwards

- Can create bespoke contract on individual bonds, thus reducing basis risk
- More flexibility in collateral arrangements

13. Continued

- (b) Calculate the forward price in 1 year per \$1,000,000 notional of the underlying to the nearest dollar.

Commentary on Question:

Candidates either received full credit on this question or received no credit. This was a straight forward calculation and most candidates did well. Full credit was given for the correct answer regardless of the method to derive it. Full credit was also given if candidates used continuous interest rates in their calculations.

Value of forward contract at inception is zero, hence present value of \$1mm payout in 30 years must equal present value of exercise price in 1 year
Hence $\$1,000,000/1.03^{30} = \text{Exercise price}/1.02$

$$\text{Thus Exercise Price} = \$1,000,000 * 1.02 / 1.03^{30} = \$420,226$$

Alternatively the 1 to 30 year forward rate equals $((1.03^{30})/1.02)^{(1/29)} - 1 = 3.0347\%$

$$\text{Thus exercise price} = \$1,000,000 / 1.030347^{29} = \$420,226$$

- (c) Calculate the change in value per \$1,000,000 notional of the forward for a 1 basis point movement in spot rates.

Commentary on Question:

Candidates did poorly on this question. Shock in "spot rate" means time zero shock. Many candidate calculate a different forward price, but the question is asking for change in forward contract value after locking in forward price calculated in part (B)

$$\begin{aligned} &\text{Dollar duration 1 basis point up shock} \\ &= \$1,000,000 / 1.0301^{30} - \$420,226/1.0201 = -\$1,157.3 \end{aligned}$$

$$\begin{aligned} &\text{Dollar duration 1 basis point down shock} \\ &= \$1,000,000 / 1.0299^{30} - \$420,226/1.0199 = \$1,161.9 \end{aligned}$$

$$\text{Final answer} = (-\$1,157.3 - \$1,161.9)/2 = -\$1,159.6$$

- (d) Recommend the position and the notional value (in millions) of forward contracts that need to be entered into to match the durations of the liability and asset portfolios.

Commentary on Question:

Many candidates received partial credit as they understood the concept that a long position on the forward was needed. However, most candidates did not correctly calculate the number of forward contracts needed.

13. Continued

Dollar duration \$1 million notional is \$1,159.6.

Dollar duration difference in liabilities is $\$20,000,000 \times (15-12) = \$60,000,000$

1 bp movement translated into \$600,000

To match duration need to go long (buy) forward contracts

Notional = $(\$600,000 / \$1,159.6) = 517.68$ mm

- (e) Estimate and interpret the impact on the forward contract value of each of the following changes in interest rates:
- (i) 0.1% increase in 1-year spot rate;
 - (ii) 0.1% increase in 30-year spot rate.

Commentary on Question:

Many candidates did not attempt this question. Some candidates received partial credit.

Estimate 1 yr KRD up shock

$= \$1,000,000 / 1.03^{30} - \$420,226 / 1.021 = \$404$

Estimate 30 yr KRD up shock

$= \$1,000,000 / 1.031^{30} - \$420,226 / 1.02 = -\$11,820$

1 year KRD: Shock 1 year spot rate, whilst holding 30 year spot rate constant.

Thus forward rate between 1 – 30 years decreased by (1/29) of the amount.

This makes the lock-in at higher forward rate more valuable (i.e. increase in value)

30 year KRD: Shock 30 year spot rate, hence shock all forward rates between 1 and 30 years.

This makes the lock-in lower forward rate less valuable (i.e. loss in value)

Observation 1) Even though both cases show the increase in rates, the profit & loss impact is opposite because they will have different impact on forward rates.

Observation 2) Loss in part (ii) is roughly 30 times more than gain in part (i).

This should be an important observation. Impact on the forward rates will be 30 times more when 30 year rate is shocked.

14. Learning Objectives:

5. The candidate will understand and identify the variety of fixed instruments available for portfolio management. This section deals with fixed income securities. As the name implies the cash flow is often predictable, however there are various risks that affect cash flows of these instruments. In general the candidates should be able to identify the cash flow pattern and the factors affecting cash flow for commonly available fixed income securities. Candidates should also be comfortable using various interest rate risk quantification measures in the valuation and managing of investment portfolios.

Learning Outcomes:

- (5b) Demonstrate an understanding of par yield curves, sport curves, and forward curves and their relationship to traded security prices.
- (5f) Evaluate different private money market instruments.
- (5h) Construct and manage portfolios of fixed income securities using the following broad categories.
 - (i) Managing funds against a target return
 - (ii) Managing funds against liabilities.

Sources:

Managing Investment Portfolios: A Dynamic Process, Maginn & Tuttle, 3rd Edition
Ch. 7 Equity Portfolio Management

Commentary on Question:

Commentary listed underneath question component.

Solution:

- (a) Identify and describe the investment approach of each manager.

Commentary on Question:

This is straightforward, most candidates did well on this part.

- Manager A – Passive management / Indexing
 - Does not attempt to reflect manager's investment expectations through changes in security holdings)
- Manager B – Semi-active management / Enhanced indexing / Risk-controlled active management
 - Manager seeks to outperform the given benchmark
 - Worries more about tracking risk than active manager
 - Tend to build a portfolio whose performance has limited volatility around the benchmark's returns

14. Continued

- Manager C – Active management
 - Manager seeks to clearly outperform the given benchmark
 - Buys stocks that will perform comparatively well versus the benchmark portfolio, and avoids stocks he believes will underperform
- (b) Identify the type of portfolio you plan to construct given the allocation between investment managers, and explain the objectives of your strategy.

Commentary on Question:

Most candidates did well on this part

- The portfolio of managers represents a core-satellite strategy
 - Objective is to anchor a strategy with either an index portfolio or an enhanced index portfolio
 - “Anchor” in this case represents more than 50% of total allocation
 - And use active managers opportunistically around that anchor to achieve an acceptable level of active return
 - While mitigating some of the active risk associated with a portfolio consisting of only active managers
 - The index or enhanced index portfolios should resemble as closely as possible the investor’s benchmark for the asset class
- (c) Evaluate whether the portfolio is expected to achieve its objective.

Commentary on Question:

Most candidates did well on this part. Partial credits are given when a few candidates use the expected return, not the active return, to calculate the Information Ratio. .

- Calculate expected alpha
- Weights:
 - Manager A: $600/1000 = 60\%$
 - Manager B: $200/1000 = 20\%$
 - Manager C: $200/1000 = 20\%$
- Active alpha = $60\%(8.0\% - 8.0\%) + 20\%(9.0\% - 8.0\%) + 20\%(15.0\% - 8.0\%) = 1.6\%$
- Portfolio’s expected tracking risk = $(60\%^2(0.0\%^2)+20\%^2(0.5\%^2)+20\%^2(9.0\%^2))0.5 = 1.80278\%$ Information ratio (IR) = $0.89 = 1.6\%/1.8\%$
- IR > 0.6 thus it meets the objective

14. Continued

- (d) Describe the purpose and characteristics of a completeness fund, and recommend whether you should include one in your portfolio.

Commentary on Question:

This part is fairly straightforward; however, most candidates did not do well. Probably because they did not study the definition of a completeness fund. Also, candidates had difficulty recommending whether a completeness fund was needed.

- The aggregate portfolio of active managers may have unintended risk exposures or biases
- A completeness fund establishes an overall portfolio with approximately the same risk exposures as the investor's equity benchmark, while attempting to retain the value added from the active manager's stock-selection ability
- The portfolio needs to be re-estimated periodically to reflect changes in active portfolios
- One drawback is that it essentially seek to eliminate misfit risk, where a nonzero amount of misfit risk may be optimal
- You may not need to include a completeness fund given that
 - The three managers have the same risk exposures as your company's equity benchmark
 - The active manager (Manager C) has relatively low allocation (20%)
 - Which makes its potential risk exposures relatively immaterial

15. Learning Objectives:

7. The candidate will understand how to develop an investment policy including governance for institutional investors and financial intermediaries.

Learning Outcomes:

- (7a) Explain how investment policies and strategies can manage risk and create value.
- (7b) Identify a fiduciary's obligations and explain how they apply in managing portfolios.
- (7c) Determine how a client's objectives, needs and constraints affect investment strategy and portfolio construction. Include capital, funding objectives, risk appetite and risk-return trade-off, tax, accounting considerations and constraints such as regulators, rating agencies, and liquidity.
- (7d) Incorporate financial and non-financial risks into an investment policy, including currency, credit, spread, liquidity, interest rate, equity, insurance product, operational, legal and political risks.

Sources:

Managing Investment Portfolios: A Dynamic Process, Maginn & Tuttle, 3rd Edition
Ch. 3

Commentary on Question:

Commentary listed underneath question component.

Solution:

- (a) List the bank's objectives in managing a securities portfolio, according to the process laid out by Maginn, et al. in "Managing investment portfolios."

Commentary on Question:

Overall, most candidates did well on part (a).

Banks' objectives in managing securities portfolios (in the order of importance):

1. To manage overall interest rate risk of the balance sheet

Bank-held securities can be bought and sold quickly to adjust for interest rate risk. Banks can manage duration risk by shortening or extending the maturities of their security portfolios.

2. To manage liquidity

Banks use their securities portfolio to assure adequate cash is available.

3. To produce income

Banks need to earn a positive spread on invested capital over the cost the capital in order to stay in business.

15. Continued

4. To manage credit risk

Portfolio is used to modify and diversify the overall credit risk exposure to a desired level.

5. To meet other needs

To meet government securities as collateral (pledge) against the uninsured portion of deposits.

Regulatory constraints: regulators impose constraints on the percentage and the credit quality of fixed-income securities that a bank should hold, or risk-based capital (RBC) requirement. Banks have to comply with regulatory requirements.

- (b) Calculate the bank's net interest margin.

Commentary on Question:

Most candidates did well on this question. Partial credit is given for various steps of the calculation in the solution below. Partial credit is given if a candidate used the average market value of invested assets.

Net interest margin is the ratio of net interest income to invested assets.

Net interest margin = (interest received – interest paid) / Average Invested Assets

Interest amount = Book Value x Book Yield per Year

Interest received (income) = [(\$60 x 3%) + (\$30 x 8%) + (\$10 x 0%)] = \$3.60 + \$2.40 + \$0 = \$6.00

Interest paid (expense) = Book Value x \$0.60 + \$0.80 = \$1.40

Average invested assets = \$60 + \$30 + \$10 = \$100 million

Net interest margin = (\$6.00 - \$1.40)/\$100 = 4.6%

[The candidate will also receive credit if he/she were to answer:

Net interest margin = (\$6.00 - \$1.40)/\$120 = 3.83%.]

- (c) Calculate the bank's leverage-adjusted duration gap.

Commentary on Question:

Full credit was given if a candidate successfully calculated the bank's leverage-adjusted gap. Partial credit was given for various steps of the calculation in the solution below.

15. Continued

Leverage-adjusted duration gap is defined as $DA - k \cdot DL$, where DA is the duration of assets, DL is the duration of liabilities, and k is the ratio of market value of liabilities to the market value of assets ($k = L/A$)

Portfolio durations of assets and liabilities are calculated using a weighted average of market values:

$$DA = [(\$70 \times 2.5) + (\$40 \times 3.0) + (\$10 \times \$0)] / [\$70 + \$40 + \$0] = 2.458$$

$$DL = [(\$20 \times 5.0) + (\$80 \times 0.5)] / [\$20 + \$80] = 1.40$$

$$k = (\$20 + \$80) / (\$70 + \$40 + \$0) = \$100 / \$120 = 0.833$$

$$\text{Leverage-adjusted duration gap} = 2.458 - 0.833 \times 1.40 = 1.29.$$

- (d) Assess the impact of an unexpected positive interest rate shock of 100 basis points on the bank's balance sheet.

Commentary on Question:

Full credit was also given if the candidate used portfolio durations of assets and liabilities to determine the impact of an unexpected positive interest rate shock.

The duration gap calculated in part (c) is a measure that takes into account a bank's overall exposure to interest rate risk. Because the bank has a positive leverage-adjusted duration gap of 1.29, its market value net worth, or economic value of equity, will decrease if interest rates suddenly rises.

$$\text{Impact on bank's net worth} = - (1\% \times \text{LADG} \times \text{MV}) = - (1\% \times 1.29 \times \$120) = -\$1.55$$

Decrease of \$1.55 million.

16. Learning Objectives:

5. The candidate will understand and identify the variety of fixed instruments available for portfolio management. This section deals with fixed income securities. As the name implies the cash flow is often predictable, however there are various risks that affect cash flows of these instruments. In general the candidates should be able to identify the cash flow pattern and the factors affecting cash flow for commonly available fixed income securities. Candidates should also be comfortable using various interest rate risk quantification measures in the valuation and managing of investment portfolios.
8. The candidate will understand the theory and techniques of portfolio asset allocation.

Learning Outcomes:

- (5c) Demonstrate understanding of the different characteristics of securities issued by government agencies.
- (8a) Explain the impact of asset allocation, relative to various investor goals and constraints.
- (8b) Propose and critique asset allocation strategies.
- (8c) Evaluate the significance of liabilities in the allocation of assets.
- (8d) Incorporate risk management principles in investment policy and strategy, including asset allocation.

Sources:

Managing Investment Portfolios: A Dynamic Process, Maginn & Tuttle, 3rd Edition
Ch. 5

Inflation-Linked Bonds, Fabozzi Handbook, Ch. 18

Commentary on Question:

Question 16 tests candidates' understandings of basic investment concepts. Question a) and b) are about TAA and question c) is on TIPS.

Question a) is an open ended question. Overall, candidates didn't do very well on this question. Partial credits were awarded if candidates provided reasonable analysis.

Candidates did better on question b) than a), but still many got confused between the factors contributing to SAA and TAA.

For question c), most candidates know that TIPS can provide inflation protection, but not many of them did very well in explaining the impacts of including TIPS together with other assets.

16. Continued

Solution:

- (a) Assess the analyst's recommendation.

Commentary on Question:

Partial credits are awarded to other reasonable statements

The analyst's suggestion is within the tactical ranges allowed by the endowment's IPS. Therefore, the suggestion is feasible.

If the analyst's beliefs are correct, the increase in the Treasury yield indicates a decrease in the probability of a U.S. recession in one year, therefore outweighing U.S. equities and shifting assets from bond into equities is logical.

High return on corporate bond index last year implies yields and spreads come down, so bonds are less attractive now and shifting out of bonds makes sense.

The fact that many funds are outweighing equities suggests investors do not perceive equities to be riskier, but that could be misleading.

- (b) Identify additional information that should be considered before the tactical asset allocation changes.

Commentary on Question:

Partial credits were also awarded if the following factors were mentioned:

- *Changes in asset's underlying risk attributes*
- *Changes in central bank policy*
- *Changes in expected inflation*
- *Position in the business cycle*

The following information to be considered:

- 1) The costs of the tactical adjustment in relation to the expected benefits.
- 2) The increase in tracking risk and the change in expected absolute risk in related to expected benefits
- 3) The economic logic of analyst's belief. If they have an economic logic, it is more likely that relationships based on past observations will hold for the future.
- 4) The strength of the expected relationship. After the allocation adjustment, the portfolio may be less well diversified than previously. Is the size of the bet justified?
- 5) The presence of any factor such as a change in the risk attributes of assets that may make past relationship fail to hold in the future.

16. Continued

- (c) Critique the validity of each of the reasons listed above.
1. TIPS permits increased flexibility in meeting investor goals affected by inflation and deflation. In an inflationary environment, its coupon is inflation protected. In a deflationary environment, its principal is deflation protected. However, nominal bonds offer better deflation protection.
 2. High within-group correlations are consistent with the criteria that asset within an asset class should be relatively homogenous, supporting the view that TIPS forms a separate asset class.
 3. All statements are true. This economic argument is consistent with the criteria that asset classes be diversified.

Given the scenario, including TIPS is a good idea.

17. Learning Objectives:

5. The candidate will understand and identify the variety of fixed instruments available for portfolio management. This section deals with fixed income securities. As the name implies the cash flow is often predictable, however there are various risks that affect cash flows of these instruments. In general the candidates should be able to identify the cash flow pattern and the factors affecting cash flow for commonly available fixed income securities. Candidates should also be comfortable using various interest rate risk quantification measures in the valuation and managing of investment portfolios.

Learning Outcomes:

- (5e) Describe the cash flow of various corporate bonds considering underlying risks such as interest rate, credit and event risks.

Sources:

The Handbook of Fixed Income Securities, Fabozzi, Frank, 8th Edition
o Ch. 12, 16-18, 21, 24-26

Commentary on Question:

The concept (part b) of convexity was not answered very well.

In part d, many candidates have not chosen the correct discount CMT rate. Also not many candidates expressed the condition of the redemption price being higher than the principal amount plus accrued interest.

Solution:

- (a) Explain the advantages and disadvantages of each type from the bond issuer's perspective

Call Provisions:

It gives the issuer the right to buy the bonds back at a fixed price either in whole or in part prior to maturity.

For the Make-Whole, the call price dependent on Treasury yields. It has a floating call price.

Advantage of Callable bond:

-Could gain if the debt is refinanced at a lower rate when rates fall.

Advantage of Callable bond with Make-whole provision:

-Lower borrowing cost than coupon rate than no-Make-whole provision while maintaining the flexibility of early retirement of debt. However, there is no advantage to retire the debt solely due to the decrease of the interest rates

17. Continued

- (b) Compare their coupon rates if issued at par.

Coupon rates for callable bond with no Make-whole provision will be substantively higher than callable bond with Make-whole provision.

Higher coupon is to compensate investor for shorting optionality (call provision at fixed scheduled price).

- (c) Compare their effective convexities.

Convexity is a measure of the sensitivity of the duration of a bond's price to changes in interest rates, i.e. the second derivative of the price with respect to interest rates.

Positive convexity of a bond is represented by a larger price increase due to a decline in the yield than price declines due to an increase in yield.

The capital appreciation potential of a callable bond may not rise as much above the price at which the issuer may call the bond.

For a fixed price call provision, the maximum price of the unprotected fixed price callable bond is bounded above by the call price. Callable bonds with fixed price call provisions have negative convexity.

With make-whole call provisions, the make-whole call price increases and decreases with interest rate changes as a standard non-callable bond would. Therefore, it has standard, positive convexity.

- (d) Calculate the current redemption price.

The Redemption price is the greater of:

- * 100% of the principal amount plus accrued interest or;
- * make-whole redemption price.

The Make-whole redemption price = sum of the present value of the remaining coupon and principal payments discounted at the Treasury rate plus make-whole call premium.

The discount rate = 4-Year CMT + make-whole call premium
 $= 1.4\% + 0.2\% = 1.6\%$

PV(coupon) @ 1.6% + PV(Principal) @ 1.6%

17. Continued

$$\begin{aligned} &\text{Since 1.6\% is BEY (or semi-annual coupon)} \\ &= \$200,000((1.008)^{-1} + (1.008)^{-2} + (1.008)^{-3} + (1.008)^{-4} + (1.008)^{-5} + (1.008)^{-6} \\ &\quad + (1.008)^{-7} + (1.008)^{-8}) + \$10 \text{ million}((1.008)^{-8}) \\ &= \$10,926,342 \end{aligned}$$

This is greater than 100% of the principal amount plus accrued interest, so this is the redemption price.