(1) Consider a European call option and a European put option on a nondividend-paying stock. You are given:

(i) The current price of the stock is \$60.
(ii) The call option currently sells for \$0.15 more than the put option.
(iii) Both the call option and put option will expire in 4 years.
(iv) Both the call option and put option have a strike price of \$70.

Calculate the continuously compounded risk-free interest rate.

(A) 0.039
(B) 0.049
(C) 0.059
(D) 0.069
(E) 0.079

Solution to (1) Answer: (A)

The put-call parity formula for a European call and a European put on a nondividend-paying stock with the same strike price and maturity date is

$$C - P = S_0 - Ke^{-rT}.$$

We are given that $C - P = 0.15$, $S_0 = 60$, $K = 70$ and $T = 4$. Then, $r = 0.039$.

Remark 1: If the stock pays n dividends of fixed amounts $D_1, D_2, \ldots, D_n$ at fixed times $t_1, t_2, \ldots, t_n$ prior to the option maturity date, T, then the put-call parity formula for European put and call options is

$$C - P = S_0 - \mathrm{PV}_{0,T}(\mathrm{Div}) - Ke^{-rT},$$

where $\mathrm{PV}_{0,T}(\mathrm{Div}) = \sum_{i=1}^{n} D_i e^{-rt_i}$ is the present value of all dividends up to time T. The difference, $S_0 - \mathrm{PV}_{0,T}(\mathrm{Div})$, is the *prepaid forward price* $F_{0,T}^{P}(S)$.

Remark 2: The put-call parity formula above does not hold for American put and call options. For the American case, the parity relationship becomes

$$S_0 - \mathrm{PV}_{0,T}(\mathrm{Div}) - K \le C - P \le S_0 - Ke^{-rT}.$$

This result is given in Appendix 9A of McDonald (2006) but is not required for Exam MFE/3F. Nevertheless, you may want to try proving the inequalities as follows:
For the first inequality, consider a portfolio consisting of a European call plus an amount of cash equal to $\mathrm{PV}_{0,T}(\mathrm{Div}) + K$.
For the second inequality, consider a portfolio of an American put option plus one share of the stock.

(2) Near market closing time on a given day, you lose access to stock prices, but some European call and put prices for a stock are available as follows:

Strike Price	Call Price	Put Price
$40	$11	$3
$50	$6	$8
$55	$3	$11

All six options have the same expiration date.

After reviewing the information above, John tells Mary and Peter that no arbitrage opportunities can arise from these prices.

Mary disagrees with John. She argues that one could use the following portfolio to obtain arbitrage profit: Long one call option with strike price 40; short three call options with strike price 50; lend $1; and long some calls with strike price 55.

Peter also disagrees with John. He claims that the following portfolio, which is different from Mary's, can produce arbitrage profit: Long 2 calls and short 2 puts with strike price 55; long 1 call and short 1 put with strike price 40; lend $2; and short some calls and long the same number of puts with strike price 50.

Which of the following statements is true?

(A) Only John is correct.

(B) Only Mary is correct.

(C) Only Peter is correct.

(D) Both Mary and Peter are correct.

(E) None of them is correct.

Solution to (2) Answer: (D)

The prices are not arbitrage-free. To show that Mary's portfolio yields arbitrage profit, we follow the analysis in Table 9.7 on page 302 of McDonald (2006).

	Time 0	Time T	Time T	Time T	Time T
		$S_T < 40$	$40 \leq S_T < 50$	$50 \leq S_T < 55$	$S_T \geq 55$
Buy 1 call Strike 40	– 11	0	$S_T - 40$	$S_T - 40$	$S_T - 40$
Sell 3 calls Strike 50	+ 18	0	0	$-3(S_T - 50)$	$-3(S_T - 50)$
Lend \$1	– 1	e^{rT}	e^{rT}	e^{rT}	e^{rT}
Buy 2 calls strike 55	– 6	0	0	0	$2(S_T - 55)$
Total	0	$e^{rT} > 0$	$e^{rT} + S_T - 40 > 0$	$e^{rT} + 2(55 - S_T) > 0$	$e^{rT} > 0$

Peter's portfolio makes arbitrage profit, because:

	Time-0 cash flow	Time-T cash flow
Buy 2 calls & sells 2 puts Strike 55	2(–3 + 11) = 16	$2(S_T - 55)$
Buy 1 call & sell 1 put Strike 40	–11 + 3 = –8	$S_T - 40$
Lend \$2	–2	$2e^{rT}$
Sell 3 calls & buy 3 puts Strike 50	3(6 – 8) = –6	$3(50 - S_T)$
Total	0	$2e^{rT}$

Remarks: Note that Mary's portfolio has no put options. The call option prices are not arbitrage-free; they do not satisfy the convexity condition (9.17) on page 300 of McDonald (2006). The time-T cash flow column in Peter's portfolio is due to the identity

$$\max[0, S - K] - \max[0, K - S] = S - K$$

(see also page 44).

In *Loss Models*, the textbook for Exam C/4, max[0, α] is denoted as α_+. It appears in the context of stop-loss insurance, $(S - d)_+$, with S being the claim random variable and d the deductible. The identity above is a particular case of

$$x = x_+ - (-x)_+,$$

which says that every number is the difference between its positive part and negative part.

(3) An insurance company sells single premium deferred annuity contracts with return linked to a stock index, the time-t value of one unit of which is denoted by $S(t)$. The contracts offer a minimum guarantee return rate of $g\%$. At time 0, a single premium of amount π is paid by the policyholder, and $\pi \times y\%$ is deducted by the insurance company. Thus, at the contract maturity date, T, the insurance company will pay the policyholder

$$\pi \times (1 - y\%) \times \text{Max}[S(T)/S(0), (1 + g\%)^T].$$

You are given the following information:

(i) The contract will mature in one year.
(ii) The minimum guarantee rate of return, $g\%$, is 3%.
(iii) Dividends are incorporated in the stock index. That is, the stock index is constructed with all stock dividends reinvested.
(iv) $S(0) = 100$.
(v) The price of a one-year European put option, with strike price of \$103, on the stock index is \$15.21.

Determine $y\%$, so that the insurance company does not make or lose money on this contract.

Solution to (3)

The payoff at the contract maturity date is

$$\pi\times(1 - y\%)\times\text{Max}[S(T)/S(0), (1 + g\%)^T]$$
$$= \pi\times(1 - y\%)\times\text{Max}[S(1)/S(0), (1 + g\%)^1] \qquad \text{because } T = 1$$
$$= [\pi/S(0)](1 - y\%)\text{Max}[S(1), S(0)(1 + g\%)]$$
$$= (\pi/100)(1 - y\%)\text{Max}[S(1), 103] \qquad \text{because } g=3 \ \&\ S(0)=100$$
$$= (\pi/100)(1 - y\%)\{S(1) + \text{Max}[0, 103 - S(1)]\}.$$

Now, Max[0, 103 – S(1)] is the payoff of a one-year European put option, with strike price \$103, on the stock index; the time-0 price of this option is given to be is \$15.21. Dividends are incorporated in the stock index (i.e., $\delta = 0$); therefore, S(0) is the time-0 price for a time-1 payoff of amount S(1). Because of the no-arbitrage principle, the time-0 price of the contract must be

$$(\pi/100)(1 - y\%)\{S(0) + 15.21\}$$
$$= (\pi/100)(1 - y\%)\times 115.21.$$

Therefore, the "break-even" equation is

$$\pi = (\pi/100)(1 - y\%)\times 115.21,$$

or

$$y\% = 100\times(1 - 1/1.1521)\% = 13.202\%$$

Remark 1 Many stock indexes, such as S&P500, do not incorporate dividend reinvestments. In such cases, the time-0 cost for receiving S(1) at time 1 is the prepaid forward price $F_{0,1}^P(S)$, which is less than S(0).

Remark 2 The identities

$$\text{Max}[S(T), K] = K + \text{Max}[S(T) - K, 0] = K + (S(T) - K)_+$$

and

$$\text{Max}[S(T), K] = S(T) + \text{Max}[0, K - S(T)] = S(T) + (K - S(T))_+$$

can lead to a derivation of the put-call parity formula. Such identities are useful for understanding Section 14.6 *Exchange Options* in McDonald (2006).

(4) For a two-period binomial model, you are given:

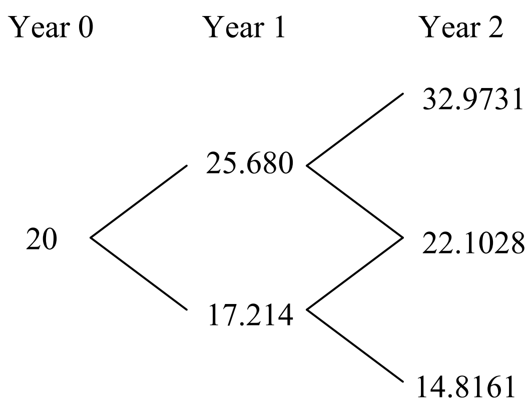
(i) Each period is one year.
(ii) The current price for a non-dividend paying stock is \$20.
(iii) $u = 1.2840$, where u is one plus the rate of capital gain on the stock per period if the stock price goes up.
(iv) $d = 0.8607$, where d is one plus the rate of capital loss on the stock per period if the stock price goes down.
(v) The continuously compounded risk-free interest rate is 5%.

Calculate the price of an American call option on the stock with a strike price of \$22.

(A) \$0
(B) \$1
(C) \$2
(D) \$3
(E) \$4

Solution to (4) Answer: (C)

First, we construct the two-period binomial tree for the stock price.



The calculations for the stock prices at various nodes are as follows:

$S_u = 20 \times 1.2840 = 25.680$
$S_d = 20 \times 0.8607 = 17.214$
$S_{uu} = 25.68 \times 1.2840 = 32.9731$
$S_{ud} = S_{du} = 17.214 \times 1.2840 = 22.1028$
$S_{dd} = 17.214 \times 0.8607 = 14.8161$

The risk-neutral probability for the stock price to go up is

$$p^* = \frac{e^{rh} - d}{u - d} = \frac{e^{0.05} - 0.8607}{1.2840 - 0.8607} = 0.4502.$$

Thus, the risk-neutral probability for the stock price to go down is 0.5498.

If the option is exercised at time 2, the value of the call would be
$C_{uu} = (32.9731 - 22)_+ = 10.9731$
$C_{ud} = (22.1028 - 22)_+ = 0.1028$
$C_{dd} = (14.8161 - 22)_+ = 0$

If the option is European, then $C_u = e^{-0.05}[0.4502C_{uu} + 0.5498C_{ud}] = 4.7530$ and $C_d = e^{-0.05}[0.4502C_{ud} + 0.5498C_{dd}] = 0.0440$.
But since the option is American, we should compare C_u and C_d with the value of the option if it is exercised at time 1, which is 3.68 and 0, respectively. Since $3.68 < 4.7530$ and $0 < 0.0440$, it is not optimal to exercise the option at time 1 whether the stock is in the up or down state. Thus the value of the option at time 1 is either 4.7530 or 0.0440.

Finally, the value of the call is
$C = e^{-0.05}[0.4502(4.7530) + 0.5498(0.0440)] = 2.0585.$

Remark: Since the stock pays no dividends, the price of an American call is the same as that of a European call. See pages 294-295 of McDonald (2006). The European option price can be calculated using the binomial probability formula. See formula (11.17) on page 358 and formula (19.1) on page 618 of McDonald (2006). The option price is

$$e^{-r(2h)}\left[\binom{2}{2}p^{*2}C_{uu} + \binom{2}{1}p^*(1-p^*)C_{ud} + \binom{2}{0}(1-p^*)^2C_{dd}\right]$$

$= e^{-0.1}\,[(0.4502)^2\times 10.9731 \;+\; 2\times 0.4502\times 0.5498\times 0.1028 \;+\; 0]$
$= 2.0507$

Formula (19.1) is in the syllabus of Exam C/4.

(5) Consider a 9-month dollar-denominated American put option on British pounds. You are given that:

(i) The current exchange rate is 1.43 US dollars per pound.
(ii) The strike price of the put is 1.56 US dollars per pound.
(iii) The volatility of the exchange rate is $\sigma = 0.3$.
(iv) The US dollar continuously compounded risk-free interest rate is 8%.
(v) The British pound continuously compounded risk-free interest rate is 9%.

Using a three-period binomial model, calculate the price of the put.

Solution to (5)

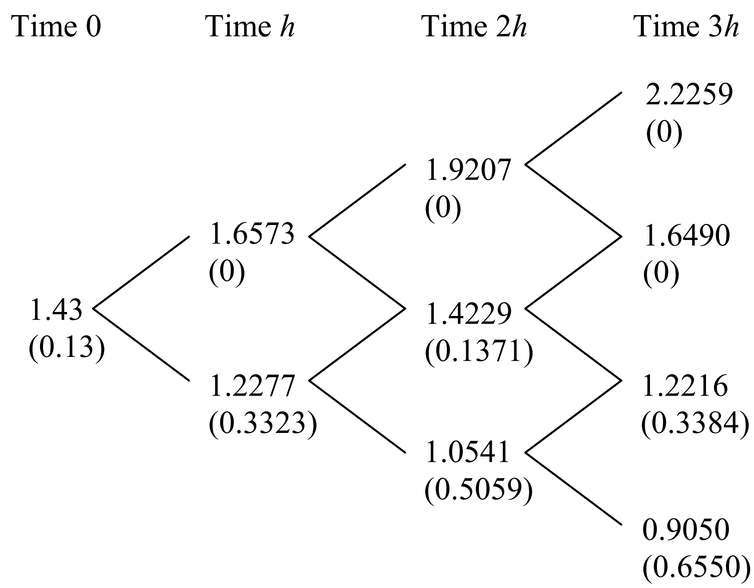
Each period is of length $h = 0.25$. Using the first two formulas on page 332 of McDonald (2006):

$$u = \exp[-0.01\times0.25 + 0.3\times\sqrt{0.25}\,] = \exp(0.1475) = 1.158933,$$
$$d = \exp[-0.01\times0.25 - 0.3\times\sqrt{0.25}\,] = \exp(-0.1525) = 0.858559.$$

Using formula (10.13), the risk-neutral probability of an up move is

$$p^* = \frac{e^{-0.01\times0.25} - 0.858559}{1.158933 - 0.858559} = 0.4626.$$

The risk-neutral probability of a down move is thus 0.5374. The 3-period binomial tree for the exchange rate is shown below. The numbers within parentheses are the payoffs of the put option if exercised.



The payoffs of the put at maturity (at time $3h$) are
$P_{uuu} = 0$, $P_{uud} = 0$, $P_{udd} = 0.3384$ and $P_{ddd} = 0.6550$.

Now we calculate values of the put at time $2h$ for various states of the exchange rate.

If the put is European, then
$P_{uu} = 0$,
$P_{ud} = e^{-0.02}[0.4626P_{uud} + 0.5374P_{udd}] = 0.1783$,
$P_{dd} = e^{-0.02}[0.\ 4626P_{udd} + 0.5374P_{ddd}] = 0.4985$.
But since the option is American, we should compare P_{uu}, P_{ud} and P_{dd} with the values of the option if it is exercised at time $2h$, which are 0, 0.1371 and 0.5059, respectively. Since $0.4985 < 0.5059$, it is optimal to exercise the option at time $2h$ if the exchange rate has gone down two times before. Thus the values of the option at time $2h$ are $P_{uu} = 0$, $P_{ud} = 0.1783$ and $P_{dd} = 0.5059$.

Now we calculate values of the put at time h for various states of the exchange rate.

If the put is European, then
$P_u = e^{-0.02}[0.4626P_{uu} + 0.5374P_{ud}] = 0.0939,$
$P_d = e^{-0.02}[0.4626P_{ud} + 0.5374P_{dd}] = 0.3474.$
But since the option is American, we should compare P_u and P_d with the values of the option if it is exercised at time h, which are 0 and 0.3323, respectively. Since 0.3474 > 0.3323, it is not optimal to exercise the option at time h. Thus the values of the option at time h are $P_u = 0.0939$ and $P_d = 0.3474$.

Finally, discount and average P_u and P_d to get the time-0 price,

$$P = e^{-0.02}[0.4626P_u + 0.5374P_d] = 0.2256.$$

Since it is greater than 0.13, it is not optimal to exercise the option at time 0 and hence the price of the put is 0.2256.

Remarks: (1) Because $\dfrac{e^{(r-\delta)h} - e^{(r-\delta)h-\sigma\sqrt{h}}}{e^{(r-\delta)h+\sigma\sqrt{h}} - e^{(r-\delta)h-\sigma\sqrt{h}}} = \dfrac{1-e^{-\sigma\sqrt{h}}}{e^{\sigma\sqrt{h}} - e^{-\sigma\sqrt{h}}} = \dfrac{1}{1+e^{\sigma\sqrt{h}}}$, we can also calculate the risk-neutral probability p^* as follows:

$$p^* = \frac{1}{1+e^{\sigma\sqrt{h}}} = \frac{1}{1+e^{0.3\sqrt{0.25}}} = \frac{1}{1+e^{0.15}} = 0.46257.$$

(2) $$1 - p^* = 1 - \frac{1}{1+e^{\sigma\sqrt{h}}} = \frac{e^{\sigma\sqrt{h}}}{1+e^{\sigma\sqrt{h}}} = \frac{1}{1+e^{-\sigma\sqrt{h}}}.$$

Because $\sigma > 0$, we have the inequalities

$$p^* < ½ < 1 - p^*.$$

(6) You are considering the purchase of 100 European call options on a stock, which pays dividends continuously at a rate proportional to its price. Assume that the Black-Scholes framework holds. You are given:

(i) The strike price is \$25.
(ii) The options expire in 3 months.
(iii) $\delta = 0.03$.
(iv) The stock is currently selling for \$20.
(v) $\sigma = 0.24$
(vi) The continuously compounded risk-free interest rate is 5%.

Calculate the price of the block of 100 options.

(A) \$0.04
(B) \$1.93
(C) \$3.50
(D) \$4.20
(E) \$5.09

Solution to (6) Answer: (C)

$$C(S,K,\sigma,r,T,\delta) = Se^{-\delta T}N(d_1) - Ke^{-rT}N(d_2) \qquad (12.1)$$

with

$$d_1 = \frac{\ln(S/K) + (r - \delta + \frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}} \qquad (12.2a)$$

$$d_2 = d_1 - \sigma\sqrt{T} \qquad (12.2b)$$

Because S = \$20, K = \$25, σ = 0.24, r = 0.05, T = 3/12 = 0.25, and δ = 0.03, we have

$$d_1 = \frac{\ln(20/25) + (0.05 - 0.03 + \frac{1}{2}0.24^2)0.25}{0.24\sqrt{0.25}} = -1.75786$$

and

$$d_2 = -1.75786 - 0.24\sqrt{0.25} = -1.87786$$

Because d_1 and d_2 are negative, use $N(d_1) = 1 - N(-d_1)$ and $N(d_2) = 1 - N(-d_2)$.
In Exam MFE/3F, round $-d_1$ to 1.76 before looking up the standard normal distribution table. Thus, $N(d_1)$ is $1 - 0.9608 = 0.0392$. Similarly, round $-d_2$ to 1.88, and $N(d_2)$ is thus $1 - 0.9699 = 0.0301$.

Formula (12.1) becomes

$$C = 20e^{-(0.03)(0.25)}(0.0392) - 25e^{-(0.05)(0.25)}(0.0301) = 0.0350$$

Cost of the block of 100 options = 100 × 0.0350 = \$3.50

(7) Company A is a U.S. international company, and Company B is a Japanese local company. Company A is negotiating with Company B to sell its operation in Tokyo to Company B. The deal will be settled in Japanese yen. To avoid a loss at the time when the deal is closed due to a sudden devaluation of yen relative to dollar, Company A has decided to buy at-the-money dollar-denominated yen put of the European type to hedge this risk.

You are given the following information:

(i) The deal will be closed 3 months from now.
(ii) The sale price of the Tokyo operation has been settled at 120 billion Japanese yen.
(iii) The continuously compounded risk-free interest rate in the U.S. is 3.5%.
(iv) The continuously compounded risk-free interest rate in Japan is 1.5%.
(v) The current exchange rate is 1 U.S. dollar = 120 Japanese yen.
(vi) The natural logarithm of the yen per dollar exchange rate is an arithmetic Brownian motion with daily volatility 0.261712%.
(vii) 1 year = 365 days; 3 months = ¼ year.

Calculate Company A's option cost.

Solution to (7)

Let X(t) be the exchange rate of U.S. dollar per Japanese yen at time t. That is, at time t,

¥1 = $X(t).

We are given that X(0) = 1/120.

At time ¼, Company A will receive ¥ 120 billion, which is exchanged to $[120 billion × X(¼)]. However, Company A would like to have

$ Max[1 billion, 120 billion × X(¼)],

which can be decomposed as

$120 billion × X(¼) + $ Max[1 billion – 120 billion × X(¼), 0],

or

$120 billion × {X(¼) + Max[120^{-1} – X(¼), 0]}.

Thus, Company A purchases 120 billion units of a put option whose payoff three months from now is

$ Max[120^{-1} – X(¼), 0].

The exchange rate can be viewed as the price, in US dollar, of a traded asset, which is the Japanese yen. The continuously compounded risk-free interest rate in Japan can be interpreted as δ, the dividend yield of the asset. See also page 381 of McDonald (2006) for the *Garman-Kohlhagen model*. Then, we have

$$r = 0.035,\ \delta = 0.015,\ S = X(0) = 1/120,\ K = 1/120,\ T = ¼.$$

Because the logarithm of the exchange rate of yen per dollar is an arithmetic Brownian motion, its negative, which is the logarithm of the exchange rate of dollar per yen, is also an arithmetic Brownian motion and has the SAME volatility. Therefore, {X(t)} is a geometric Brownian motion, and the put option can be priced using the Black-Scholes formula for European put options. It remains to determine the value of σ, which is given by the equation

$$\sigma\sqrt{\frac{1}{365}} = 0.261712\ \%.$$

Hence,

$$\sigma = 0.05.$$

Therefore,

$$d_1 = \frac{(r - \delta + \sigma^2/2)T}{\sigma\sqrt{T}} = \frac{(0.035 - 0.015 + 0.05^2/2)/4}{0.05\sqrt{1/4}} = 0.2125$$

and

$$d_2 = d_1 - \sigma\sqrt{T} = 0.2125 - 0.05/2 = 0.1875.$$

By (12.3) of McDonald (2006), the time-0 price of 120 billion units of the put option is

$$\$120 \text{ billion} \times [Ke^{-rT}N(-d_2) - X(0)e^{-\delta T}N(-d_1)]$$
$$= \$\ [e^{-rT}N(-d_2) - e^{-\delta T}N(-d_1)] \text{ billion} \qquad \text{because } K = X(0) = 1/120$$
$$= \$\ \{e^{-rT}[1 - N(d_2)] - e^{-\delta T}[1 - N(d_1)]\} \text{ billion}$$

In Exam MFE/3F, you will be given a standard normal distribution table. Use the value of $N(0.21)$ for $N(d_1)$, and $N(0.19)$ for $N(d_1)$.

Because $N(0.21) = 0.5832$, $N(0.19) = 0.5753$, $e^{-rT} = e^{-0.035\times0.25} = 0.9913$, and $e^{-\delta T} = e^{-0.015\times0.25} = 0.9963$, Company A's option cost is
$\$0.9913\times0.4247 - 0.9963\times0.4168 = 0.005747$ billion $\approx$ \$5.75 million.

Remarks: (1) Suppose that the problem is to be solved using options on the exchange rate of Japanese yen per US dollar, i.e., using yen-denominated options. Let

$$\$1 = ¥U(t)$$

at time t, i.e., $U(t) = 1/X(t)$.

Because Company A is worried that the dollar may increase in value with respect to the yen, it buys 1 billion units of a 3-month yen-denominated European *call* option, with exercise price ¥120. The payoff of the option at time ¼ is

$$¥\ \text{Max}[U(¼) - 120, 0].$$

To apply the Black-Scholes call option formula (12.1) to determine the time-0 price in yen, use

$$r = 0.015,\ \delta = 0.035,\ S = U(0) = 120,\ K = 120,\ T = ¼, \text{ and } \sigma = 0.05.$$

Then, divide this price by 120 to get the time-0 option price in dollars. We get the same price as above, because d_1 here is $-d_2$ of above.

The above is a special case of formula (9.7) on page 292 of McDonald (2006).

(2) There is a cheaper solution for Company A. At time 0, borrow

$$¥\ 120\times\exp(-\ ¼\ r_{¥}) \text{ billion},$$

and immediately convert this amount to US dollars. The loan is repaid with interest at time ¼ when the deal is closed.
On the other hand, with the option purchase, Company A will benefit if the yen increases in value with respect to the dollar.

(8) You are considering the purchase of an American call option on a nondividend-paying stock. Assume the Black-Scholes framework. You are given:

(i) The stock is currently selling for \$40.
(ii) The strike price of the option is \$41.5
(iii) The option expires in 3 months.
(iv) The stock's volatility is 30%.
(v) The current call option delta is 0.5.

Determine the current price of the option.

(A) $20 - 20.453\int_{-\infty}^{0.15} e^{-x^2/2}\mathrm{d}x$

(B) $20 - 16.138\int_{-\infty}^{0.15} e^{-x^2/2}\mathrm{d}x$

(C) $20 - 40.453\int_{-\infty}^{0.15} e^{-x^2/2}\mathrm{d}x$

(D) $16.138\int_{-\infty}^{0.15} e^{-x^2/2}\mathrm{d}x - 20.453$

(E) $40.453\int_{-\infty}^{0.15} e^{-x^2/2}\mathrm{d}x - 20.453$

Solution to (8) Answer: (D)

Since it is never optimal to exercise an American call option before maturity if the stock pays no dividends, we can price the call option using the European call option formula

$$C = SN(d_1) - Ke^{-rT}N(d_2),$$

where $d_1 = \dfrac{\ln(S/K) + (r + \frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}}$ and $d_2 = d_1 - \sigma\sqrt{T}$.

Because the call option delta is $N(d_1)$ and it is given to be 0.5, we have $d_1 = 0$. Hence,

$$d_2 = -0.3 \times \sqrt{0.25} = -0.15 .$$

To find the continuously compounded risk-free interest rate, use the equation

$$d_1 = \frac{\ln(40/41.5) + (r + \frac{1}{2} \times 0.3^2) \times 0.25}{0.3\sqrt{0.25}} = 0,$$

which gives $r = 0.1023$.

Thus,

$$\begin{aligned} C &= 40N(0) - 41.5e^{-0.1023 \times 0.25}N(-0.15) \\ &= 20 - 40.453[1 - N(0.15)] \\ &= 40.453N(0.15) - 20.453 \\ &= \frac{40.453}{\sqrt{2\pi}} \int_{-\infty}^{0.15} e^{-x^2/2} \mathrm{d}x - 20.453 \\ &= 16.138 \int_{-\infty}^{0.15} e^{-x^2/2} \mathrm{d}x - 20.453 \end{aligned}$$

(9) Consider the Black-Scholes framework. A market-maker, who delta-hedges, sells a three-month at-the-money European call option on a nondividend-paying stock. You are given that:

(i) The current stock price is $50.
(ii) The continuously compounded risk-free interest rate is 10%.
(iii) The call option delta is 0.6179.
(iv) There are 365 days in the year.

If, after one day, the market-maker has zero profit or loss, determine the stock price move over the day.

(A) 0.41
(B) 0.52
(C) 0.63
(D) 0.75
(E) 1.11

Solution to (9)

According to the first paragraph on page 429 of McDonald (2006), such a stock price move is given by plus or minus of

$$\sigma\, S(0)\sqrt{h}\,,$$

where h = 1/365 and $S(0)$ = 50. It remains to find σ.

Because the stock pays no dividends (i.e., $\delta = 0$), it follows from the bottom of page 383 that $\Delta = N(d_1)$. By the condition $N(d_1) = 0.6179$, we get $d_1 = 0.3$. Because $S = K$ and $\delta = 0$, formula (12.2a) is

$$d_1 \;=\; \frac{(r+\sigma^2/2)T}{\sigma\sqrt{T}}$$

or

$$\tfrac{1}{2}\sigma^2 \;-\; \frac{d_1}{\sqrt{T}}\sigma \;+\; r \;=\; 0.$$

With $d_1 = 0.3$, $r = 0.1$, and $T = 1/4$, the quadratic equation becomes

$$\tfrac{1}{2}\sigma^2 \;-\; 0.6\sigma \;+\; 0.1 \;=\; 0,$$

whose roots can be found by using the quadratic formula or by factorization,

$$\tfrac{1}{2}(\sigma \;-\; 1)(\sigma \;-\; 0.2) \;=\; 0.$$

We reject $\sigma = 1$ because such a volatility seems too large (and none of the five answers fit). Hence,

$$\sigma\, S(0)\sqrt{h} \;=\; 0.2 \times 50 \times 0.052342 \;\approx\; 0.52.$$

Remarks: The Itô Lemma in Chapter 20 of McDonald (2006) can help us understand Section 13.4. Let $C(S, t)$ be the price of the call option at time t if the stock price is S at that time. We use the following notation

$$C_S(S, t) \;=\; \frac{\partial}{\partial S}C(S,t)\,,\;\; C_{SS}(S, t) \;=\; \frac{\partial^2}{\partial S^2}C(S,t)\,,\;\; C_t(S, t) \;=\; \frac{\partial}{\partial t}C(S,t)\,,$$

$$\Delta_t \;=\; C_S(S(t), t),\;\; \Gamma_t \;=\; C_{SS}(S(t), t),\;\; \theta_t \;=\; C_t(S(t), t).$$

At time t, the so-called market-maker sells one call option, and he delta-hedges, i.e., he buys delta, Δ_t, shares of the stock. At time $t + dt$, the stock price moves to $S(t + dt)$, and option price becomes $C(S(t + dt), t + dt)$. The interest expense for his position is

$$[\Delta_t S(t) \;-\; C(S(t), t)](rdt).$$

Thus, his profit at time $t + dt$ is

$$\begin{aligned}
&\Delta_t[S(t + dt) - S(t)] \;-\; [C(S(t + dt), t + dt) \;-\; C(S(t), t)] \;-\; [\Delta_t S(t) \;-\; C(S(t), t)](rdt)\\
&= \Delta_t dS(t) \;-\; dC(S(t), t) \;-\; [\Delta_t S(t) \;-\; C(S(t), t)](rdt). \qquad (*)
\end{aligned}$$

We learn from Section 20.6 that

$$\begin{aligned}
dC(S(t), t) \;&=\; C_S(S(t), t)dS(t) \;+\; \tfrac{1}{2}C_{SS}(S(t), t)[dS(t)]^2 \;+\; C_t(S(t), t)dt \qquad (20.28)\\
&=\; \Delta_t\, dS(t) \;+\; \tfrac{1}{2}\Gamma_t\,[dS(t)]^2 \;+\; \theta_t\, dt. \qquad (**)
\end{aligned}$$

Because $dS(t) = S(t)[\alpha\, dt + \sigma\, dZ(t)]$, it follows from the multiplication rules (20.17) that

$$[dS(t)]^2 = [S(t)]^2 \sigma^2\, dt, \qquad (***)$$

which should be compared with (13.8). Substituting (***) in (**) yields

$$dC(S(t), t) = \Delta_t\, dS(t) + \tfrac{1}{2}\Gamma_t [S(t)]^2 \sigma^2\, dt + \theta_t\, dt,$$

application of which to (*) shows that the market-maker's profit at time $t + dt$ is

$$-\{\tfrac{1}{2}\Gamma_t [S(t)]^2 \sigma^2\, dt + \theta_t\, dt\} - [\Delta_t S(t) - C(S(t), t)](r dt)$$
$$= -\{\tfrac{1}{2}\Gamma_t [S(t)]^2 \sigma^2 + \theta_t + [\Delta_t S(t) - C(S(t), t)]r\}dt, \qquad (****)$$

which is the same as (13.9) if dt can be h.

Now, at time t, the value of stock price, $S(t)$, is known. Hence, expression (****), the market-maker's profit at time $t+dt$, is not stochastic. If there are no riskless arbitrages, then quantity within the braces in (****) must be zero,

$$C_t(S, t) + \tfrac{1}{2}\sigma^2 S^2 C_{SS}(S, t) + rSC_S(S, t) - rC(S, t) = 0,$$

which is the celebrated Black-Scholes equation (13.10) for the price of an option on a nondividend-paying stock. Equation (21.11) in McDonald (2006) generalizes (13.10) to the case where the stock pays dividends continuously and proportional to its price.

Let us consider the substitutions

$$dt \to h$$
$$dS(t) = S(t + dt) - S(t) \to S(t + h) - S(t),$$
$$dC(S(t), t) = C(S(t + dt), t + dt) - C(S(t), t) \to C(S(t + h), t + h) - C(S(t), t).$$

Then, equation (**) leads to the approximation formula

$$C(S(t + h), t + h) - C(S(t), t) \approx \Delta_t [S(t + h) - S(t)] + \tfrac{1}{2}\Gamma_t [S(t + h) - S(t)]^2 + \theta_t\, h,$$

which is given near the top of page 665. Figure 13.3 on page 426 is an illustration of this approximation. Note that in formula (13.6) on page 426, the equal sign, =, should be replaced by an approximately equal sign such as ≈.

Although (***) holds because $\{S(t)\}$ is a geometric Brownian motion, the analogous equation,

$$[S(t + h) - S(t)]^2 = [\sigma S(t)]^2 h, \qquad h > 0,$$

which should be compared with (13.8) on page 429, almost never holds. If it turns out that it holds, then the market maker's profit is approximated by the right-hand side of (13.9). The expression is zero because of the Black-Scholes partial differential equation.

(10) Consider the Black-Scholes framework. Let $S(t)$ be the stock price at time t, $t \geq 0$. Define $X(t) = \ln[S(t)]$. You are given the following three statements concerning $X(t)$.

(i) $\{X(t), t \geq 0\}$ is an arithmetic Brownian motion.

(ii) $\text{Var}[X(t+h) - X(t)] = \sigma^2 h, \qquad t \geq 0, h > 0.$

(iii) $\lim_{n\to\infty} \sum_{j=1}^{n} [X(jT/n) - X((j-1)T/n)]^2 = \sigma^2 T.$

A Only (i) is true

B Only (ii) is true

C Only (i) and (ii) are true

D Only (i) and (iii) are true

E (i), (ii) and (iii) are true

Solution to (10) Answer: (E)

(i) is true. That $\{S(t)\}$ is a geometric Brownian motion means exactly that its logarithm is an arithmetic Brownian motion. (Also see the solution to problem (11).)

(ii) is true. Because $\{X(t)\}$ is an arithmetic Brownian motion, the increment, $X(t + h) - X(t)$, is a normal random variable with variance $\sigma^2 h$. This result can also be found at the bottom of page 605.

(iii) is true. Because $X(t) = \ln S(t)$, we have

$$X(t + h) - X(t) = \mu h + \sigma[Z(t + h) - Z(t)],$$

where $\{Z(t)\}$ is a (standard) Brownian motion and $\mu = \alpha - \delta - ½\sigma^2$. (Here, we assume the stock price follows (20.25), but the actual value of μ is not important.) Then,

$$[X(t + h) - X(t)]^2 = \mu^2 h^2 + 2\mu h\sigma[Z(t + h) - Z(t)] + \sigma^2[Z(t + h) - Z(t)]^2.$$

With $h = T/n$,

$$\sum_{j=1}^{n}[X(jT/n) - X((j-1)T/n)]^2$$

$$= \mu^2T^2/n + 2\mu(T/n)\sigma[Z(T) - Z(0)] + \sigma^2 \sum_{j=1}^{n}[Z(jT/n) - Z((j-1)T/n)]^2 .$$

As $n \to \infty$, the first two terms on the last line become 0, and the sum becomes T according to formula (20.6) on page 653.

Remarks: What is called "arithmetic Brownian motion" is the textbook is called "Brownian motion" by many other authors. What is called "Brownian motion" is the textbook is called "standard Brownian motion" by others.

Statement (iii) is a non-trivial result: The limit of sums of stochastic terms turns out to be deterministic. A consequence is that, if we can observe the prices of a stock over a time interval, no matter how short the interval is, we can determine the value of σ by evaluating the quadratic variation of the natural logarithm of the stock prices. Of course, this is under the assumption that the stock price follows a geometric Brownian motion. This result is a reason why the true stock price process (20.25) and the risk-neutral stock price process (20.26) must have the same σ. A discussion on realized quadratic variation can be found on page 755 of McDonald (2006).

A quick "proof" of the quadratic variation formula (20.6) can be obtained using the multiplication rule (20.17c). The left-hand side of (20.6) can be seen as $\int_0^T [dZ(t)]^2$. Formula (20.17c) states that $[dZ(t)]^2 = dt$. Thus,

$$\int_0^T [dZ(t)]^2 = \int_0^T dt = T.$$

(11) Consider the Black-Scholes framework. You are given the following three statements on variances, conditional on knowing $S(t)$, the stock price at time t.

(i) $\mathrm{Var}[\ln S(t+h) \mid S(t)] = \sigma^2 h, \qquad h > 0.$

(ii) $\mathrm{Var}\left[\frac{dS(t)}{S(t)} \middle| S(t)\right] = \sigma^2\, dt$

(iii) $\mathrm{Var}[S(t+dt) \mid S(t)] = S(t)^2\, \sigma^2\, dt$

(A) Only (i) is true

(B) Only (ii) is true

(C) Only (i) and (ii) are true

(D) Only (ii) and (iii) are true

(E) (i), (ii) and (iii) are true

Here are some facts about geometric Brownian motion. The solution of the stochastic differential equation

$$\frac{dS(t)}{S(t)} = \alpha dt + \sigma dZ(t) \tag{20.1}$$

is

$$S(t) = S(0) \exp[(\alpha - ½\sigma^2)t + \sigma Z(t)]. \qquad (*)$$

Formula (*), which can be verified to satisfy (20.1) by using Itô's Lemma, is equivalent to formula (20.29), which is the solution of the stochastic differential equation (20.25). It follows from (*) that

$$S(t + h) = S(t) \exp[(\alpha - ½\sigma^2)h + \sigma[Z(t + h) - Z(t)]], \quad h \geq 0. \qquad (**)$$

From page 650, we know that the random variable $[Z(t + h) - Z(t)]$ has the same distribution as $Z(h)$, i.e., it is normal with mean 0 and variance h.

Solution to (11) Answer: (E)

(i) is true: The logarithm of equation (**) shows that given the value of $S(t)$, $\ln[S(t + h)]$ is a normal random variable with mean $(\ln[S(t)] + (\alpha - ½\sigma^2)h)$ and variance $\sigma^2 h$. See also the top paragraph on page 650 of McDonald (2006).

(ii) is true:
$$\text{Var}\left[\frac{dS(t)}{S(t)}\middle| S(t)\right] = \text{Var}[\alpha dt + \sigma dZ(t)|S(t)]$$
$$= \text{Var}[\alpha dt + \sigma dZ(t)|Z(t)],$$

because it follows from (*) that knowing the value of $S(t)$ is equivalent to knowing the value of $Z(t)$. Now,

$$\begin{aligned}\text{Var}[\alpha dt + \sigma dZ(t)|Z(t)] &= \text{Var}[\sigma dZ(t)|Z(t)] \\ &= \sigma^2 \text{ Var}[dZ(t)|Z(t)] \\ &= \sigma^2 \text{ Var}[dZ(t)] \qquad \because \text{ independent increments} \\ &= \sigma^2 dt.\end{aligned}$$

Remark: The unconditional variance also has the same answer: $\text{Var}\left[\frac{dS(t)}{S(t)}\right] = \sigma^2 dt.$

(iii) is true because (ii) is the same as

$$\text{Var}[dS(t) \mid S(t)] = S(t)^2 \sigma^2 dt,$$

and

$$\begin{aligned}\text{Var}[dS(t) \mid S(t)] &= \text{Var}[S(t+dt) - S(t) \mid S(t)] \\ &= \text{Var}[S(t+dt) \mid S(t)].\end{aligned}$$

(12) Consider two nondividend-paying assets *X* and *Y*. There is a single source of uncertainty which is captured by a standard Brownian motion $\{Z(t)\}$. The prices of the assets satisfy the stochastic differential equations

$$\frac{dX(t)}{X(t)} = 0.07dt + 0.12dZ(t)$$

and

$$\frac{dY(t)}{Y(t)} = Adt + BdZ(t),$$

where *A* and *B* are constants. You are also given:

(i) $d[\ln Y(t)] = \mu dt + 0.085dZ(t)$;

(ii) The continuously compounded risk-free interest rate is 0.04.

Determine *A*.

(A) 0.0604
(B) 0.0613
(C) 0.0650
(D) 0.0700
(E) 0.0954

Solution to (12) Answer: (B)

If $f(x)$ is a twice-differentiable function of one variable, then Itô's Lemma (page 664) simplifies as

$$df(Y(t)) = f'(Y(t))dY(t) + ½ f''(Y(t))[dY(t)]^2,$$

because $\frac{\partial}{\partial t} f(x) = 0$.

If $f(x) = \ln x$, then $f'(x) = 1/x$ and $f''(x) = -1/x^2$. Hence,

$$d[\ln Y(t)] = \frac{1}{Y(t)} dY(t) + \frac{1}{2}\left(-\frac{1}{[Y(t)]^2}\right)[dY(t)]^2. \quad (1)$$

We are given that

$$dY(t) = Y(t)[Adt + BdZ(t)]. \quad (2)$$

Thus,

$$[dY(t)]^2 = \{Y(t)[Adt + BdZ(t)]\}^2 = [Y(t)]^2 B^2 dt, \quad (3)$$

by applying the multiplication rules (20.17) on pages 658 and 659. Substituting (2) and (3) in (1) and simplifying yields

$$d[\ln Y(t)] = (A - \frac{B^2}{2})dt + BdZ(t).$$

It thus follows from condition (i) that $B = 0.085$.

It is pointed out in Section 20.4 that two assets having the same source of randomness must have the same Sharpe ratio. Thus,

$$(0.07 - 0.04)/0.12 = (A - 0.04)/B = (A - 0.04)/0.085$$

Therefore, $A = 0.04 + 0.085(0.25) = 0.06125 \approx 0.0613$

(13) Let $\{Z(t)\}$ be a Brownian motion. You are given:

(i) $U(t) = 2Z(t) - 2$

(ii) $V(t) = [Z(t)]^2 - t$

(iii) $W(t) = t^2 Z(t) - 2\int_0^t sZ(s)\mathrm{d}s$

Which of the processes defined above has / have zero drift?

A. $\{V(t)\}$ only
B. $\{W(t)\}$ only
C. $\{U(t)\}$ and $\{V(t)\}$ only
D. $\{V(t)\}$ and $\{W(t)\}$ only
E. All three processes have zero drift.

Solution to (13) Answer: (E)

Apply Itô's Lemma.

(i) $\mathrm{d}U(t) = 2\mathrm{d}Z(t) - 0 = 0\mathrm{d}t + 2\mathrm{d}Z(t)$.

Thus, the stochastic process $\{U(t)\}$ has zero drift.

(ii) $\mathrm{d}V(t) = \mathrm{d}[Z(t)]^2 - \mathrm{d}t.$

$$\mathrm{d}[Z(t)]^2 = 2Z(t)\mathrm{d}Z(t) + \frac{2}{2}[\mathrm{d}Z(t)]^2$$

$$= 2Z(t)\mathrm{d}Z(t) + \mathrm{d}t$$

by the multiplication rule (20.17c) on page 659. Thus,

$$\mathrm{d}V(t) = 2Z(t)\mathrm{d}Z(t).$$

The stochastic process $\{V(t)\}$ has zero drift.

(iii) $\mathrm{d}W(t) = \mathrm{d}[t^2 Z(t)] - 2t\, Z(t)\mathrm{d}t$

Because

$$\mathrm{d}[t^2 Z(t)] = t^2\mathrm{d}Z(t) + 2tZ(t)\mathrm{d}t,$$

we have

$$\mathrm{d}W(t) = t^2\mathrm{d}Z(t).$$

The process $\{W(t)\}$ has zero drift.

(14) You are using the Vasicek one-factor interest-rate model with the short-rate process calibrated as

$$\mathrm{d}r(t) = 0.6[b - r(t)]\mathrm{d}t + \sigma \mathrm{d}Z(t).$$

For $t \le T$, let $P(r, t, T)$ be the price at time t of a zero-coupon bond that pays \$1 at time T, if the short-rate at time t is r. The price of each zero-coupon bond in the Vasicek model follows an Itô process,

$$\frac{dP[r(t),t,T]}{P[r(t),t,T]} = \alpha[r(t), t, T]\,\mathrm{d}t - q[r(t), t, T]\,\mathrm{d}Z(t), \qquad t \le T.$$

You are given that $\alpha(0.04, 0, 2) = 0.04139761$.

Find $\alpha(0.05, 1, 4)$.

Solution to (14)

Because all bond prices are driven by a single source of uncertainties, $\{Z(t)\}$, the no-arbitrage condition implies that the ratio, $\frac{\alpha(r,t,T)-r}{q(r,t,T)}$, does not depend on T. See (24.16) on page 782 and (20.24) on page 660. In the Vasicek model, the ratio is set to be ϕ, a constant. Thus, we have

$$\frac{\alpha(0.05,1,4)-0.05}{q(0.05,1,4)}=\frac{\alpha(0.04,0,2)-0.04}{q(0.04,0,2)}. \qquad (*)$$

To finish the problem, we need to know q, which is the coefficient of $\mathrm{d}Z(t)$ in $\frac{dP[r(t),t,T]}{P[r(t),t,T]}$. To evaluate the numerator, we apply Itô's Lemma:

$$\mathrm{d}P[r(t), t, T] \;=\; P_t[r(t), t, T]\mathrm{d}t + P_r[r(t), t, T]\mathrm{d}r(t) + \tfrac{1}{2}P_{rr}[r(t), t, T][\mathrm{d}r(t)]^2,$$

which is a portion of (20.10). Because $\mathrm{d}r(t) = a[b - r(t)]\mathrm{d}t + \sigma \mathrm{d}Z(t)$, we have $[\mathrm{d}r(t)]^2 = \sigma^2\mathrm{d}t$, which has no $\mathrm{d}Z$ term. Thus, we see that

$$q(r, t, T) = -\sigma P_r(r, t, T)/P(r, t, T) \qquad \text{which is a special case of (24.12)}$$

$$= -\sigma\frac{\partial}{\partial r}\ln[P(r, t, T)].$$

In the Vasicek model and in the Cox-Ingersoll-Ross model, the zero-coupon bond price is of the form

$$P(r, t, T) = A(t, T)\, \mathrm{e}^{-B(t, T)r};$$

hence,

$$q(r, t, T) = -\sigma\frac{\partial}{\partial r}\ln[P(r, t, T)] = \sigma B(t, T).$$

In fact, both $A(t, T)$ and $B(t, T)$ are functions of the time to maturity, $T - t$. In the Vasicek model, $B(t, T) = [1 - \mathrm{e}^{-a(T-t)}]/a$. Thus, equation (*) becomes

$$\frac{\alpha(0.05,1,4)-0.05}{1-e^{-a(4-1)}}=\frac{\alpha(0.04,0,2)-0.04}{1-e^{-a(2-0)}}.$$

Because $a = 0.6$ and $\alpha(0.04, 0, 2) = 0.04139761$, we get $\alpha(0.05, 1, 4) = 0.05167$.

Remark 1: The second equation in the problem is equation (24.1) [or (24.13)] of MacDonald (2006). In its first printing, the minus sign on the right-hand side is a plus sign. In the earlier version of this problem, we followed that convention.

Remark 2: Unfortunately, zero-coupon bond prices are denoted as $P(r, t, T)$ and also as $P(t, T, r)$ in McDonald (2006).

Remark 3: One can remember the formula,

$$B(t, T) = [1 - e^{-a(T-t)}]/a,$$

in the Vasicek model as $\bar{a}_{\overline{T-t|}\,\delta=a}$, the present value of a continuous annuity-certain of rate 1, payable for $T - t$ years, and evaluated at force of interest a, where a is the "speed of mean reversion" for the associated short-rate process.

Remark 4: If the zero coupon prices are of the so-called *affine form*,

$$P(r, t, T) = A(t, T)\, e^{-B(t, T)r},$$

where $A(t, T)$ and $B(t, T)$ are independent of r, then (24.12) becomes

$$q(r, t, T) = \sigma(r)B(t, T).$$

Thus, (24.17) is

$$\phi(r, t) = \frac{\alpha(r,t,T) - r}{q(r,t,T)} = \frac{\alpha(r,t,T) - r}{\sigma(r)B(t,T)},$$

from which we see that the instantaneous expected return of the zero-coupon bond is

$$\alpha(r, t, T) = r + \phi(r, t)\sigma(r)\, B(t, T).$$

In the Vasicek model, $\sigma(r) = \sigma$, $\phi(r, t) = \phi$, and

$$\alpha(r, t, T) = r + \phi\sigma B(t, T).$$

In the CIR model, $\sigma(r) = \sigma\sqrt{r}$, $\phi(r, t) = \dfrac{\bar{\phi}\sqrt{r}}{\sigma}$, and

$$\alpha(r, t, T) = r + \bar{\phi} r B(t, T).$$

In either model, $A(t, T)$ and $B(t, T)$ depend on the variables t and T by means of their difference $T - t$, which is the time to maturity.

Remark 4: Formula (24.20) on page 783 of McDonald (2006) is

$$P(r, t, T) = \mathrm{E}^*[\exp(-\int_t^T r(s)\,ds) \mid r(t) = r],$$

where E* represents the expectation taken with respect to the risk-neutral probability measure. Under the risk-neutral probability measure, the expected return of each asset is the risk-free interest rate. Now, (24.13) is

$$\frac{dP[r(t),t,T]}{P[r(t),t,T]} = \alpha[r(t), t, T]\,\mathrm{d}t - q[r(t), t, T]\,\mathrm{d}Z(t)$$

$$= r(t)\,\mathrm{d}t - q[r(t), t, T]\,\mathrm{d}Z(t) + \{\alpha[r(t), t, T] - r(t)\}\mathrm{d}t$$

$$= r(t)\,\mathrm{d}t - q[r(t), t, T]\left\{\mathrm{d}Z(t) - \frac{\alpha[r(t),t,T]-r(t)}{q[r(t),t,T]}\,\mathrm{d}t\right\}$$

$$= r(t)\,\mathrm{d}t - q[r(t), t, T]\{\mathrm{d}Z(t) - \phi[r(t), t]\mathrm{d}t\}. \qquad (**)$$

Let us define the stochastic process $\{\tilde{Z}(t)\}$ by

$$\tilde{Z}(t) = Z(t) - \int_0^t \phi[r(s), s]\mathrm{d}s.$$

Then, applying

$$\mathrm{d}\tilde{Z}(t) = \mathrm{d}Z(t) - \phi[r(t), t]\mathrm{d}t \qquad (***)$$

to (**) yields

$$\frac{dP[r(t),t,T]}{P[r(t),t,T]} = r(t)\mathrm{d}t - q[r(t), t, T]\mathrm{d}\tilde{Z}(t),$$

which is analogous to (20.26) on page 661. The risk-neutral probability measure is such that $\{\tilde{Z}(t)\}$ is a standard Brownian motion.

Applying (***) to equation (24.2) yields

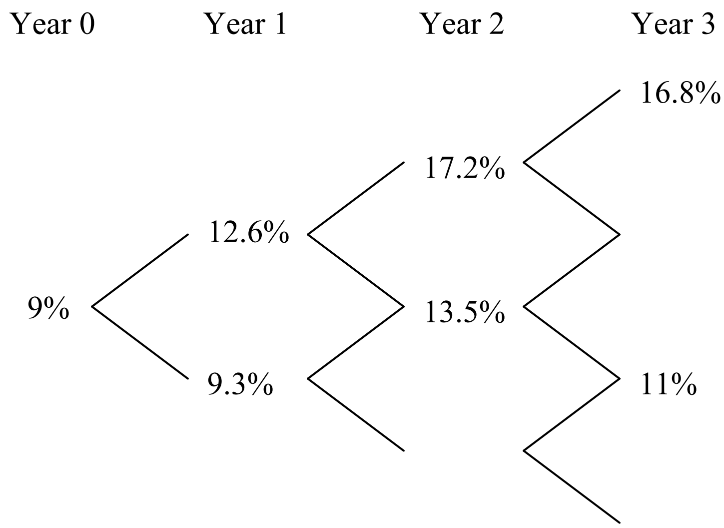
$$\mathrm{d}r(\mathrm{t}) = a[r(t)]\mathrm{d}t + \sigma[r(t)]\mathrm{d}Z(t)$$

$$= a[r(t)]\mathrm{d}t + \sigma[r(t)]\{\mathrm{d}\tilde{Z}(t) + \phi[r(t), t]\mathrm{d}t\}$$

$$= \{a[r(t)] + \sigma[r(t)]\phi[r(t), t]\}\mathrm{d}t + \sigma[r(t)]\mathrm{d}\tilde{Z}(t),$$

which is (24.19) on page 783 of McDonald (2006).

(15) You are given the following incomplete Black-Derman-Toy interest rate tree model for the effective annual interest rates:



Calculate the price of a year-4 caplet for the notional amount of $100. The cap rate is 10.5%.

Solution to (15)

First, let us fill in the three missing interest rates in the B-D-T binomial tree. In terms of the notation in Figure 24.4 of McDonald (2006), the missing interest rates are r_d, r_{ddd}, and r_{duu}. We can find these interest rates, because in each period, the interest rates in different states are terms of a *geometric progression*.

$$\frac{0.135}{r_{dd}} = \frac{0.172}{0.135} \Rightarrow r_{dd} = 10.6\%$$

$$\frac{r_{ddu}}{0.11} = \frac{0.168}{r_{ddu}} \Rightarrow r_{ddu} = 13.6\%$$

$$\left(\frac{0.11}{r_{ddd}}\right)^2 = \frac{0.168}{0.11} \Rightarrow r_{ddd} = 8.9\%$$

The payment of a year-4 caplet is made at year 4 (time 4), and we consider its discounted value at year 3 (time 3). At year 3 (time 3), the binomial model has four nodes; at that time, a year-4 caplet has one of four values:

$\frac{16.8-10.5}{1.168} = 5.394$, $\frac{13.6-10.5}{1.136} = 2.729$, $\frac{11-10.5}{1.11} = 0.450$, and 0 because $r_{ddd} = 8.9\%$ which is less than 10.5%.

For the Black-Derman-Toy model, the risk-neutral probability for an up move is ½. We now calculate the caplet's value in each of the three nodes at time 2:

$$\frac{(5.394+2.729)/2}{1.172} = 3.4654\,,\quad \frac{(2.729+0.450)/2}{1.135} = 1.4004\,,\quad \frac{(0.450+0)/2}{1.106} = 0.2034\,.$$

Then, we calculate the caplet's value in each of the two nodes at time 1:

$$\frac{(3.4654+1.4004)/2}{1.126} = 2.1607\,,\qquad \frac{(1.40044+0.2034)/2}{1.093} = 0.7337\,.$$

Finally, the time-0 price of the year-4 caplet is $\frac{(2.1607+0.7337)/2}{1.09} = 1.3277\,.$

Remarks: (1) The discussion on caps and caplets on page 805 of McDonald (2006) involves a loan. This is not necessary. (2) If your copy of McDonald was printed before 2008, then you need to correct the typographical errors on page 805; see http://www.kellogg.northwestern.edu/faculty/mcdonald/htm/typos2e_01.html (3) In the earlier version of this problem, we mistakenly used the term "year-3 caplet" for "year-4 caplet."

Alternative Solution The payoff of the year-4 caplet is made at year 4 (at time 4). In a binomial lattice, there are 16 paths from time 0 to time 4.

For the *uuuu* path, the payoff is $(16.8 - 10.5)_+$

For the *uuud* path, the payoff is also $(16.8 - 10.5)_+$

For the *uudu* path, the payoff is $(13.6 - 10.5)_+$

For the *uudd* path, the payoff is also $(13.6 - 10.5)_+$

:

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We discount these payoffs by the one-period interest rates (annual interest rates) along interest-rate paths, and then calculate their average with respect to the risk-neutral probabilities. In the Black-Derman-Toy model, the risk-neutral probability for each interest-rate path is the same. Thus, the time-0 price of the caplet is

$$\frac{1}{16}\left\{\frac{(16.8-10.5)_+}{1.09\times1.126\times1.172\times1.168} + \frac{(16.8-10.5)_+}{1.09\times1.126\times1.172\times1.168}\right.$$

$$\left. + \frac{(13.6-10.5)_+}{1.09\times1.126\times1.172\times1.136} + \frac{(13.6-10.5)_+}{1.09\times1.126\times1.172\times1.136} + \ldots\ldots\ldots\ldots\right\}$$

$$= \frac{1}{8}\left\{\frac{(16.8-10.5)_+}{1.09\times1.126\times1.172\times1.168}\right.$$

$$+ \frac{(13.6-10.5)_+}{1.09\times1.126\times1.172\times1.136} + \frac{(13.6-10.5)_+}{1.09\times1.126\times1.135\times1.136} + \frac{(13.6-10.5)_+}{1.09\times1.093\times1.135\times1.136}$$

$$+ \frac{(11-10.5)_+}{1.09\times1.126\times1.135\times1.11} + \frac{(11-10.5)_+}{1.09\times1.093\times1.135\times1.11} + \frac{(11-10.5)_+}{1.09\times1.093\times1.106\times1.11}$$

$$\left. + \frac{(9-10.5)_+}{1.09\times1.093\times1.106\times1.09}\right\} = 1.326829.$$

Remark: In this problem, the payoffs are path-independent. The "backward induction" method in the earlier solution is more efficient. However, if the payoffs are path-dependent, then the price will need to be calculated by the "path-by-path" method illustrated in this alternative solution.

(16) Assume that the Black-Scholes framework holds. Let S(t) be the price of a nondividend-paying stock at time t, $t \geq 0$. The stock's volatility is 20%, and the continuously compounded risk-free interest rate is 4%.

You are interested in claims with payoff being the stock price raised to some power. For $0 \leq t < T$, consider the equation

$$F^P_{t,T}[S(T)^x] = S(t)^x,$$

where the left-hand side is the prepaid forward price at time t of a claim that pays $S(T)^x$ at time T. A solution for the equation is x = 1.

Determine another x that solves the equation.

(A) −4
(B) −2
(C) −1
(D) 2
(E) 4

Solution to (16) Answer (B)

It follows from (20.30) in Proposition 20.3 that

$$F^P_{t,T}[S(T)^x] = S(t)^x \exp\{[-r + x(r - \delta) + ½x(x - 1)\sigma^2](T - t)\},$$

which equals $S(t)^x$ if and only if

$$-r + x(r - \delta) + ½x(x - 1)\sigma^2 = 0.$$

This is a quadratic equation of x. With $\delta = 0$, the quadratic equation becomes

$$\begin{aligned} 0 &= -r + xr + ½x(x - 1)\sigma^2 \\ &= (x - 1)(½\sigma^2 x + r). \end{aligned}$$

Thus, the solutions are 1 and $-2r/\sigma^2 = -2(4\%)/(20\%)^2 = -2$, which is (B).

Remarks:

(1) Three derivations for (20.30) can be found in Section 20.7 of McDonald (2006). Here is a fourth. Define $Y = \ln[S(T)/S(t)]$. Then,

$$\begin{aligned} F^P_{t,T}[S(T)^x] &= E^*_t[e^{-r(T-t)} S(T)^x] && \because \text{ Prepaid forward price} \\ &= E^*_t[e^{-r(T-t)} (S(t)e^Y)^x] && \because \text{ Definition of Y} \\ &= e^{-r(T-t)} S(t)^x E^*_t[e^{xY}]. && \because \text{ The value of S(t) is not random at time t} \end{aligned}$$

The problem is to find x such that $e^{-r(T-t)} E^*_t[e^{xY}] = 1$. The expectation $E^*_t[e^{xY}]$ is the *moment-generating function* of the random variable Y at the value x. Under the risk-neutral probability measure, Y is normal with mean $(r - \delta - ½\sigma^2)(T - t)$ and variance $\sigma^2(T - t)$. Thus, by the moment-generating function formula for a normal r.v.,

$$E^*_t[e^{xY}] = \exp[x(r - \delta - ½\sigma^2)(T - t) + ½x^2\sigma^2(T - t)],$$

and the problem becomes finding x such that

$$0 = -r(T - t) + x(r - \delta - ½\sigma^2)(T - t) + ½x^2\sigma^2(T - t),$$

which is the same quadratic equation as above.

(2) Applying the quadratic formula, one finds that the two solutions of

$$-r + x(r - \delta) + ½x(x - 1)\sigma^2 = 0$$

are $x = h_1$ and $x = h_2$ of Section 12.6 in McDonald (2006). A reason for this "coincidence" is that $x = h_1$ and $x = h_2$ are the values for which the stochastic process $\{e^{-rt} S(t)^x\}$ becomes a *martingale*. Martingales are defined on page 651.

(17) You are to estimate a nondividend-paying stock's annualized volatility using its prices in the past nine months.

Month	Stock Price ($/share)
1	80
2	64
3	80
4	64
5	80
6	100
7	80
8	64
9	80

Calculate the historical volatility for this stock over the period.

(A) 83%

(B) 77%

(C) 24%

(D) 22%

(E) 20%

Solution to (17) Answer (A)

This problem is based on Sections 11.3 and 11.4 of McDonald (2006), in particular, Table 11.1 on page 361.

Let $\{r_j\}$ denote the continuously compounded monthly returns. Thus, $r_1 = \ln(80/64)$, $r_2 = \ln(64/80)$, $r_3 = \ln(80/64)$, $r_4 = \ln(64/80)$, $r_5 = \ln(80/100)$, $r_6 = \ln(100/80)$, $r_7 = \ln(80/64)$, and $r_8 = \ln(64/80)$. Note that four of them are ln(1.25) and the other four are –ln(1.25); in particular, their mean is zero.

The (unbiased) sample variance of the non-annualized monthly returns is

$$\frac{1}{n-1}\sum_{j=1}^{n}(r_j - \bar{r})^2 = \frac{1}{7}\sum_{j=1}^{8}(r_j - \bar{r})^2 = \frac{1}{7}\sum_{j=1}^{8}(r_j)^2 = \frac{8}{7}[\ln(1.25)]^2.$$

The annual standard deviation is related to the monthly standard deviation by formula (11.5),

$$\sigma = \frac{\sigma_h}{\sqrt{h}},$$

where $h = 1/12$. Thus, the historical volatility is

$$\sqrt{12} \times \sqrt{\frac{8}{7}} \times \ln(1.25) = 82.6\%.$$

Remarks: Further discussion is given in Section 23.2 of McDonald (2006). (Chapter 23 is not in the syllabus of Exam MFE/3F.) Suppose that we observe n continuously compounded returns over the time period $[\tau, \tau + T]$. Then, $h = T/n$, and the historical annual variance of returns is estimated as

$$\frac{1}{h}\frac{1}{n-1}\sum_{j=1}^{n}(r_j - \bar{r})^2 = \frac{1}{T}\frac{n}{n-1}\sum_{j=1}^{n}(r_j - \bar{r})^2 .$$

Now,

$$\bar{r} = \frac{1}{n}\sum_{j=1}^{n} r_j = \frac{1}{n}\ln\frac{S(\tau+T)}{S(\tau)},$$

which is around zero when n is large. Thus, a simpler estimation formula is

$\frac{1}{h}\frac{1}{n-1}\sum_{j=1}^{n}(r_j)^2$ which is formula (23.2) on page 744, or equivalently,

$\frac{1}{T}\frac{n}{n-1}\sum_{j=1}^{n}(r_j)^2$ which is the formula in footnote 9 on page 756. The last formula is related to #10 in this set of sample problems: With probability 1,

$$\lim_{n\to\infty}\sum_{j=1}^{n}[\ln S(jT/n) - \ln S((j-1)T/n)]^2 = \sigma^2 T.$$

(18) A market-maker sells 1,000 1-year European gap call options, and delta-hedges the position with shares. You are given:

(i) Each gap call option is written on 1 share of a nondividend-paying stock.

(ii) The current price of the stock is $100.

(iii) The stock's volatility is 100%.

(iv) Each gap call option has a strike price of $130.

(v) Each gap call option has a payment trigger of $100.

(vi) The risk-free interest rate is 0%.

Under the Black-Scholes framework, determine the initial number of shares in the delta-hedge.

(A) 586

(B) 594

(C) 684

(D) 692

(E) 797

Solution to (18) Answer: (A)

Note that, in this problem, r = 0 and δ = 0.

By formula (14.15), the time-0 price of the gap option is

$$C_{gap} = SN(d_1) - 130N(d_2) = [SN(d_1) - 100N(d_2)] - 30N(d_2) = C - 30N(d_2),$$

where d_1 and d_2 are calculated with K = 100 (and r = δ = 0), and C denotes the time-0 price of the plain-vanilla call option with exercise price 100.

In the Black-Scholes framework, delta of a derivative security of a stock is the partial derivative of the security price with respect to the stock price. Thus,

$$\Delta_{gap} = \frac{\partial}{\partial S} C_{gap} = \frac{\partial}{\partial S} C - 30 \frac{\partial}{\partial S} N(d_2) = \Delta_C - 30N'(d_2) \frac{\partial}{\partial S} d_2$$

$$= N(d_1) - 30N'(d_2) \frac{1}{S\sigma\sqrt{T}},$$

where $N'(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}$ is the density function of the standard normal.

Now, with S = K = 100, T = 1, and σ = 1,

$$d_1 = [\ln(S/K) + \sigma^2 T/2]/(\sigma\sqrt{T}) = (\sigma^2 T/2)/(\sigma\sqrt{T}) = ½\sigma\sqrt{T} = ½,$$

and $d_2 = d_1 - \sigma\sqrt{T} = -½$. Hence,

$$\Delta_{gap} = N(d_1) - 30N'(d_2) \frac{1}{100} = N(½) - 0.3N'(-½) = N(½) - 0.3 \frac{1}{\sqrt{2\pi}} e^{-(-½)^2/2}$$

$$= 0.6915 - 0.3 \frac{0.8825}{\sqrt{2\pi}} = 0.6915 - 0.3\times0.352 = 0.6915 - 0.1056 = 0.5859$$

Remark: The formula of the standard normal density function, $\frac{1}{\sqrt{2\pi}} e^{-x^2/2}$, will be found in the Normal Table distributed to students.

(19) Consider a *forward start option* which, 1 year from today, will give its owner a 1-year European call option with a strike price equal to the stock price at that time.

You are given:

(i) The European call option is on a stock that pays no dividends.

(ii) The stock's volatility is 30%.

(iii) The forward price for delivery of 1 share of the stock 1 year from today is $100.

(iv) The continuously compounded risk-free interest rate is 8%.

Under the Black-Scholes framework, determine the price today of the forward start option.

(A) $11.90

(B) $13.10

(C) $14.50

(D) $15.70

(E) $16.80

Solution to (19): Answer: C

This problem is based on Exercise 14.21 on page 465 of McDonald (2006).

Let S_1 denote the stock price at the end of one year. Apply the Black-Scholes formula to calculate the price of the at-the-money call one year from today, conditioning on S_1.

$d_1 = [\ln (S_1/S_1) + (r + \sigma^2/2)T]/(\sigma\sqrt{T}) = (r + \sigma^2/2)/\sigma = 0.417$, which turns out to be independent of S_1.

$d_2 = d_1 - \sigma\sqrt{T} = d_1 - \sigma = 0.117$

The value of the forward start option at time 1 is

$$
\begin{aligned}
C(S_1) &= S_1N(d_1) - S_1e^{-r}N(d_2) \\
&= S_1[N(0.417) - e^{-0.08}N(0.117)] \\
&\approx S_1[N(0.42) - e^{-0.08}N(0.12)] \\
&= S_1[0.6628 - e^{-0.08}\times 0.5438] \\
&= 0.157S_1.
\end{aligned}
$$

(Note that, when viewed from time 0, S_1 is a random variable.)

Thus, the time-0 price of the forward start option must be 0.157 multiplied by the time-0 price of a security that gives S_1 as payoff at time 1, i.e., multiplied by the prepaid forward price $F_{0,1}^P(S)$. Hence, the time-0 price of the forward start option is

$$0.157\times F_{0,1}^P(S) = 0.157\times e^{-0.08}\times F_{0,1}(S) = 0.157\times e^{-0.08}\times 100 = 14.5$$

Remark: A key to pricing the forward start option is that d_1 and d_2 turn out to be independent of the stock price. This is the case if the strike price of the call option will be set as a fixed percentage of the stock price at the issue date of the call option.

(20) Assume the Black-Scholes framework. Consider a stock, and a European call option and a European put option on the stock. The stock price, call price, and put price are 45.00, 4.45, and 1.90, respectively.

Investor A purchases two calls and one put. Investor B purchases two calls and writes three puts.

The elasticity of Investor A's portfolio is 5.0. The delta of Investor B's portfolio is 3.4.

Calculate the put option elasticity.

(A) –0.55
(B) –1.15
(C) –8.64
(D) –13.03
(E) –27.24

Solution to (20): Answer: D

Applying the formula

$$\Delta_{\text{portfolio}} = \frac{\partial}{\partial S}\text{portfolio value}$$

to Investor B's portfolio yields

$$3.4 = 2\Delta_C - 3\Delta_P. \qquad (1)$$

Applying the elasticity formula

$$\Omega_{\text{portfolio}} = \frac{\partial}{\partial \ln S}\ln[\text{portfolio value}] = \frac{S}{\text{portfolio value}} \times \frac{\partial}{\partial S}\text{portfolio value}$$

to Investor A's portfolio yields

$$5.0 = \frac{S}{2C+P}(2\Delta_C + \Delta_P) = \frac{45}{8.9+1.9}(2\Delta_C + \Delta_P),$$

or

$$1.2 = 2\Delta_C + \Delta_P. \qquad (2)$$

Now, $(2) - (1) \quad \Rightarrow \quad -2.2 = 4\Delta_P.$

Hence, put option elasticity $= \Omega_P = \frac{S}{P}\Delta_P = \frac{45}{1.9} \times -\frac{2.2}{4} = -13.03$, which is (D).

Remarks: (i) Although not explicitly stated, you are asked to calculate the *current* option delta. The quantities, delta and elasticity, are not independent of time.
(ii) If the stock pays not dividends, and if the European call and put options have the same expiration date and strike price, then $\Delta_C - \Delta_P = 1$. In this problem, the put and call do not have the same expiration date and strike price, so this relationship does not hold.
(iii) If your copy of McDonald was printed before 2008, then you need to replace the last paragraph of Section 12.3 on page 395 by
http://www.kellogg.northwestern.edu/faculty/mcdonald/htm/erratum395.pdf
The n_i in the new paragraph corresponds to the ω_i on page 389.
(iv) The statement on page 395 in McDonald (2006) that "[t]he elasticity of a portfolio is the *weighted average* of the elasticities of the portfolio components" may remind students, who are familiar with fixed income mathematics, the concept of *duration*. Formula (3.5.8) on page 101 of *Financial Economics*: *With Applications to Investments, Insurance and Pensions* (edited by H.H. Panjer and published by The Actuarial Foundation in 1998) shows that the so-called *Macaulay duration* is an elasticity.
(v) A cleaner explanation of some of the results on page 394 of McDonald (2006) can be found on page 687, which is not part of the syllabus for Exam MFE/3F.
(vi) In the Black-Scholes framework, the hedge ratio or delta of a portfolio is the partial derivative of the portfolio price with respect to the stock price. In other continuous-time frameworks (which are not in the syllabus of Exam MFE/3F), the hedge ratio may not be given by a derivative; for an example, see formula (10.5.7) on page 478 of *Financial Economics*: *With Applications to Investments, Insurance and Pensions*.