INSTRUCTIONS TO CANDIDATES

General Instructions

1. This examination has a total of 100 points. It consists of a morning session (worth 60 points) and an afternoon session (worth 40 points).

   a) The morning session consists of 9 questions numbered 1 through 9.

   b) The afternoon session consists of 8 questions numbered 10 through 17.

   The points for each question are indicated at the beginning of the question.

2. Failure to stop writing after time is called will result in the disqualification of your answers or further disciplinary action.

3. While every attempt is made to avoid defective questions, sometimes they do occur. If you believe a question is defective, the supervisor or proctor cannot give you any guidance beyond the instructions on the exam booklet.

Written-Answer Instructions

1. Write your candidate number at the top of each sheet. Your name must not appear.

2. Write on only one side of a sheet. Start each question on a fresh sheet. On each sheet, write the number of the question that you are answering. Do not answer more than one question on a single sheet.

3. The answer should be confined to the question as set.

4. When you are asked to calculate, show all your work including any applicable formulas. When you are asked to recommend, provide proper justification supporting your recommendation.

5. When you finish, insert all your written-answer sheets into the Essay Answer Envelope. Be sure to hand in all your answer sheets because they cannot be accepted later. Seal the envelope and write your candidate number in the space provided on the outside of the envelope. Check the appropriate box to indicate morning or afternoon session for Exam QFICORE.

6. Be sure your written-answer envelope is signed because if it is not, your examination will not be graded.

Tournez le cahier d'examen pour la version française.

1. (*5 points*) Assume that the value of an investment at time *n* is given by:

$$S_n = s + \sum_{k=1}^{n} \Delta_k$$

where $S_0 = s$ is the value at time 0 and is constant. Each Δ_k is independent and identically distributed, and satisfies the following:

$$\Delta_k = \begin{cases} +0.1, & \text{with probability } 0.75 \\ -0.3, & \text{with probability } 0.25 \end{cases}$$

Denote by $\{I_k\}$, the filtration generated by $\{\Delta_k\}$ where $I_1 \subseteq I_2 \subseteq I_3...$

(a) (*1.5 points*) Calculate:

   (i) $E[S_5 | I_1]$

   (ii) $E[S_5 | I_5]$

(b) (*2 points*) Show that the process $M_t = E[S_{t+j} | I_t]$ is a martingale for a fixed and known $j > 0$.

(c) (*1.5 points*) Explain why financial assets need not follow a martingale and how to convert them to be a martingale.

2. (*8 points*) Let W_t be a Wiener process with respect to information sets I_t and the probability measure $\mathbb{P}$. Define the process $\xi_t = \exp\left(-\alpha W_t - \frac{1}{2}\alpha^2 t\right)$ for $\alpha > 0$ where α is a nonzero constant.

(a) (*1 point*) Using the Ito's Lemma, show that $d\xi_t = -\alpha\xi_t dW_t$ under the measure $\mathbb{P}$.

(b) (*1.5 points*) Show that ξ_t is a martingale process under the measure $\mathbb{P}$ based on the definition of a martingale.

Let $\mathbb{Q}$ be the probability measure defined by:

$$\mathbb{Q}(A) = \int_A \xi_T d\mathbb{P}$$

Let $B_t = W_t + \alpha t$.

(c) (*2.5 points*) Calculate the expectation $E^{\mathbb{Q}}\left[e^{uB_t}\right]$ for any constant u.

(d) (*1 point*) Show that for all s, $t \geq 0$ the increments $B_{t+s} - B_s$ have a normal distribution $N(0,t)$ with mean zero and variance t under the measure $\mathbb{Q}$.

(e) (*2 points*) Show that B_t is a Wiener process under the measure $\mathbb{Q}$.

3. (*8 points*) Your colleague made the following statements regarding martingales:

Statement I:

If $S(t)$ is a positive and non-constant martingale under filtration H_t with $E\left[S(t)^n\right] < \infty$ for any $n > 0$, then $S(t)^n$ is not a sub-martingale for any $n > 0$.

(a) (*1.5 points*) Assess whether Statement I is correct.

Statement II:

If W_t is a Brownian motion for $t \geq 0$, then $E\left[W_{t+1}^2 \mid I_t\right] = W_t^2 + 1$ where I_t is the filtration generated by W_t.

(b) (*2 points*) Assess whether Statement II is correct.

(c) (*1 point*) Derive an expression for $E\left[E\left[W_3^2 \mid I_2\right] \mid I_1\right]$ in terms of W_1.

(d) (*2 points*) Construct a non-constant quadratic function of W_t and t that is a martingale.

Statement III:

Investment managers A and B each start out with $100 under management. Customers come along in a sequence, each with $100. The first customer gives $100 to A or B at random. Each of the subsequent customers gives A or B another $100, but the customer chooses the investment manager with probabilities that are proportional to the amount of money that each of the managers already has collected. Let A_n and B_n denote the total amounts that A and B have collected respectively including the initial seed money after the n-th customer. Let M_n denote the fraction of money that is invested with A.

$$M_n = \frac{A_n}{A_n + B_n}$$

Your colleague claims M_n is a martingale.

(e) (*1.5 points*) Assess whether Statement III is correct.

4. (*6 points*) You are implementing a Delta-neutral dynamic hedging for a short position of a call option on a single stock A. You assume that Stock A follows a geometric Brownian motion with constant volatility, and pays no dividend. The following table presents data related to the stock and the option based on the current market conditions:

Stock A Price	1000
Option Value	100
Risk-Free Rate	2% (per annum, continuous compounding)
Implied Volatility (σ)	0.20 (per annum)
Delta	0.6
Gamma	0.002

(a) (*0.5 points*) Calculate the amount to be borrowed or lent for the Delta-neutral strategy at its inception.

(b) (*2 points*) Derive the following Black-Scholes PDE using the Delta-neutral riskless hedge:

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + rS\frac{\partial V}{\partial S} - rV = 0$$

(c) (*1.5 points*) Calculate the Theta of the option from the PDE using the option value, Delta, and Gamma.

You want to add an option position to make the hedge portfolio Gamma-neutral.

(d) (*1 point*) Explain the benefits of making the dynamic hedge Gamma-neutral.

You find the actual realized volatility is 0.22, different from the implied volatility.

(e) (*1 point*) Calculate the P&L of the Delta-neutral hedging for the next time period assuming a finite time step of $dt = 0.01$.

5. (*7 points*) Consider a Heath-Jarrow-Morton (HJM) forward interest rate model given by:

$$dF(t,T) = .0001\left(e^{-(T-t)} - e^{-2(T-t)}\right)dt + .01e^{-(T-t)}dW_t$$

where W_t is a standard Brownian motion under the risk-neutral measure.

(a) (*1 point*) Show that the absence of the arbitrage condition of the HJM model is satisfied in this equation.

(b) (*1.5 points*) Derive $F(t,T)$ by solving the above equation.

(c) (*1 point*) Derive an expression for the short rate r_t in terms of $F(0,t)$, t, and $\int_0^t e^s dW_s$.

(d) (*2 points*) Derive an expression for dr_t in terms of $F(0,t)$, $F_t(0,t)$, t, dt, and dW_t.

(e) (*1.5 points*) List the challenges of implementing the HJM model.

6. (*4 points*) Bank ABC is pricing an option on DEF Insurance Co. using volatility estimated from historical data. The following information based on stock prices during 31 consecutive trading days is provided:

- Number of business days in a year = 255

- $u_i = \ln\left(\dfrac{S_i}{S_{i-1}}\right)$

- $\sum u_i = -0.014251$

- $\sum u_i^2 = 0.005235$

(a) (*1.5 points*) Calculate an estimate of the annualized volatility (σ) and its standard error.

DEF Insurance Co. will make important announcements over the next few months. Bank ABC believes that the announcements will lead to sudden changes in the stock price that are larger than implied by option prices.

(b) (*1 point*) Recommend two trading strategies for Bank ABC based on its belief.

The implied volatilities calculated using the Black-Scholes formula for far-out-of-the-money or far-in-the-money options are much higher than those for at-the-money options.

(c) (*1.5 points*) Describe reasons for this relationship of implied volatilities.

7. (*10 points*) Consider the $n \times (N-n)$ interest rate payer swap, per which at times, $n+1, \ldots, N$, the holder will receive the LIBOR rates $L_n, \ldots, L_{N-1}$ (fixed at times $n, \ldots, N-1$) and will pay a rate K fixed at inception. For simplicity, assume that the payments are annual and no day-count adjustment is necessary.

Let $p(t,i)$ be the price at time t of the zero-coupon bond maturing at time i. Assume that a liquid market for Forward Rate Agreements (FRAs) and Zero-Coupon Bonds (ZCBs) exists and that rates are stochastic unless told otherwise.

(a) (*1.5 points*) Derive an expression for the fair strike K^* of the 0×5 swap at time 0 in terms of the expected value of the LIBOR rates using the standard risk-neutral measure.

(b) (*2.5 points*) Derive an expression for the fair strike K^* of the 0×5 swap at time 0 in terms of bond prices and FRAs using the n-period forward measures ($n = 1, 2, 3, 4, 5$).

(c) (*1.5 points*)

    (i) Describe the differences between the two approaches in (a) and (b).

    (ii) Specify, for each approach, the asset used for normalization for asset pricing purposes. Assume that interest rates are deterministic.

(d) (*1.5 points*) Show how the 0×5 interest rate swap struck at K can be valued using the prices of a default-free fixed-coupon bond and a floating-coupon bond (coupons paid annually).

Consider the payer swaption, which at time 2 pays the greater of 0 and the value at that time of the 2×3 payer swap and assume a liquid swap market exists. Let $A(p,m)$ denote the annuity, paying \$1 at times $p+1, \ldots, p+m$ (p, m are in years) and let $A_t(p,m)$ denote its price at time t.

(e) (*3 points*) Show that the Black-76 method can be used to value the swaption with strike K if one were to assume that the fair strike of the $t \times 3$ constant-maturity swap follows a log-normal process.

    Hint: Normalize asset prices by $A_t(2,3)$.

8. (*7 points*) You are modelling the monthly log returns for XYZ stock using the following ARCH(2) model:

$$r_t = \mu + a_t, \quad \sigma_t^2 = \alpha_0 + \alpha_1 a_{t-1}^2 + \alpha_2 a_{t-2}^2$$

where r_t is the log return of XYZ's stock, and $a_t = \sigma_t \epsilon_t$ where ϵ_t is a sequence of independent and identically distributed (i.i.d.) random variables with mean 0 and variance 1.

You obtained the following data for your fitted model:

Table 1

Parameters	Estimates	Standard Errors	p-value
μ	0.0234	0.0042	0.00526531
α_0	0.0038	0.0001	1.8186E-05
α_1	0.1728	0.0338	0.00670017
α_2	0.0502	0.1194	0.31137402

Over the 15 years of XYZ stock's history, it experienced four major price change phases:

- First 8 years: Price increased gradually overtime
- Next 0.5 years: Price dropped very significantly
- Next 6 years: Price stayed at the low level with minimal fluctuation
- Most recent 0.5 years: Price rose back to the pre-declining level

(a) (*2 points*) Critique the use of an ARCH(2) model to predict future XYZ stock returns, based on the above information.

You are given the following statistics for the Ljung-Box test where $\tilde{a}_t = a_t/\sigma_t$:

Table 2

Residual Series	lag	Ljung-Box Test Statistics Q(lag) Value	p-value
$\tilde{a}_t$	12	13.15	0.3548
$\tilde{a}_t^2$	12	21.07	0.0258

8. **Continued**

(b)   (*1.5 points*) Define Null Hypotheses for the Ljung-Box test and conclude whether ARCH(2) is appropriate at the 5% significance level using the test results of Table 1 and Table 2.

Your colleague suggests using an EGARCH model.

(c)   (*1 point*) Critique your colleague's suggestion.

Another colleague used an ARCH(1) model and based on his analysis, he determined the following parameters fit the XYZ stock performance well:

Table 3

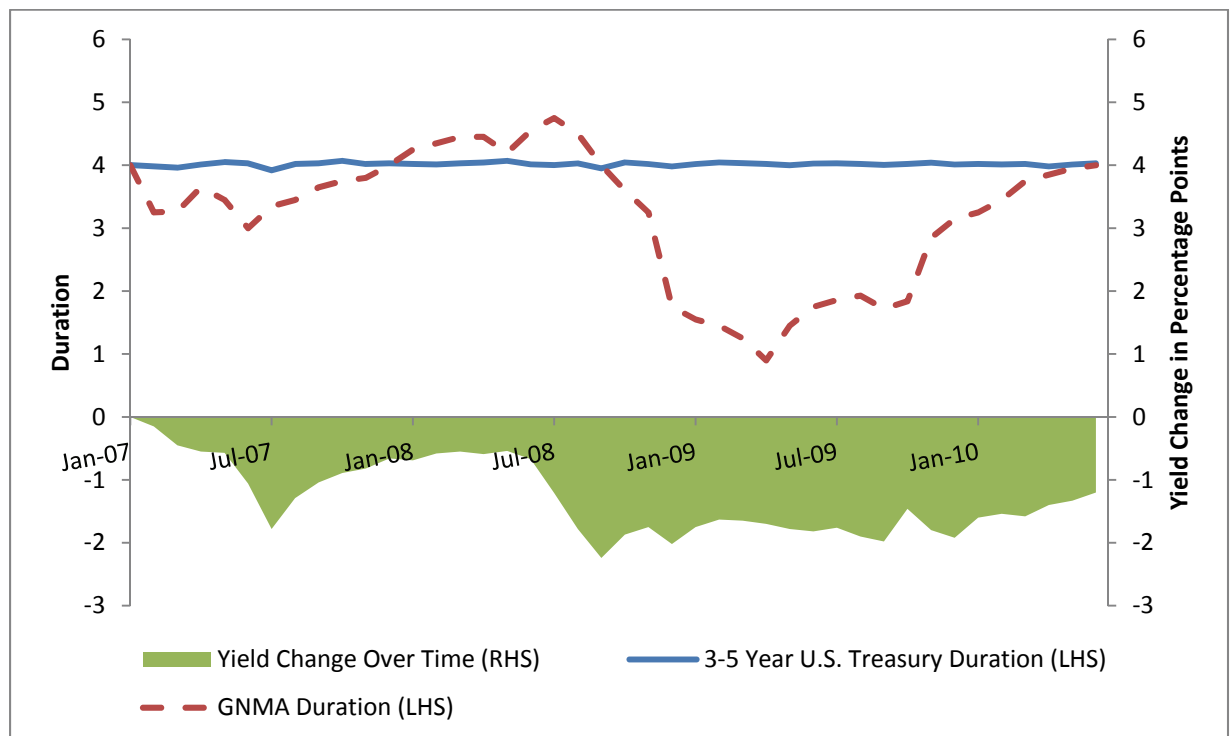| Parameters	Estimates	Standard Errors	p-value
μ	0.0157	0.0031	0.00687818
α_0	0.0148	0.0012	5.23E-04
α_1	0.6800	0.0958	0.00263761

He then made the following three statements:

- Using ARCH(1), r_t is a stationary process and we can define the volatility with just two parameters.
- ARCH(1) is less likely to produce outliers in forecasting future volatilities than an i.i.d. normal random variable – which improves the stability of the forecast.
- For ARCH(1), all moments are finite.

(d)   (*2.5 points*) Assess the correctness of the above three statements based on Table 3 as applicable.

9. (*5 points*)

    (a)   (*0.5 points*) Explain how convexity impacts the difference of yields between agency MBS and U.S. Treasuries with the same duration.

You are tracking the U.S. GNMA Bond Index and the U.S. 3-5 Year Treasury Bond Index. Changes in yields, shown as shaded areas on the chart below and represented on the right vertical axis, are based on monthly data for the U.S. 3-5 Year Treasury Bond Index. Represented on the left vertical axis are the durations for the U.S. GNMA Bond Index (dotted line) and the U.S. 3-5 Year Treasury Bond Index (solid line).

    (b)   (*1 point*) Explain why GNMA Bond's duration (dotted line) has been more volatile than U.S. Treasury's (solid line).

    (c)   (*1 point*) Draw a chart showing the shape of the price-yield relationship curve for each of the indices. State clearly titles of both axes of the chart.

    (d)   (*1 point*) Explain the shape of the price-yield relationship curve in part (c).

    (e)   (*1.5 points*) Design a hedge strategy to maintain a stable duration of a bond portfolio which is invested in GNMA bonds.

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END OF EXAMINATION
Morning Session

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