SOCIETY OF ACTUARIES
Quantitative Finance and Investment Core

Exam QFICORE

MORNING SESSION

Date: Wednesday, April 29, 2015
Time: 8:30 a.m. – 11:45 a.m.

INSTRUCTIONS TO CANDIDATES

General Instructions

1. This examination has a total of 100 points. It consists of a morning session (worth 60 points) and an afternoon session (worth 40 points).

   a) The morning session consists of 9 questions numbered 1 through 9.

   b) The afternoon session consists of 8 questions numbered 10 through 17.

   The points for each question are indicated at the beginning of the question.

2. Failure to stop writing after time is called will result in the disqualification of your answers or further disciplinary action.

3. While every attempt is made to avoid defective questions, sometimes they do occur. If you believe a question is defective, the supervisor or proctor cannot give you any guidance beyond the instructions on the exam booklet.

Written-Answer Instructions

1. Write your candidate number at the top of each sheet. Your name must not appear.

2. Write on only one side of a sheet. Start each question on a fresh sheet. On each sheet, write the number of the question that you are answering. Do not answer more than one question on a single sheet.

3. The answer should be confined to the question as set.

4. When you are asked to calculate, show all your work including any applicable formulas. When you are asked to recommend, provide proper justification supporting your recommendation.

5. When you finish, insert all your written-answer sheets into the Essay Answer Envelope. Be sure to hand in all your answer sheets because they cannot be accepted later. Seal the envelope and write your candidate number in the space provided on the outside of the envelope. Check the appropriate box to indicate morning or afternoon session for Exam QFICORE.

6. Be sure your written-answer envelope is signed because if it is not, your examination will not be graded.

Tournez le cahier d’examen pour la version française.

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****BEGINNING OF EXAMINATION****
Morning Session

1. (*5 points*) In a one-period binomial model, the current price of a stock is S and the prices at the end of the period will be:

$$\begin{cases} Su(1+\delta), & \text{on the upper path} \\ Sd(1+\delta), & \text{on the lower path} \end{cases}$$

where $u > d$ are constants, and δ is the dividend accrual rate.

The current price, B, of a risk-free bond will increase to $B(1+r)$ at the end of the period. An option on the stock has a payoff of C^u on the upper path, and C^d on the lower path.

(a) (*3 points*) Derive the price of the option assuming $d(1+\delta) < 1+r < u(1+\delta)$.

(b) (*1 point*) Explain why it must be assumed that $d(1+\delta) < 1+r < u(1+\delta)$.

(c) (*1 point*) Analyze the impact on the price of the option if the stock dividend is increased to $\delta^* > \delta$.

2. (*5 points*) Suppose that the stock price S_t follows the following stochastic differential equations (SDE):

$$\frac{dS_t}{S_t} = \mu dt + \sqrt{v_t}\, dZ_t,$$

$$dv_t = \kappa(\theta - v_t)dt + \sigma\sqrt{v_t}\, dW_t,$$

where

- v_t is the volatility,
- W_t and Z_t are Brownian processes such that $dZ_t\, dW_t = \rho dt$ with $|\rho| < 1$, and
- $\mu, \kappa, \theta, \sigma,$ and ρ are constants.

We are to price a derivative with payoff contingent on volatility.

Consider the following self-financing portfolio Π:

$$\Pi(S_t, v_t, t) = f(S_t, v_t, t) + \delta S_t + \gamma U(S_t, v_t, t),$$

where δ and γ are parameters to be determined;

$f(S_t, v_t, t) =$ the value at time *t* of a European derivative with expiration *T* and terminal payoff $g(S_T)$;

$U(S_t, v_t, t) =$ the value at time *t* of a European derivative with expiration *T* and terminal payoff $h(v_T)$.

(a) (*2 points*) Derive $d\Pi(S_t, v_t, t)$, the change in the portfolio value, in terms of $dt, dS,$ and dv_t.

(b) (*3 points*) Solve for the weights δ and γ so as to make the portfolio Π riskless. Express the resulting $d\Pi(S_t, v_t, t)$ with the parameters δ and γ obtained.

3. (*8 points*) Suppose we have a market containing the following 2 risky portfolios:

Portfolio 1:
Two non-dividend-paying assets whose prices satisfy the following:

$$dU_1(t) = \mu_1 U_1(t)\,dt + \sigma_1 U_1(t)\,dW(t)$$
$$dU_2(t) = \mu_2 U_2(t)\,dt + \sigma_2 U_2(t)\,dW(t)$$

where $W(t)$ is a Brownian motion under a probability measure $\mathbb{P}_1$, and μ_1, μ_2 and $\sigma_1 > 0$, $\sigma_2 > 0$ are constants.

Portfolio 2:
Two non-dividend-paying assets whose prices satisfy the following stochastic processes:

$$dS_1(t) = m_1 S_1(t)\,dt + v_1 S_1(t)\,dW_1(t)$$
$$dS_2(t) = m_2 S_2(t)\,dt + v_2 S_2(t)\,dW_2(t)$$
$$W_2(t) = \rho W_1(t) + \sqrt{1-\rho^2}\, W_0(t)$$

where $W_0(t)$ and $W_1(t)$ are independent Brownian motions with probability measure $\mathbb{P}_2$, $m_1, m_2, v_1 > 0, v_2 > 0$ are constants, and ρ is a constant such that $|\rho| \le 1$.

Let k be the value such that $\ln\left[e^{-kt} U_2(t)\right]$ is a martingale under the probability measure $\mathbb{P}_1$.

(a) (*1 point*) Derive an expression for k in terms of μ_2 and σ_2.

Let r be the instantaneous constant risk-free rate.

(b) (*1 point*) Write an expression for the relationship among μ_1, σ_1, and μ_2, σ_2, and r necessary for the no-arbitrage condition to hold.

(c) (*1 point*) Prove that $W_2(t)$ is a Brownian motion under the probability measure $\mathbb{P}_2$.

3. Continued

(d) (*2 points*) Show that when $|\rho| < 1$, there is exactly one probability measure equivalent to $\mathbb{P}_2$ for portfolio 2 such that:

$$e^{-rt} S_1(t) \text{ and } e^{-rt} S_2(t)$$

are martingales.

For Portfolio 2, assume that $\rho = 1$ and that no arbitrage exists.

(e) (*3 points*) Construct a self-financing trading strategy with non-zero weights using the two assets in Portfolio 2 and the risk-free savings account such that the resulting portfolio has zero volatility.

4. (*8 points*) The price process of a stock follows the geometric Brownian motion given by

$$dS_t = \mu S_t dt + \sigma S_t dW_t.$$

Let r be the continuously compounded risk-free interest rate. An option on this stock with maturity T has the following payoff:

$$V_T = \begin{cases} S_T, & \text{if } S_T \geq K \\ 0, & \text{if } S_T < K \end{cases}$$

(a) (*0.5 points*) Sketch the payoff diagram of this option at maturity.

Denote the payoff of a European call option on this same stock by:

$$C_T = \begin{cases} S_T - K, & \text{if } S_T \geq K \\ 0, & \text{if } S_T < K \end{cases}$$

(b) (*0.5 points*) Show that V_T can be restated as $C_T + K\,I\left(S_T \geq K\right)$ where $I(A)$ denotes the indicator function for an event A.

(c) (*2.5 points*) Derive a formula for the value of this option at time 0. $\left(\text{Hint: } \Pr\left(S_T \geq K\right) = N\left(d_2\right)\right)$

(d) (*0.5 points*) Show that

$$\frac{\partial d_1}{\partial S} = \frac{1}{S\sigma\sqrt{T}}$$

(e) (*3 points*) Show that the Gamma for this option can be expressed as

$$\Gamma = \frac{N'\left(d_1\right)}{S\sigma\sqrt{T}}\left(1 - \frac{d_1}{\sigma\sqrt{T}}\right)$$

using the result in part (d) where $N'(\bullet)$ is the density of a standard normal random variable.

4. **Continued**

Your colleague made the following observation: "*Because the payoff for this option is always larger than the payoff of a call option with the same exercise price, this requires less frequent rebalancing in order to maintain a Delta-neutral position*."

(f) (*1 point*) Critique your colleague's statement.

5. (*8 points*) ABC, a bank with a big capital base, has written a 5-year European call option on a dividend bearing market index. The option has a strike of 1,900.

Current market conditions are as follows:

- Index value at time 0, $S(0) = 1{,}400$
- Risk-free rate $r = 4\%$ per annum, compounded continuously
- Dividend yield q on the index = 2% per annum, compounded continuously
- There is a liquid market in one-year futures on the index.

The bank wishes to delta hedge its position exploring the following volatilities assumptions:

Assumption 1: Volatilities estimated from actual historical data

Assumption 2: Use implied fixed volatilities $\left(\sigma_{fixed}\right)$ of 20% for all strike prices

Assumption 3: Use the following adjusted implied volatilities formula:

$$\sigma_{adj}(S) = \sigma_{fixed} - \frac{(S - 1{,}400)}{10{,}000}$$

(a) (*1 point*) Describe the disadvantages if ABC uses Assumption 1 to price the option.

Assume the multiplier for index options is 10 times the multiplier for index futures.

(b) (*1 point*) Calculate the number of futures needed at $t = 0$ to hedge the position using Assumption 2, stating whether a long or short position should be created.

You are given the following equation:

$$S e^{-q\tau} N'\left(d_1\right) - K e^{-r\tau} N'\left(d_2\right) = 0$$

$$\text{where } N'(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}.$$

Let the superscript "Assumption *n*" indicate a calculation under Assumption *n* ($n = 2$ or $n = 3$).

5. Continued

(c) (*2 points*) Prove using the above equation that the Vega of a European call option under Assumption 2 is

$$\frac{\partial C^{\text{Assumption 2}}}{\partial \sigma} = S\,e^{-q\tau}\,N'\left(d_1\right)\sqrt{\tau}$$

where $\tau =$ time to maturity.

(d) (*1.5 points*) Show that the Delta of the call option on the index under Assumption 3 is equal to

$$\frac{\partial C^{\text{Assumption 3}}}{\partial S} = \frac{\partial C^{\text{Assumption 2}}}{\partial S} - \frac{1}{10{,}000}\frac{\partial C^{\text{Assumption 2}}}{\partial \sigma}.$$

(e) (*1.5 points*) Calculate the number of futures needed to hedge the delta at $t = 0$ under Assumption 3.

(f) (*1 point*) Explain the rationale and the impact of using Assumption 3.

6. (*9 points*) Your manager is considering using the following model to price an interest rate derivative:

$$dX_t = \mu dt + \alpha X_t dW_t,$$

where X_t is the interest rate with initial value $X_0 = x_0 > 0$, W_t is a Brownian motion under a probability measure $\mathbb{Q}$, and both μ and α are constant.

(a) (*1.5 points*) Describe briefly the advantages and disadvantages of using the above model.

Denote by $F_t = e^{-\alpha W_t + \frac{1}{2}\alpha^2 t}$ and $G_t = F_t^{-1} = e^{\alpha W_t - \frac{1}{2}\alpha^2 t}$.

(b) (*3 points*)

(i) Derive the SDE for G_t and show that G_t is a martingale;

(ii) Prove that $d(X_t F_t) = \mu F_t dt$.

(c) (*1 point*) Solve for X_t in terms of X_0, μ, α, and W_S for all $s \le t$.

(d) (*1 point*) Derive the expected value of X_t.

(e) (*2.5 points*) Prove that $\mathbb{E}\left[X_t^2\right] = X_0^2 e^{\alpha^2 t} + 2\mu X_0 \left(\frac{e^{\alpha^2 t} - 1}{\alpha^2}\right) + \mu^2 \mathbb{E}\left[F_t^{-2}\left(\int_0^t F_s ds\right)^2\right]$.

7. (*5 points*) You are given the following:

$B(t,T)$: Price at time t of a zero-coupon bond with $1 principal maturing at time T.

$F(t,T)$: Instantaneous forward rate at time t with maturity at time T.

$r(t)$: Instantaneous risk-free interest rate at time t.

W_t: Wiener process under risk-neutral measure.

The respective risk-neutral stochastic processes governing $B(t,T)$ and $F(t,T)$ are:

$$dB(t,T) = r(t)B(t,T)dt + \alpha(T-t)^{\beta} B(t,T)dW_t,$$

where α and β are constants, and $\beta > 0.5$;

$$dF(t,T) = m\big(t,T,B(t,T)\big)dt + v\big(t,T,B(t,T)\big)dW_t.$$

(a) (*1.5 points*) Derive formulae for $m\big(t,T,B(t,T)\big)$ and $v\big(t,T,B(t,T)\big)$ in terms of $\alpha, \beta, t,$ and T.

(b) (*2.5 points*) Derive the risk-neutral expected value and variance of $r(t)$, predicted at time 0 for $t > 0$, in terms of $\alpha, \beta, t,$ and $F(0,\ t)$.

(c) (*1 point*) Describe drawbacks of this model.

8. (*6 points*) You are currently at time *t* and given the following interest rate scenarios in the economy. The given spot rates are one-period rates and LIBOR rates are multiple-period rates. Assume that no arbitrage is allowed and state prices are unique.

At $t+1$

State	State Price	Spot Rate	LIBOR
up	0.3960	2.0%	3.0%
down	0.5941	1.5%	2.0%

At $t+2$

State	State Price
(up, up)	0.2330
(down, up)	0.2536
(up, down)	0.2621
(down, down)	To be calculated

(a) (*1.5 points*)

(i) Calculate the current spot rate *r* known at the present time *t*.

(ii) Calculate the risk neutral probabilities for the economy at time $t+2$.

(iii) Calculate the state price of (down, down) for the economy at time $t+2$.

(b) (*0.5 points*) Calculate the arbitrage-free price of a 2-year zero-coupon bond issued currently at time *t*.

(c) (*1.5 points*) Show that for a bond with the maturity longer than 2 years, the price after normalization by the 2-year bond is a martingale under the forward measure.

(d) (*2.5 points*) Calculate the arbitrage-free forward LIBOR rate using:

(i) The forward measure;

(ii) The risk-neutral measure.

9. (*6 points*) Your client is setting up a college tuition endowment fund for underprivileged youth. The initial investment is $5,000,000 and future contributions are uncertain. The goal of the fund is to cover college tuition for 10 students per year. The current college tuition is about $30,000 per year.

He asks for your investment advice. You follow the portfolio management process as proposed in Maggin, et al., "Managing Investment Portfolio."

(a) (*1 point*) Outline the steps that you would follow.

(b) (*1.5 points*) Assess the merits of the following for your client:

(i) Fixed income

(ii) Equity investment

(c) (*1 point*) Identify additional information you need to make a recommendation of an investment strategy for your client.

Your client decides to use an equity portfolio and suggests using Major Stock Average 40 (MSA40) Index as a benchmark. All dividends received will be reinvested. There are 1000 companies available in the investment universe. Details of MSA40 are listed below:

- Oldest and most widely followed index
- Widely quoted in news media
- Composed of 40 major companies determined by a committee
- Price Return Computational Method
- Price-weighted weighting method

(d) (*1.5 points*) Recommend whether or not to use MSA40 index as a benchmark.

You consider four choices for style orientation of your portfolio.

- Size
- Value/Growth
- Momentum
- Liquidity

(e) (*1 point*) Recommend which of these four styles should be selected and justify your answer.

****END OF EXAMINATION****
Morning Session

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