FALL 2006
EXAM C SOLUTIONS

Question #1
Key: E

With $n + 1 = 16$, we need the 0.3(16) = 4.8 and 0.65(16) = 10.4 smallest observations. They are 0.2(280) + 0.8(350) = 336 and 0.6(450) + 0.4(490) = 466.
The equations to solve are:

$$0.3 = 1 - \left(\frac{\theta^\gamma}{\theta^\gamma + 336^\gamma}\right)^2 \text{ and } 0.65 = 1 - \left(\frac{\theta^\gamma}{\theta^\gamma + 466^\gamma}\right)^2$$

$$(0.7)^{-1/2} = 1 + (336/\theta)^\gamma \text{ and } (0.35)^{-1/2} = 1 + (466/\theta)^\gamma$$

$$\frac{(0.7)^{-1/2} - 1}{(0.35)^{-1/2} - 1} = \frac{(336/\theta)^\gamma}{(466/\theta)^\gamma}$$

$$0.282814 = (336/466)^\gamma$$

$$\ln(0.282814) = \gamma \ln(336/466)$$

$$\gamma = 3.8614.$$

Question #2
Key: D

Let E be the even of having 1 claim in the first four years. In four years, the total number of claims is Poisson(4λ).

$$\Pr(Type\,I \mid E) = \frac{\Pr(E \mid Type\,I)\Pr(Type\,I)}{\Pr(E)} = \frac{e^{-1}(0.05)}{\Pr(E)} = \frac{0.01839}{\Pr(E)} = 0.14427$$

$$\Pr(Type\,II \mid E) = \frac{e^{-2}(2)(0.2)}{\Pr(E)} = \frac{0.05413}{\Pr(E)} = 0.42465$$

$$\Pr(Type\,III \mid E) = \frac{e^{-4}(4)(0.75)}{\Pr(E)} = \frac{0.05495}{\Pr(E)} = 0.43108$$

$$Note: \Pr(E) = 0.01839 + .05413 + .05495 = 0.12747$$

The Bayesian estimate of the number of claims in Year 5 is:
0.14427(0.25) + 0.42465(0.5) + 0.43108(1) = 0.67947.

Question #3
Key: B

The sample mean is 0.2(400) + 0.7(800) + 0.1(1600) = 800.
The sample variance is $0.2(400-800)^2 + 0.7(800-800)^2 + 0.1(1600-800)^2 = 96,000$.
The sample third central moment is
$0.2(400-800)^3 + 0.7(800-800)^3 + 0.1(1600-800)^3 = 38,400,000$.

The skewness coefficient is $38,400,000 / 96,000^{1.5} = 1.29$.

Question #4
Key: C

Because 0.656 < 0.7654 < 0.773, the simulated number of losses is 4. To simulate a loss by inversion, use

$$F(x) = 1 - e^{-(x/\theta)^\tau} = u$$

$$1 - u = e^{-(x/\theta)^\tau}$$

$$\ln(1-u) = -(x/\theta)^\tau$$

$$x = \theta(-\ln(1-u))^{1/\tau} = 200(-\ln(1-u))^{1/2}$$

$$u_1 = 0.2738,\ x_1 = 113.12$$

$$u_2 = 0.5152,\ x_2 = 170.18$$

$$u_3 = 0.7537,\ x_3 = 236.75$$

$$u_4 = 0.6481,\ x_4 = 204.39$$

With a deductible of 150, the first loss produces no payments and 113.12 toward the 500 limit. The second loss produces a payment of 20.18 and the insured is now out-of-pocket 263.12. The third loss produces a payment of 86.75 and the insured is out 413.12. The deductible on the fourth loss is then 86.88 for a payment of 204.29 – 86.88 = 117.51.

The total paid by the insurer is 20.18 + 86.75 + 117.51 = 224.44.

Question #5
Key: A

The density function is $f(x)=\theta x^{-2}e^{-\theta/x}$ and the likelihood function is

$$L(\theta)=\theta(186^{-2})e^{-\theta/186}\theta(91^{-2})e^{-\theta/91}\theta(66^{-2})e^{-\theta/66}(e^{-\theta/60})^{7}$$

$$\propto\theta^{3}e^{-0.148184\theta}$$

$$l(\theta)=\ln L(\theta)=3\ln(\theta)-0.148184\theta$$

$$l'(\theta)=3\theta^{-1}-0.148184=0$$

$$\theta=3/0.148184=20.25.$$

The mode is $\theta/2=20.25/2=10.125$.

Question #6
Key: D

We have $\mu(\theta)=4\theta$ and $\mu=4\mathrm{E}(\theta)=4(600)=2400$. The average loss for Years 1 and 2 is 1650 and so $1800=Z(1650)+(1-Z)(2400)$ which gives $Z=0.8$. Because there were two years, $Z=0.8=2/(2+k)$ which gives $k=0.5$.

For three years, the revised value is $Z=3/(3+0.5)=6/7$ and the revised credibility estimate (using the new sample mean of 2021), $(6/7)(2021)+(1/7)(2400)=2075.14$.

Question #7
Key: B

The uncensored observations are 4 and 8 (values beyond 11 are not needed). The two r values are 10 and 5 and the two s values are 2 and 1. The Kaplan-Meier estimate is $\hat{S}(11)=(8/10)(4/5)=0.64$ and Greenwood's estimate is $(0.64)^{2}\left(\frac{2}{10(8)}+\frac{1}{5(4)}\right)=0.03072$.

Question #8
Key: C

There are two ways to approach this problem. One is LaGrange's formula:

$$f(3)=\frac{(3-4)(3-5)}{(2-4)(2-5)}25+\frac{(3-2)(3-5)}{(4-2)(4-5)}20+\frac{(3-2)(3-4)}{(5-2)(5-4)}30=18.33.$$

Or, if the equation is $f(x)=a+bx+cx^2$ then three equations must be satisfied:

$$25=a+2b+4c$$

$$20=a+4b+16c$$

$$30=a+5b+25c$$

The solutions is a = 63.3333, b = -27.5, and c = 4.1667. The answer is
$63.3333-27.5(3)+4.1667(9)=18.33$.

Question #9
Key: E

$S_m\,|\,Q\sim bin(m,Q)$ and $Q\sim beta(1,99)$. Then
$E(S_m)=E[E(S_m\,|\,Q)]=E(mQ)=m\frac{1}{1+99}=0.01m$. For the mean to be at least 50, m must be at least 5,000.

Question #10
Key: D

The posterior distribution is
$\pi(\lambda\,|\,data)\propto(e^{-\lambda})^{90}(\lambda e^{-\lambda})^{7}(\lambda^{2}e^{-\lambda})^{2}(\lambda^{3}e^{-\lambda})\frac{\lambda^{4}e^{-50\lambda}}{\lambda}=\lambda^{17}e^{-150\lambda}$ which is a gamma distribution with parameters 18 and 1/150. For one risk, the estimated value is the mean, 18/150. For 100 risks it is 100(18)/150 = 12.

Alternatively,

The prior distribution is gamma with α = 4 and β = 50. The posterior will be continue to be gamma, with $\alpha^{/}$ = α + no. of claims = 4 + 14 = 18 and $\beta^{/}$ = β + no. of exposures = 50 + 100 = 150. Mean of the posterior = α / β = 18/150 = 0.12.
Expected number of claims for the portfolio = 0.12 (100) = 12.

Question #11
Key: E

$$0.95 = \Pr(0.95\mu < \bar{X} < 1.05\mu)$$
$$\bar{X} \sim N(\mu, \sigma^2 / n = 1.44\mu^2 / n)$$
$$0.95 = \Pr\left(\frac{0.95\mu - \mu}{1.2\mu / \sqrt{n}} < Z < \frac{1.05\mu - \mu}{1.2\mu / \sqrt{n}}\right)$$
$$0.95 = \Pr(-0.05\sqrt{n} / 1.2 < Z < 0.05\sqrt{n} / 1.2)$$
$$0.05\sqrt{n} / 1.2 = 1.96$$
$$n = 2212.76.$$

Question #12
Key: B

$$L(q) = \left[\binom{2}{0}(1-q)^2\right]^{5000}\left[\binom{2}{1}q(1-q)\right]^{5000} = 2^{5000} q^{5000} (1-q)^{15000}$$
$$l(q) = 5000\ln(2) + 5000\ln(q) + 15000\ln(1-q)$$
$$l'(q) = 5000q^{-1} - 15000(1-q)^{-1} = 0$$
$$\hat{q} = 0.25$$
$$l(0.25) = 5000\ln(2) + 5000\ln(0.25) + 15000\ln(0.75) = -7780.97.$$

Question #13
Key: C

The estimate of the overall mean, μ, is the sample mean, per vehicle, which is 7/10 = 0.7. With the Poisson assumption, this is also the estimate of v = EPV. The means for the two insureds are 2/5 = 0.4 and 5/5 = 1.0. The estimate of a is the usual non-parametric estimate,

$$\text{VHM} = \hat{a} = \frac{5(0.4-0.7)^2 + 5(1.0-0.7)^2 - (2-1)(0.7)}{10 - \frac{1}{10}(25+25)} = 0.04$$

(The above formula: Loss Models page 596, Herzog page 116, Dean page 25)

Then, k = 0.7/0.04 = 17.5 and so Z = 5/(5+17.5) = 2/9. The estimate for insured A is (2/9)(0.4) + (7/9)(0.7) = 0.6333.

Question #14
Key: A

Item (i) indicates that X must one of the four given values.
Item (ii) indicates that X cannot be 200
Item (iii) indicates that X cannot be 400.
First assume $X = 100$. Then the values of r are 5, 3, 2, and 1 and the values of s are 2, 1, 1, and 1. Then $\hat{H}(410) = \frac{2}{5} + \frac{1}{3} + \frac{1}{2} + \frac{1}{1} = 2.23$ and thus the answer is 100. As a check, if $X = 300$, the r values are 5, 4, 3, and 1 and the s values are 1, 1, 2, and 1. Then, $\hat{H}(410) = \frac{1}{5} + \frac{1}{4} + \frac{2}{3} + \frac{1}{1} = 2.12$.

Question #15
Key: B

The estimator of the Poisson parameter is the sample mean. Then,

$$E(\hat{\lambda}) = E(\bar{X}) = \lambda$$

$$Var(\hat{\lambda}) = Var(\bar{X}) = \lambda / n$$

$$c.v. = \sqrt{\lambda / n} / \lambda = 1 / \sqrt{n\lambda}$$

It is estimated by $1/\sqrt{n\lambda} = 1/\sqrt{39} = 0.1601$.

Question #16
Key: E

$$\Pr(\theta = 8 \mid X_1 = 5) = \frac{\Pr(X_1 = 5 \mid \theta = 8)\Pr(\theta = 8)}{\Pr(X_1 = 5 \mid \theta = 8)\Pr(\theta = 8) + \Pr(X_1 = 5 \mid \theta = 2)\Pr(\theta = 2)}$$

$$= \frac{0.125e^{-5(0.125)}(0.8)}{0.125e^{-5(0.125)}(0.8) + 0.5e^{-5(0.5)}(0.2)} = 0.867035.$$

Then,

$$E(X_2 \mid X_1 = 5) = E(\theta \mid X_1 = 5) = 0.867035(8) + 0.132965(2) = 7.202.$$

Question #17
Key: D

We have $q^{(T)} = 1 - (1 - q'^{(1)})(1 - q'^{(2)})$ and so $q'^{(1)} = 1 - \frac{1 - q^{(T)}}{1 - q'^{(2)}} = 1 - \frac{1 - q^{(T)}}{1 - 0.05} = \frac{q^{(T)} - 0.05}{0.95}$. Then,
${}_{20}q_0'^{(1)} = 0.05/0.95 = 0.05263$, ${}_{20}q_{20}'^{(1)} = 0.132/0.95 = 0.1389$, and
${}_{40}p_0'^{(1)} = 0.9474(0.8611) = 0.8158$. Out of 1000 at age 0, 816 are expected to survive to age 40.

Question #18
Key: D

$$L(\omega)=\frac{\frac{1}{\omega}\frac{1}{\omega}\frac{1}{\omega}\left(\frac{\omega-4-p}{\omega}\right)^2}{\left(\frac{\omega-4}{\omega}\right)^5}=\frac{(\omega-4-p)^2}{(\omega-4)^5}$$

$$l(\omega)=2\ln(\omega-4-p)-5\ln(\omega-4)$$

$$l'(\omega)=\frac{2}{\omega-4-p}-\frac{5}{\omega-4}=0$$

$$0=l'(29)=\frac{2}{25-p}-\frac{5}{25}$$

$p=15.$

The denominator in the likelihood function is $S(4)$ to the power of five to reflect the fact that it is known that each observation is greater than 4.

Question #19
Key: B

$$\mu(\lambda)=v(\lambda)=\lambda$$

$$\mu=v=E(\lambda)=0.1\Gamma(1+1/2)=0.088623$$

VHM = $a=Var(\lambda)=(0.1)^2\Gamma(1+2/2)-0.088623^2=0.002146$

$$Z=\frac{500}{500+0.088623/0.002146}=0.92371.$$

The estimate for one insured for one month is $0.92371(35/500)+0.07629(0.088623)=0.07142.$ For 300 insureds for 12 months it is $(300)(12)(0.07142)=257.11.$

Question #20
Key: D

With no censoring the r values are 12, 9, 8, 7, 6, 4, and 3 and the s values are 3, 1, 1, 1, 2, 1, 1 (the two values at 7500 are not needed). Then,

$$\hat{H}_1(7000)=\frac{3}{12}+\frac{1}{9}+\frac{1}{8}+\frac{1}{7}+\frac{2}{6}+\frac{1}{4}+\frac{1}{3}=1.5456.$$

With censoring, there are only five uncensored values with r values of 9, 8, 7, 4, and 3 and all five s values are 1. Then,

$\hat{H}_2(7000)=\frac{1}{9}+\frac{1}{8}+\frac{1}{7}+\frac{1}{4}+\frac{1}{3}=0.9623.$ The absolute difference is 0.5833.

Question #21
Key: A

The simulated paid loss is $\exp[0.494\Phi^{-1}(u)+13.294]$ where $\Phi^{-1}(u)$ is the inverse of the standard normal distribution function. The four simulated paid losses are 450,161, 330,041, 939,798, and 688,451 for an average of 602,113. The multiplier for unpaid losses is $0.801(2006-2005)^{0.851}e^{-0.747(2006-2005)}=0.3795$ and the answer is $0.3795(602,113)=228,502$

Question #22
Key: A

The deduction to get the SBC is $(r/2)\ln(n)=(r/2)\ln(260)=2.78r$ where *r* is the number of parameters. The SBC values are then -416.78, -417.56, -419.34, -420.12, and -425.68. The largest value is the first one, so model I is to be selected.

Question #23
Key: E

$$\Pr\left(\lambda=1 \mid X_1=r\right)=\frac{\Pr(X_1=r\mid\lambda=1)\Pr(\lambda=1)}{\Pr(X_1=r\mid\lambda=1)\Pr(\lambda=1)+\Pr(X_1=r\mid\lambda=3)\Pr(\lambda=3)}$$

$$=\frac{\frac{e^{-1}}{r!}(0.75)}{\frac{e^{-1}}{r!}(0.75)+\frac{e^{-3}3^r}{r!}(0.25)}=\frac{0.2759}{0.2759+0.1245(3^r)}.$$

Then,

$$2.98=\frac{0.2759}{0.2759+0.1245(3^r)}(1)+\frac{0.1245(3^r)}{0.2759+0.1245(3^r)}(3)$$

$$=\frac{0.2759+0.3735(3^r)}{0.2759+0.1245(3^r)}.$$

Rearrange to obtain

$$0.82218+0.037103(3^r)=0.2759+0.03735(3^r)$$

$$0.54628=0.00025(3^r)$$

$$r=7.$$

Because the risks are Poisson, (μ = EPV, a = VHM):

$$\mu=v=E(\lambda)=0.75(1)+0.25(3)=1.5$$

$$a=Var(\lambda)=0.75(1)+0.25(9)-2.25=0.75$$

$$Z=\frac{1}{1+1.5/0.75}=1/3$$

and the estimate is (1/3)(7) + (2/3)(1.5) = 3.33.

Question #24
Key: E

The uniform kernel spreads the probability of 0.1 to 10 units each side of an observation. So the observation at 25 contributes a density of 0.005 from 15 to 35, contributing nothing to survival past age 40. The same applies to the point at 30. For the next 7 points:
35 contributes probability from 25 to 45 for 5(0.005) = 0.025 above age 40.
35 contributes probability from 25 to 45 for 5(0.005) = 0.025 above age 40.
37 contributes probability from 27 to 47 for 7(0.005) = 0.035 above age 40.
39 contributes probability from 29 to 49 for 9(0.005) = 0.045 above age 40.
45 contributes probability from 35 to 55 for 15(0.005) = 0.075 above age 40.
47 contributes probability from 37 to 57 for 17(0.005) = 0.085 above age 40.
49 contributes probability from 39 to 59 for 19(0.005) = 0.095 above age 40.
The observation at 55 contributes all 0.1 of probability. The total is 0.485.

Question #25
Key: D

$$f(3) = 2 + (3/2)(4) - (1/4)(8) = 6$$
$$f'(3) = (3/2)(2)(2) - (1/4)(3)(4) = 3$$
$$f(4) = f(3) + (4-3)f'(3) = 6 + 1(3) = 9.$$

Question #26
Key: A

$$X \sim Exp(\theta)$$
$$\sum_{i=1}^{n} X_i \sim \Gamma(n,\theta)$$
$$\bar{X} \sim \Gamma(n,\theta/n)$$
$$E\left(\bar{X}^2\right) = (\theta/n)^2(n)(n+1) = (n+1)\theta^2/n.$$

The second line follows because an exponential distribution is a gamma distribution with $\alpha = 1$ and the sum of independent gamma random variables is gamma with the "α" parameters added. The third line follows because the gamma distribution is a scale distribution. Multiplying by $1/n$ retains the gamma distribution with the "θ" parameter multiplied by $1/n$.

Question #27
Key: C

The sample means are 3, 5, and 4 and the overall mean is 4. Then,

$$\hat{v} = \frac{1+0+0+1+0+0+1+1+1+1+1+1}{3(4-1)} = \frac{8}{9}$$

$$\hat{a} = \frac{(3-4)^2+(5-4)^2+(4-4)^2}{3-1} - \frac{8/9}{4} = \frac{7}{9} = 0.78.$$

Question #28
Key: C

The ordered values are:
22t, 25t, 27h, 28t, 31h, 33t, 35h, 39t, 42h, and 45h where t is a traditional car and h is a hybrid car. The *s* values are all 1 because there are no duplicate values. The *c* values are 1 for traditional cars and e^{-1} for hybrid cars. Then

$$\hat{H}_0(32) = \frac{1}{5+5e^{-1}} + \frac{1}{4+5e^{-1}} + \frac{1}{3+5e^{-1}} + \frac{1}{3+4e^{-1}} + \frac{1}{2+4e^{-1}} = 1.0358.$$

Question #29
Key: C

$$\pi(q\,|\,2) = 6q^2(1-q)^2 6q(1-q) \propto q^3(1-q)^3$$

The mode can be determined by setting the derivative equal to zero.

$$\pi'(q\,|\,2) \propto 3q^2(1-q)^3 - 3q^3(1-q)^2 = 0$$

$$(1-q) - q = 0$$

$$q = 0.5.$$

Question #30
Key: B

For the severity distribution the mean is 5,000 and the variance is $10{,}000^2/12$. For credibility based on accuracy with regard to the number of claims,

$$2000 = \left(\frac{z}{0.03}\right)^2, \quad z^2 = 1.8$$

where *z* is the appropriate value from the standard normal distribution. For credibility based on accuracy with regard to the total cost of claims, the number of claims needed is

$$\frac{z^2}{0.05^2}\left(1 + \frac{10000^2/12}{5000^2}\right) = 960.$$

Question #31
Key: C

$$\hat{S}(10) = e^{-\hat{H}(10)} = 0.575$$

$$\hat{H}(10) = -\ln(0.575) = 0.5534 = \frac{1}{50} + \frac{3}{49} + \frac{5}{k} + \frac{7}{12}.$$

The solution is $k = 36$.

Question #32
Key: A

The annual dental charges are simulated from

$$u = 1 - e^{-x/1000}$$

$$x = -1000\ln(1-u).$$

The four simulated values are 356.67, 2525.73, 1203.97, and 83.38. The reimbursements are 205.34 (80% of 256.67), 1000 (the maximum), 883.18 (80% of 1103.97), and 0. The total is 2088.52 and the average is 522.13.

Question #33
Key: B

$$L(\theta) = \left(1 - \frac{\theta}{10}\right)^9 \left(\frac{\theta}{10} - \frac{\theta}{25}\right)^6 \left(\frac{\theta}{25}\right)^5 \propto (10-\theta)^9 \theta^{11}$$

$$l(\theta) = 9\ln(10-\theta) + 11\ln(\theta)$$

$$l'(\theta) = -\frac{9}{10-\theta} + \frac{11}{\theta} = 0$$

$$11(10-\theta) = 9\theta$$

$$110 = 20\theta$$

$$\theta = 110/20 = 5.5.$$

Question #34
Key: A

The maximum likelihood estimate is $\hat{\theta} = \bar{x} = 1000$. The quantity to be estimated is $S(\theta) = \exp(-1500/\theta)$ and $S'(\theta) = 1500\theta^{-2}\exp(-1500/\theta)$. For the delta method,

$$Var[S(\hat{\theta})] \cong [S'(\hat{\theta})]^2 Var(\hat{\theta})$$
$$= [1500(1000)^{-2}\exp(-1500/1000)]^2(1000^2/6)$$
$$= 0.01867.$$

This is based on $Var(\hat{\theta}) = Var(\bar{X}) = Var(X)/n = \theta^2/n$.

Question #35
Key: A

Based on the information given

$$0.21 = \frac{36}{n} + \frac{0.4x}{n}$$
$$0.51 = \frac{36}{n} + \frac{x}{n} + \frac{0.6y}{n}$$
$$n = 200 + x + y.$$

Then,

$$0.21(200 + x + y) = 36 + 0.4x$$
$$0.51(200 + x + y) = 36 + x + 0.6y$$

and these linear equations can be solved for x = 119.37.