SOCIETY OF ACTUARIES

EXAM FM FINANCIAL MATHEMATICS

EXAM FM SAMPLE SOLUTIONS

This page indicates changes made to Study Note FM-09-05.

April 28, 2014:
Question and solution 61 added.

January 14, 2014:
Questions and solutions 58–60 were added.

Some of the questions in this study note are taken from past SOA/CAS examinations.

EXAM 2/FM SAMPLE QUESTIONS *SOLUTIONS*

The following model solutions are presented for educational purposes. Alternate methods of solution are, of course, acceptable.

1. Solution: C

Given the same principal invested for the same period of time yields the same accumulated value, the two measures of interest $i^{(2)}$ and δ must be equivalent, which means: $(1+\frac{i^{(2)}}{2})^2 = e^{\delta}$ over one interest measurement period (a year in this case).

Thus, $(1+\frac{.04}{2})^2 = e^{\delta}$ or $(1+.02)^2 = e^{\delta}$ and $\delta = \ln(1.02)^2 = 2\ln(1.02) = .0396$ or 3.96%.

----

2. Solution: E

Accumulated value end of 40 years =
100 $[(1+i)^4 + (1+i)^8 + \ldots\ldots(1+i)^{40}]$= 100 $((1+i)^4)[1-((1+i)^4)^{10}]/[1 - (1+i)^4]$
("Sum of finite geometric progression =
1st term times [1 – (common ratio) raised to the number of terms] divided by [1 –common ratio]")
and accumulated value end of 20 years =
100 $[(1+i)^4 + (1+i)^8 + \ldots\ldots(1+i)^{20}]$=100 $((1+i)^4)[1-((1+i)^4)^5]/[1 - (1+i)^4]$

But accumulated value end of 40 years = 5 times accumulated value end of 20 years
Thus, 100 $((1+i)^4)[1-((1+i)^4)^{10}]/[1 - (1+i)^4]$ = 5 {100 $((1+i)^4)[1-((1+i)^4)^5]/[1 - (1+i)^4]$}
Or, for i > 0, $1-((1+i)^{40} = 5 [1-((1+i)^{20}]$ or $[1-((1+i)^{40}]/[1-((1+i)^{20}] = 5$

But $x^2 - y^2 = [x-y] [x+y]$, so $[1-((1+i)^{40}]/[1-((1+i)^{20}]= [1+((1+i)^{20}]$ Thus, $[1+((1+i)^{20}] = 5$ or $(1+i)^{20} = 4$.
So X = Accumulated value at end of 40 years = 100 $((1+i)^4)[1-((1+i)^4)^{10}]/[1 - (1+i)^4]$
=100 $(4^{1/5})[1-((4^{1/5})^{10}]/[1 – 4^{1/5}]$ = 6194.72

<u>Alternate solution using annuity symbols</u>: End of year 40, accumulated value = $100(s_{\overline{40|}}/a_{\overline{4|}})$, and end of year 20 accumulated value = $100(s_{\overline{20|}}/a_{\overline{4|}})$. Given the ratio of the values equals 5, then

5 = $(s_{\overline{40|}}/s_{\overline{20|}}) = [(1+i)^{40}-1]/[(1+i)^{20}-1] = [(1+i)^{20}+1]$. Thus, $(1+i)^{20} = 4$ and the accumulated value at the end of 40 years is $100(s_{\overline{40|}}/a_{\overline{4|}}) = 100[(1+i)^{40}-1]/[1-(1+i)^{-4}] = 100[16-1]/[1-4^{-1/5}] = 6194.72$

<u>Note</u>: if i = 0 the conditions of the question are not satisfied because then the accumulated value at the end of 40 years = 40 (100) = 4000, and the accumulated value at the end of 20 years = 20 (100) = 2000 and thus accumulated value at the end of 40 years is not 5 times the accumulated value at the end of 20 years.

3. Solution: C

Eric's interest (compound interest), last 6 months of the 8th year: $100(1+\frac{i}{2})^{15}(\frac{i}{2})$

Mike's interest (simple interest), last 6 months of the 8th year: $200(\frac{i}{2})$. Thus, $100(1+\frac{i}{2})^{15}(\frac{i}{2}) = 200(\frac{i}{2})$

or $(1+\frac{i}{2})^{15} = 2$, which means i/2 = .047294 or

i = .094588 = 9.46%

----

4. Solution: A

The payment using the amortization method is 1627.45.

The periodic interest is .10(10000) = 1000. Thus, deposits into the sinking fund are 1627.45-1000 = 627.45

Then, the amount in sinking fund at end of 10 years is 627.45 $s_{\overline{10}|.14}$

Using BA II Plus calculator keystrokes: 2nd FV (to clear registers) 10 N, 14 I/Y, 627.45 PMT, CPT FV +/- - 10000= yields 2133.18 (Using BA 35 Solar keystrokes are AC/ON (to clear registers) 10 N 14 %i 627.45 PMT CPT FV +/- – 10000 =)

----

5. Solution: E

Key formulas for estimating dollar-weighted rate of return:

Fund January 1 + deposits during year – withdrawals during year + interest = Fund December 31.

Estimate of dollar–weighted rate of return = amount of interest divided by the weighted average amount of fund exposed to earning interest

$$\text{total deposits} = 120$$

$$\text{total withdrawals} = 145$$

$$\text{Investment income} = 60+145-120-75 = 10$$

$$\text{Rate of return} = \frac{10}{75+\left(\frac{1}{12}+\cdots+\frac{11}{12}\right)\cdot 10-\frac{10}{12}5-\frac{6}{12}25-\frac{2.5}{12}80-\frac{2}{12}35}$$

= 10/90.833 = 11%

----

6. Solution: C

Cost of the perpetuity $= v \cdot \left(Ia\right)_{\overline{n|}} + \dfrac{n \cdot v^{n+1}}{i}$

$$= v \cdot \left[\frac{\ddot{a}_{\overline{n|}} - nv^n}{i}\right] + \frac{n \cdot v^{n+1}}{i}$$

$$= \frac{a_{\overline{n|}}}{i} - \frac{nv^{n+1}}{i} + \frac{nv^{n+1}}{i}$$

$$= \frac{a_{\overline{n|}}}{i}$$

Given $i = 10.5\%$,

$$\frac{a_{\overline{n|}}}{i} = \frac{a_{\overline{n|}}}{0.105} = 77.10 \Rightarrow a_{\overline{n|}} = 8.0955, \text{ at } 10.5\%$$

$$\therefore n = 19$$

Tips:
Helpful analysis tools for varying annuities: draw picture, identify “layers” of level payments, and add values of level layers.
In this question, first layer gives a value of 1/i (=PV of level perpetuity of 1 = sum of an infinite geometric progression with common ratio v, which reduces to 1/i) at 1, or v (1/i) at 0
2nd layer gives a value of 1/i at 2, or v^2 (1/i) at 0
…….
nth layer gives a value of 1/i at n, or v^n (1/i) at 0
Thus 77.1 = PV = (1/i) (v + v^2 + …. v^n) = (1/.105) $a_{\overline{n|}.105}$
n can be easily solved for using BA II Plus or BA 35 Solar calculator

7. Solution: C

$$6(Ds)_{\overline{10}|0.09} + 100\ s_{\overline{10}|0.09}$$

$$6\left(\frac{10(1.09)^{10} - s_{\overline{10}|0.09}}{0.09}\right) + 100(15.19293)$$

$$565.38 + 1519.29$$

$$2084.67$$

Helpful general result for obtaining PV or Accumulated Value (AV) of arithmetically varying sequence of payments with interest conversion period (ICP) equal to payment period (PP):
Given: Initial payment P at end of 1st PP; increase per PP = Q (could be negative); number of payments = n; effective rate per PP = i (in decimal form). Then
PV = P $a_{\overline{n}|i}$ + Q [($a_{\overline{n}|i}$ – n v^n)/i] (if first payment is at beginning of first PP, just multiply this result by (1+i))

To efficiently use special calculator keys, simplify to: (P + Q/i) $a_{\overline{n}|i}$ – n Q v^n/ i = (P + Q/i) $a_{\overline{n}|i}$ – n (Q/i) v^n.

Then for BA II Plus: select 2nd FV, enter value of n select N, enter value of 100i select I/Y, enter value of (P+(Q/i)) select PMT, enter value of (–n (Q/i)) select FV, CPT PV +/-
For accumulated value: select 2nd FV, enter value of n select N, enter value of 100i select I/Y, enter value of (P+(Q/i)), select PMT, CPT FV select +/- select – enter value of (n (Q/i)) =
For this question: Initial payment into Fund Y is 160, increase per PP = - 6
BA II Plus: 2nd FV, 10 N, 9 I/Y, (160 – (6/.09)) PMT, CPT FV +/- + (60/.09) = yields 2084.67344
(For BA 35 Solar: AC/ON, 10 N, 9 %i, (6/.09 = +/- + 160 =) PMT, CPT FV +/- STO, 60/.09 + RCL (MEM) =)
----

8. Solution: D

$$P = 1000(1.095)(1.095)(1.096) = 1314.13$$

$$Q = 1000(1.0835)(1.086)(1.0885) = 1280.82$$

$$R = 1000(1.095)(1.10)(1.10) = 1324.95$$

Thus, $R > P > Q$.

9. Solution: D

For the first 10 years, each payment equals 150% of interest due. The lender charges 10%, therefore 5% of the principal outstanding will be used to reduce the principal.

At the end of 10 years, the amount outstanding is $1000(1-0.05)^{10} = 598.74$

Thus, the equation of value for the last 10 years using a comparison date of the end of year 10 is

598.74 = X $a_{\overline{10}|10\%}$. So X = $\dfrac{598.74}{a_{\overline{10}|10\%}} = 97.4417$

Alternatively, derive answer from basic principles rather than intuition.
Equation of value at time 0:
1000 = 1.5(.1)(1000) (v +.95 v^2 + $.95^2$ v^3 + ……+ $.95^9$ v^{10}) + X v^{10} $a_{\overline{10}|.1}$.

Thus X = [1000 - {1.5(.1)(1000) (v +.95 v^2 + $.95^2$ v^3 + ……+ $.95^9$ v^{10})}]/ (v^{10} $a_{\overline{10}|.1}$)

= {1000 –[150 v (1 – (.95 v)^{10})/(1-.95 v)]}/ (v^{10} $a_{\overline{10}|.1}$)= 97.44

----

10. Solution: B

$i = 6\%$

$$BV_6 = 10,000v^4 + 800a_{\overline{4}|0.06} = 7920.94 + 2772.08 = 10,693$$

$$I_7 = i \times BV_6 = 0.06 \times 10,693 = 641.58$$

----

11. Solution: A

Value of initial perpetuity immediately after the 5th payment (or any other time) = 100 (1/i) = 100/.08 = 1250.
Exchange for 25-year annuity-immediate paying X at the end of the first year, with each subsequent payment increasing by 8%, implies
1250 (value of the perpetuity) must =
X (v + 1.08 v^2 + 1.08^2 v^3 + …..1.08^{24} v^{25}) (value of 25-year annuity-immediate)
= X (1.08^{-1} + 1.08 $(1.08)^{-2}$ + 1.08^2 $(1.08)^{-3}$ + 1.08^{24} $(1.08)^{-25}$)
(because the annual effective rate of interest is 8%)
= X (1.08^{-1} +1.08^{-1} +….. 1.08^{-1}) = X [25(1.08^{-1})].
So, 1250 (1.08) = 25 X or X = 54

12. Solution: C

Equation of value at end of 30 years:

$$10\left(1-\tfrac{d}{4}\right)^{-40}(1.03)^{40}+20(1.03)^{30}=100$$

$$10\left(1-\tfrac{d}{4}\right)^{-40}=15.77$$

$$1-\tfrac{d}{4}=0.98867052$$

$$\therefore d=0.0453$$

----

13. Solution: E

$$\int \frac{t^2}{100}dt=\frac{t^3}{300}$$

So accumulated value at time 3 of deposit of 100 at time 0 is:

$$100e^{\left.t^3/300\right]_0^3}=109.41743$$

The amount of interest earned from time 3 to time 6 equals the accumulated value at time 6 minus the accumulated value at time 3. Thus

$$(109.41743+X)e^{\left.t^3/300\right]_3^6}-(109.41743+X)=X$$

$$(109.41743+X)(1.8776106)-109.41743-X=X$$

96.025894 = 0.1223894 X

X = 784.59

----

14. Solution: A

167.50 = Present value = $10a_{\overline{5}|9.2}+10v_{9.2}^{5}\sum_{t=1}^{\infty}[\frac{(1+k)}{1.092}]^t$

= 38.70 + $10v_{9.2}^{5}\frac{1+k}{1.092}(\frac{1}{1-\frac{1+k}{1.092}})$ because the summation is an infinite geometric progression, which simplifies to (1/(1-common ratio)) as long as the absolute value of the common ratio is less than 1 (i.e. in this case common ratio is (1+k)/1.092 and so k must be less than .092)

So 167.50 = 38.70 + $\frac{(6.44)(1+k)}{0.092-k}$ or 128.80 = $\frac{(6.44)(1+k)}{0.092-k}$ or 20 = (1+k)/(0.092-k)

and thus 0.84 = 21 k or k = 0.04. Answer is 4.0.

15. Solution: B

$$\text{Option 1: } 2000 = Pa_{\overline{10}|0.0807}$$

$$P = 299 \Rightarrow \text{Total payments} = 2990$$

$$\text{Option 2: Interest needs to be } 990$$

$$990 = i\left[2000 + 1800 + 1600 + \cdots + 200\right]$$

$$= i\left[11{,}000\right]$$

$$i = 0.09$$

Tip:
For an arithmetic progression, the sum equals the average of the first and last terms times the number of terms. Thus in this case, 2000 + 1800 + 1600 + ….. + 200 = (1/2) (2000 + 200) 10 = 11000. Of course, with only 10 terms, it's fairly quick to just add them on the calculator!

----

16. Solution: B

The point of this question is to test whether a student can determine the outstanding balance of a loan when the payments are not level.

Monthly payment at time t = $1000(0.98)^{t-1}$

Since the actual amount of the loan is not given, the outstanding balance must be calculated prospectively,

OB_{40} = present value of payments at time 41 to time 60

= $1000(0.98)^{40}(1.0075)^{-1} + 1000(0.98)^{41}(1.0075)^{-2} + \ldots + 1000(0.98)^{59}(1.0075)^{-20}$

This is the sum of a finite geometric series, with

first term, a = $1000(0.98)^{40}(1.0075)^{-1}$

common ratio, r = $(0.98)(1.0075)^{-1}$

number of terms, n = 20

Thus, the sum

= $a(1 - r^n)/(1 - r)$

= $1000(0.98)^{40}(1.0075)^{-1}\,[1 - (0.98/1.0075)^{20}]/[1 - (0.98/1.0075)]$

= 6889.11

----

17. Solution: C

The payments can be separated into two “layers” of 98 and the equation of value at 3n is

$$98\, S_{\overline{3n}|} + 98\, S_{\overline{2n}|} = 8000$$

$$\frac{(1+i)^{3n}-1}{i} + \frac{(1+i)^{2n}-1}{i} = 81.63$$

$$(1+i)^n = 2$$

$$\frac{8-1}{i} + \frac{4-1}{i} = 81.63$$

$$\frac{10}{i} = 81.63$$

$$i = 12.25\%$$

----

18. Solution: B

Convert 9% convertible quarterly to an effective rate per month, the payment period. That is, solve for j such that $(1+j)^3 = (1+\frac{.09}{4})$ or j = .00744 or .744%

Then

$$2(Ia)_{\overline{60}|0.00744} = 2[\frac{\ddot{a}_{\overline{60}|0.00744} - 60v^{60}}{.00744}] = 2729.7$$

Alternatively, use result listed in solution to question 7 above with P = Q = 2, i = 0.00744 and n = 60.
Then (P + Q/i) = (2 + 2/.00744) = 270.8172043 and – n Q/i = - 16129.03226
Using BA II Plus calculator: select 2nd FV, enter 60 select N, enter .744 select I/Y, enter 270.8172043 select PMT, enter -16129.03226 select FV, CPT PV +/- yields 2729.68

----

19. Solution: C
Key formulas for estimating dollar-weighted rate of return:
Fund January 1 + deposits during year – withdrawals during year + interest = Fund December 31.
Estimate of dollar–weighted rate of return = amount of interest divided by the weighted average amount of fund exposed to earning interest
Then for Account K, dollar-weighted return:
Amount of interest I = 125 – 100 – 2x + x = 25 – x

i = $\dfrac{25-x}{100-x\left(\dfrac{1}{2}\right)+2x\left(\dfrac{1}{4}\right)}$ = (25 – x)/100; or $(1 + i)^K$ = (125 – x)/100

Key concepts for time-weighted rate of return:
Divide the time period into subintervals for each time there is a deposit or withdrawal
For each subinterval, calculate the ratio of the amount in the fund at the end of the subinterval (*before* the deposit or withdrawal at the end of the subinterval) to the amount in the fund at the beginning of the subinterval (*after* the deposit or withdrawal)
Multiply the ratios together to cover the desired time period
Then for Account L time-weighted return:
(1 + i) = 125/100 · 105.8/(125 – x) = 132.25/(125 – x)
But (1 + i) = (1 + i) for Account K. So 132.25/(125 – x) = (125 – x)/100 or $(125 – x)^2$ = 13,225
∴ x = 10 and i = (25 – x)/100 = 15%
----
20. Solution: A
Equate present values:
$100 + 200\, v^n + 300\, v^{2n} = 600\, v^{10}$
v^n=.76
100 + 152 + 173.28
= 425.28. Thus, v^{10} = 425.28/600 = 0.7088 ⇒ i = 3.5%
----
21. Solution: A
Use equation of value at end of 10 years:

$$(1+i)^{10-n} = e^{\int_n^{10}\frac{1}{8+t}dt} = e^{\ln(8+t)\big|_n^{10}} = \frac{18}{(8+n)}$$

$$\therefore 20,000 = \int_0^{10}(8k+t\cdot k)\cdot(1+i)^{10-t}\,dt = \int_0^{10}k\cdot(8+t)\cdot\frac{18}{8+t}dt$$

$$= 18k\cdot t\Big|_0^{10} = 180k \Rightarrow k = \frac{20,000}{180} = 111$$

22. Solution: D
Price for any bond is the present value at the yield rate of the coupons plus the present value at the yield rate of the redemption value. Given r = semi-annual coupon rate and i = the semi-annual yield rate. Let C = redemption value.
Then Price for bond X = P^X = 1000 r $a_{\overline{2n}|i}$ + C v^{2n} (using a semi-annual yield rate throughout)

= 1000 $\dfrac{r}{i}(1 - v^{2n})$ + 381.50 because $a_{\overline{2n}|i} = \dfrac{1-v^{2n}}{i}$ and the present value of the redemption value, C v^{2n}, is given as 381.50.

We are also given $\dfrac{r}{i}$ = 1.03125 so 1000$\dfrac{r}{i}$ = 1031.25. Thus, P^X = 1031.25 $(1 – v^{2n})$ + 381.50.
Now only need v^{2n}. Given v^n = 0.5889, $v^{2n} = (0.5889)^2$.
Thus P^X = 1031.25 $(1 – (0.5889)^2)$ + 381.50 = 1055.10

----
23. Solution: D
Equate net present values:

$$-4000+2000v+4000v^2 = 2000+4000v-xv^2$$

$$\left(\frac{4000+x}{1.21}\right)=6000+\frac{2000}{1.1}$$

$$x=5460$$

----
24. Solution: E
For the amortization method, payment P is determined by 20000 = X $a_{\overline{20}|0.065}$, which yields (using calculator) X = 1815.13.
For the sinking fund method, interest is .08 (2000) = 1600 and total payment is given as X, the same as for the amortization method. Thus the sinking fund deposit = X – 1600 = 1815.13 – 1600 = 215.13.
The sinking fund, at rate j, must accumulate to 20000 in 20 years. Thus, 215.13 $s_{\overline{20}|j}$ = 20000. which yields (using calculator) j = 14.18.

25. Solution: D

The present value of the perpetuity = X/i. Thus, the given information yields:

$$B = X\, a_{\overline{n}|} = 0.4 \cdot \frac{X}{i}$$

$$C = v^n\ X a_{\overline{n}|}$$

$$J = v^{2n}\ \frac{X}{i}$$

$$a_{\overline{n}|} = \frac{0.4}{i} \Rightarrow v^n = 0.6$$

$$J = 0.36 \frac{X}{i}$$

That is, Jeff's share is 36% of the perpetuity's present value.

----

26. Solution: D

The given information yields the following amounts of interest paid:

$$\text{Seth} = 5000\left(\left(1+\frac{0.12}{2}\right)^{10} - 1\right) = 8954.24 - 5000 = 3954.24$$

$$\text{Janice} = 5000\,(0.06)(10) = 3000.00$$

$$\text{Lori} = P(10) - 5000 = 1793.40 \text{ where } P = \frac{5000}{a_{\overline{10}|6\%}} = 679.35$$

The sum is 8747.64.

----

27. Solution: E

X = Bruce's interest is i times the accumulated value at the end of 10 years = i 100 $(1+i)^{10}$.
X = Robbie's interest is i times the accumulated value at the end of 16 years = i 50 $(1+i)^{16}$
Because both amounts equal X, taking the ratio yields: X/X = 2 v^6 or v^6 = 1/2.
Thus, $(1+i)^6$ = 2 and i = $2^{1/6}$ – 1 = .122462. So X = .122462 [100 $(1.122462)^{10}$]= 38.88.

----

28. Solution: D

Year (t + 1) principal repaid = v^{n-t}

Year t interest repaid = $i \cdot a_{\overline{n-t+1}|}$ = 1 – v^{n-t+1}

Total = 1 – v^{n-t+1} + v^{n-t} = 1 – v^{n-t} (v – 1) = 1 – v^{n-t} (–(1 – v)) = 1 + v^{n-t} (d)

----

29. Solution: B
32 is given as PV of perpetuity paying 10 at end of each 3-year period, with first payment at the end of 3 years. Thus, $32 = 10\,(v^3 + v^6 + ,,,,,,,\ ) = 10\,v^3\,(1/1- v^3)$ (infinite geometric progression), and $v^3 = 32/42$ or $(1+i)^3 = 42/32$. Thus, i = .094879785.

X is given as the PV, at the same interest rate, of a perpetuity paying 1 at the end of each 4 months, with the first payment at the end of 4 months. Thus, $X = 1\,(v^{1/3} + v^{2/3} + ,,,,,,,) = v^{1/3}\,(1/(1- v^{1/3})) = 32.6$

----
30. Solution: D
The present value of the liability at 5% is $822,702.48 ($1,000,000/ (1.05^4)).
The future value of the bond, including coupons reinvested at 5%, is $1,000,000.
If interest rates drop by ½%, the coupons will be reinvested at an interest rate 4.5%. Annual coupon payments = 822,703 x .05 = 41,135. Accumulated value at 12/31/2007 will be
41,135 + [41,135 x (1.045)] + [41,135 x (1.045^2)] + [41,135 x (1.045^3)] + 822,703 = $998,687. The amount of the liability payment at 12/31/2007 is $1,000,000, so the shortfall = 998,687 – 1,000,000 = -1,313 (loss)
If interest rates increase, the coupons could be reinvested at an interest rate of 5.5%, leading to an accumulation of more than the $1,000,000 needed to fund the liability. Accumulated value at 12/31/2007 will be 41,135 + [41,135 x (1.055)] + [41,135 x (1.055^2)] + [41,135 x (1.055^3)] + 822,703 = $ 1,001,323. The amount of the liability is $1,000,000, so the surplus or profit = 1,001,323 – 1,000,000 = +1,323 profit.
----
31. Solution: D.
Present value = $5000\,(1.07v + 1.07^2\,v^2 + 1.07^3\,v^3 + \ldots\ldots\ldots + 1.07^{19}\,v^{19} + 1.07^{20}\,v^{20})$

$= 5000\ 1.07\ v\ \left(\dfrac{1-(1.07v)^{20}}{1-(1.07v)}\right)$

simplifying to: $5{,}000\,(1.07)\,[\,1-(1.07/1.05)^{20}] / (.05 - .07) = 122{,}634$
----

32. Solution: C.
NPV = -100000 + $(1.05)^{-4}$$(60000(1.04)^{1} + 60000)$ = -100000 + $(1.05)^{-4}$(122400) = 698.72

Time		0	1	2	3	4
Cash Flow	Initial Investment	-100,000				
	Investment Returns				60,000	60,000
	Reinvestment Returns					60,000*.04 = 2400
	Total amount to be discounted	-100,000	0	0	0	60000+ 62400 =122400
Discount Factor		1				1/(1.05)^4 = .822702
	698.72	-100,000	0	0		100,698.72

----
33. Solution: B.
Using spot rates, the value of the bond is:
$60/(1.07) + 60/((1.08)^2) + 1060/((1.09)^3) = 926.03$
----
34. Solution: E.
Using spot rates, the value of the bond is:
$60/(1.07) + 60/((1.08)^2) + 1060/((1.09)^3) = 926.03$.

Thus, the annual effective yield rate, i, for the bond is such that 926.03 = $60a_{\overline{3|}} + 1000v^3$ at i. This can be easily calculated using one of the calculators allowed on the actuarial exam. For example, using the BA II PLUS the keystrokes are: 3 N, 926.03 PV, 60 +/- PMT, 1000 +/- FV, CPT I/Y = and the result is 8.9% (rounded to one decimal place).
----

35. Solution: C.

Duration is defined as $\dfrac{\sum_{t=1}^{n} tv^t R_t}{\sum_{t=1}^{n} v^t R_t}$, where v is calculated at 8% in this problem.

(Note: There is a minor but important error on page 228 of the second edition of Broverman's text. The reference "The quantity in brackets in Equation (4.11) is called the duration of the investment or cash flow" is not correct because of the minus sign in the brackets**.** There is an errata list for the second edition. Check http://www.actexmadriver.com/client/client_images/pdfs/Math_Inv_Credit_2ED.pdf if you do not have a copy).

The current price of the bond is $\sum_{t=1}^{n} v^t R_t$, the denominator of the duration expression, and is given as 100. The derivative of price with respect to the yield to maturity is $-\sum_{t=1}^{n} tv^{t+1} R_t$ = - v times the numerator of the duration expression. Thus, the numerator of the duration expression is - (1.08) times the derivative. But the derivative is given as -700. So the numerator of the duration expression is 756. Thus, the duration = 756/100 = 7.56.

----

36. Solution: C

Duration is defined as $\dfrac{\sum_{t=1}^{\infty} tv^t R_t}{\sum_{t=1}^{\infty} v^t R_t}$, where for this problem v is calculated at i = 10% and R_t is a constant D, the dividend amount. Thus, the duration = $\dfrac{\sum_{t=1}^{\infty} tv^t D}{\sum_{t=1}^{\infty} v^t D} = \dfrac{\sum_{t=1}^{\infty} tv^t}{\sum_{t=1}^{\infty} v^t}$.

Using the mathematics of infinite geometric progressions (or just remembering the present value for a 1 unit perpetuity immediate), the denominator = v (1/(1-v)) (first term times 1 divided by the quantity 1 minus the common ratio; converges as long as the absolute value of the common ratio, v in this case, is less than 1). This simplifies to 1/i because 1- v = d = i v.

The numerator may be remembered as the present value of an increasing perpetuity immediate beginning at 1 unit and increasing by I unit each payment period, which equals $\dfrac{1}{i}+\dfrac{1}{i^2}$ = $\dfrac{1+i}{i^2}$. So duration = S^{Num}/denominator =((1+i)/i^2)/(1/i) = (1+i)/i = 1.1/.1 = 11

----

37. Solution: B

Duration is defined as $\dfrac{\sum_{t=1}^{\infty} tv^t R_t}{\sum_{t=1}^{\infty} v^t R_t}$, where for this problem v is calculated at i = 5% and R_t is D, the initial dividend amount, times $(1.02)^{t-1}$. Thus, the duration = $\dfrac{\sum_{t=1}^{\infty} tv^t D(1.02)^{t-1}}{\sum_{t=1}^{\infty} v^t D(1.02)^{t-1}} = \dfrac{\sum_{t=1}^{\infty} tv^t (1.02)^{t-1}}{\sum_{t=1}^{\infty} v^t (1.02)^{t-1}}$.

Using the mathematics of infinite geometric progressions (or just remembering the present value for a 1 unit geometrically increasing perpetuity immediate), the denominator = $v\dfrac{1}{(1-v(1.02))}$**,** which simplifies to $\dfrac{1}{i-.02}$. It can be shown* that the numerator simplifies to $\dfrac{1+i}{(i-.02)^2}$**.** So duration = numerator/denominator

= $\dfrac{1+i}{(i-.02)^2} / \dfrac{1}{i-.02} = \dfrac{1+i}{i-.02}$.

Thus, for i = .05, duration = (1.05)/.03 = 35.

Alternative solution:

A shorter alternative solution uses the fact that the definition of duration can be can be shown to be equivalent to – (1+i) P'(i)/P(i) where P(i) = $\sum_{t=1}^{\infty} v^t R_t$. Thus, in this case P(i) = $D\sum_{t=1}^{\infty} v^t (1.02)^{t-1}$ = $D\dfrac{1}{i-.02}$ and P'(i) (the derivative of P(i) with respect to i) = $D(-\dfrac{1}{(i-.02)^2})$. Thus, the duration = $-(1+i)\dfrac{-D(\dfrac{1}{(i-.02)^2})}{D\dfrac{1}{i-.02}}$ = $\dfrac{1+i}{i-.02}$, yielding the same result as above.

----

*Note: The process for obtaining the value for the numerator using the mathematics of series simplification is:

Let S^{Num} denote the sum in the numerator.

Then $S^{Num} = 1\,v + 2\,(1.02)\,v^2 + 3\,(1.02)^2 v^3 + \ldots + n\,(1.02)^{n-1} v^n + \ldots$ and $(1.02)v\,S^{Num} = 1\,(1.02)v^2 + 2\,(1.02)^2 v^3 + \ldots + (n-1)\,(1.02)^{n-1}v^n + \ldots$

Thus, $(1-(1.02)v)\,S^{Num} = 1\,v + 1\,(1.02)v^2 + 1\,(1.02)^2 v^3 + \ldots + 1\,(1.02)^{n-1}v^n + \ldots = v\dfrac{1}{(1-v(1.02))} = \dfrac{1}{(i-.02)}$

and $S^{Num} = \dfrac{1}{(i-.02)}/(1-(1.02)v) = \dfrac{1}{i-.02}/\dfrac{1+i-1.02}{1+i} = \dfrac{1}{i-.02}/\dfrac{i-.02}{1+i} = \dfrac{1+i}{(i-.02)^2}$.

38. – 44. skipped
----
45. Solution: A
Key concepts for time-weighted rate of return:
Divide the time period into subintervals for each time there is a deposit or withdrawal
For each subinterval, calculate the ratio of the amount in the fund at the end of the subinterval (*before* the deposit or withdrawal at the end of the subinterval) to the amount in the fund at the beginning of the subinterval (*after* the deposit or withdrawal)
Multiply the ratios together to cover the desired time period
Thus, for this question, time-weighted return = 0% means: 1+0 = (12/10) (X/(12+X) or 120 + 10 X = 12 X and X = 60
Key formulas for estimating dollar-weighted rate of return:
Fund January 1 + deposits during year – withdrawals during year + interest = Fund December 31.
Estimate of dollar–weighted rate of return = amount of interest divided by the weighted average amount of fund exposed to earning interest
Thus, for this question, amount of interest I = X – X – 10 = - 10 and dollar-weighted rate of return is given by Y = [-10/(10 + ½ (60)] = - 10/40 = - .25 = -25%
----
46. Solution: A
Given the term of the loan is 4 years, and the outstanding balance at end of third year = 559.12, the amount of principal repaid in the 4^{th} payment is 559.12. But given level payments, the principal repaid forms a geometric progression and thus the principal repaid in the first year is v^3 times the principal repaid in the fourth year = v^3 559.12. Interest on the loan is 8%, thus principal repaid in first year is $(1/(1.08)^3)$*559.12 = 443.85
----
47. Solution: B
Price of bond = 1000 because the bond is a par value bond and the coupon rate equals the yield rate.
At the end of 10 years, the equation of value on Bill's investment is the price of the bond accumulated at 7% equals the accumulated value of the investment of the coupons plus the redemption value of 1000. However, the coupons are invested semiannually and interest i is an annual effective rate. So the equation of value is:

$1000\,(1.07)^{10} = 30\; s_{\overline{20}|j} + 1000$ where j is such that $(1+j)^2=1+i$

Rearranging, $30\; s_{\overline{20}|j} = 1000\,(1.07)^{10} - 1000 = 967.1513573$. Solving for j (e.g. using one of the approved calculators) yields j = 4.759657516%, and thus $i = (1+j)^2 - 1 = .097458584$
----
48. Solution: A
3,000/9.65 = is the number of thousands required to provide the desired monthly retirement benefit because each 1000 provides 9.65 of monthly benefit and the desired monthly retirement benefit is 3000. Thus, 310,881 is the capital required at age 65 to provide the desired monthly retirement benefit.
Using the BA II Plus calculator, select 2^{nd} BGN, select 2^{ND} SET until BGN appears on the screen (monthly contributions start today),select CE, enter 12*25 = 300 (the total number of monthly contributions) select N, enter 8/12 = 0.6667(8% compounded monthly) select I/Y, enter 310,881 select +/- select FV, select CPT PMT to obtain 324.73.
49. Solution: D
Using the daughter's age 18 as the comparison date and equating the value at age 18 of the contributions to the value at age 18 of the four 50,000 payments results in:

$$X[(1.05)^{17} + (1.05)^{16} + ...(1.05)^{1}] = 50{,}000[1 + ...v_{.05}^{3}]$$

----
50. Solution: D
The problem tests the ability to determine the purchase price of a bond between bond coupon dates.
Find the price of the bond on the previous coupon date of April 15, 2005. On that date, there are 31 coupons (of $30 each) left. So the price on April 15, 2005 is:

P = 1000 v^{31} + 30 $a_{\overline{31}|}$ all at j = 0.035 or P = 1000 + (30-35) $a_{\overline{31}|}$ at j = 0.035.

Thus P = \$906.32
Then Price (June 28) = 906.32[1+(74/183)(0.035)] = \$919.15
----
51. Solution: D
The following table summarizes what is required by the liabilities and what is provided by one unit of each of Bonds I and II.

	In 6 months	In one year
Liabilities require:	\$1,000	\$1,000
One unit of Bond I provides:	\$1,040	
One unit of Bond II provides:	\$ 25	\$1,025

Thus, to match the liability cash flow required in one year, (1/1.025) = .97561 units of Bond II are required. .97561 units of Bond II provide (.97561*25) = 24.39 in 6 months. Thus, (1000-24.39)/1040 = .93809 units of Bond I are required.
Note: Checking answer choices is another approach but takes longer!
----
52. Solution: B
Total cost = cost of .93809 units of Bond I + cost of .97561 units of Bond II =
$.93809*1040\ v_{.03} + .97561*(25\ v_{.035} + 1025\ v_{.035}^2) = 1904.27$
----

53. Solution: D
Investment contribution = 1904; investment returns = 1000 in 6 months and 1000 in one year. Thus, the effective yield rate per 6 months is that rate of interest j such that 1904 = 1000 v_j + 1000 v_j^2 = 1000 $a_{\overline{2}|j}$. Using

BA II Plus calculator keys: select 2^{nd} FV; enter 1904, select +/-, select PV; enter 1000, select PMT; enter 2, select N; select CPT, select I/Y yields 3.343 in % format. Thus, the annual effective rate = $(1.03343)^2$ – 1 = .0678.
<u>Note</u>: Even if 1904.27 is used as PV, the resulting annual effective interest rate is 6.8% when rounded to one decimal point.
----
54. Solution: C
Given the coupon rate is greater than the yield rate, the bond sells at a premium. Thus, the minimum yield rate for this callable bond is calculated based on a call at the <u>earliest</u> possible date because that is most disadvantageous to the bond holder (earliest time at which a loss occurs). Thus, X, the par value, which equals the redemption value because the bond is a par value bond, must satisfy:
Price = $1722.25 = .04Xa_{\overline{30}|.03} + Xv_{.03}^{30}$ or X = 1722.25/ $(.04a_{\overline{30}|.03} + v_{.03}^{30})$ = 1722.25/1.196 = 1440.01

----
55. Solution: A
Given the price is greater than the par value, which equals the redemption value in this case, the minimum yield rate for this callable bond is calculated based on a call at the <u>earliest</u> possible date because that is most disadvantageous to the bond holder (earliest time at which a loss occurs). Thus, the effective yield rate per coupon period, j, must satisfy:
Price = $1722.25 = 44a_{\overline{30}|j} + 1100v_j^{30}$ or, using calculator, j = 1.608%. Thus, the yield, expressed as a nominal annual rate of interest convertible semiannually, is 3.216%
----
56. Solution: E
Given the coupon rate is less than the yield rate, the bond sells at a discount. Thus, the minimum yield rate for this callable bond is calculated based on a call at the <u>latest</u> possible date because that is most disadvantageous to the bond holder (latest time at which a gain occurs). Thus, X, the par value, which equals the redemption value because the bond is a par value bond, must satisfy:
Price = $1021.50 = .02Xa_{\overline{20}|.03} + Xv_{.03}^{20}$ or X = 1021.50/ $(.02a_{\overline{20}|.03} + v_{.03}^{20})$ = 1021.50/.8512 = 1200.07

----
57. Solution: B
Given the price is less than the par value, which equals the redemption value in this case, the minimum yield rate for this callable bond is calculated based on a call at the <u>latest</u> possible date because that is most disadvantageous to the bond holder (latest time at which a gain occurs). Thus, the effective yield rate per coupon period, j, must satisfy:
Price = $1021.50 = 22a_{\overline{20}|j} + 1100v_j^{20}$ or, using calculator, j = 2.45587%. Thus, the yield, expressed as a nominal annual rate of interest convertible semiannually, is 4.912%

58. Solution: E

The transaction costs are 2 (1 for the forward and 1 for the stock)

The price of the forward is therefore: (50 + 2) * (1.06) = 55.12

----

59. Solution: C

First, the *PV* of the liability is:

$$PV = 35{,}000 a_{\overline{15}|6.2\%} = 335{,}530.30$$

The duration of the liability is:

$$\bar{d} = \frac{\sum t v^t R_t}{\sum v^t R_t} = \frac{35{,}000v + 2*35{,}000v^2 + \ldots + 15*35{,}000v^{15}}{335{,}530.30} = \frac{2{,}312{,}521.95}{335{,}530.30} = 6.89214$$

Let X denote the amount invested in the 5 year bond.

Then, $\frac{X}{335{,}530.30}(5) + (1 - \frac{X}{335{,}530.30})10 = 6.89214 \Rightarrow X = 208{,}556$

----

60. Solution: A

The present value of the first eight payments is:

$$PV = 2000v + 2000(1.03)v^2 + \ldots + 2000(1.03)^7 v^8 = \frac{2000v - 2000*1.03^8 * v^9}{1 - 1.03v} = 13{,}136.41$$

The present value of the last eight payments is:

$$PV = 2000(1.03)^7 0.97 v^9 + 2000(1.03)^7 0.97^2 v^{10} + \ldots + 2000(1.03)^7 0.97^8 v^{16} =$$
$$= \frac{2000(1.03)^7 0.97 v^9 - 2000*1.03^7 * 0.97^9 v^{17}}{1 - 0.97v} = 7{,}552.22$$

Therefore, the total loan amount is *L* = 20,688.63

61. Solution: E

Since the 2-year forward price is higher than the 1-year forward price, the buyer, relative to the forward prices, overall pays more at the end of the first year but less at the end of the second year. So this means that the buyer pays the swap counterparty at the end of the first year but receives money back from the swap counterparty at the end of the second year. So the buyer lends to the swap counterparty at the 1-year effective forward interest rate, from the end of the first year to the end of the second year, namely 6%.