May 2007 Exam MFE Solutions

1. Answer = (B)

Let D = the quarterly dividend.
Using formula (9.2), put-call parity adjusted for deterministic dividends, we have

$$4.50 = 2.45 + \left[52.00 - D \times e^{-0.01} - D \times e^{-0.025}\right] - \left[50 \times e^{-0.03}\right]$$

$$= 54.45 - D \times (0.99005 + 0.97531) - 50 \times 0.970446.$$

Rearranging the equation yields

$$D \times 1.96356 = 54.45 - 4.50 - 48.5223 = 1.4277,$$

or

$$D = 0.73.$$

2. Answer = (A)

Let p be the true probability of the stock going up. Thus,

$$puS + (1-p)dS = e^{\alpha h} S \qquad \text{(which is equation (11.3) on p. 347),}$$

yielding

$$p = \frac{e^{\alpha h} - d}{u - d}.$$

Because $\alpha = 0.1$, $h = 1$, $u = 1.433$, and $d = 0.756$, we have $p = 0.52$.

3. Answer = (C)

Let P denote the price of the European put option. Then,

$$P = 98e^{-0.055 \times ½} N(-d_2) - 100e^{-0.01 \times ½} N(-d_1)$$

by formula (12.3) with $S = 100$, $K = 98$, $\delta = 1\%$, $\sigma = 50\%$, $r = 5.5\%$, and $T = ½$.

Here, d_1 is calculated using formula (12.2a) and is equal to 0.29755819 ≈ 0.30; d_2 is from formula (12.2b) and is equal to –0.0559952 ≈ –0.06. From the normal cdf table, $N(0.06) = 0.5239$ and $N(-0.30) = 1 - 0.6179 = 0.3821$.

Thus, $P \approx 98e^{-0.055/2} \times 0.5239 - 100e^{-0.01/2} \times 0.3821 = 11.93 \approx 11.90$.

4. Answer = (E)

For a special put option with strike price K, the payoff upon immediate exercise is

$$K - 50.$$

This value should be compared with P, the price of the corresponding one-period European put option. The value of P can be determined using put-call parity:

$$P = Ke^{-r} - Se^{-\delta} + C.$$

With $S = 50, r = 4\%,$ and $\delta = 8\%,$

$$P = K \times e^{-0.04} - 50 \times e^{-0.08} + C = 0.9608K - 46.1558 + C.$$

K	C	P	$K-50$
40	9.12	1.3962	−10
50	4.91	6.7942	0
60	0.71	12.2022	10
70	0.00	21.1002	20

From the table above, we see that it is not optimal to exercise any of these special put options immediately.

5. Answer = (D)

By (12.9),

$$\sigma_{\text{option}} = \sigma_{\text{stock}} \times |\Omega| = 0.50 \times |\Omega|,$$

where Ω is the *option elasticity*.
By (12.8),

$$\Omega = S\Delta/C,$$

where Δ is the *option delta*,

$$\Delta = e^{-\delta T} N(d_1) \qquad \text{(see page 383).}$$

By (12.1),

$$C = Se^{-\delta T} N(d_1) - Ke^{-rT} N(d_2)$$
$$= S\Delta - Ke^{-rT} N(d_2).$$

Thus,

$$\Omega = S\Delta/C$$
$$= S\Delta/[S\Delta - Ke^{-rT} N(d_2)]$$
$$= 1/[1 - Ke^{-rT} N(d_2)/(S\Delta)]$$
$$= 1/\{1 - [Ke^{-rT} N(d_2)]/[Se^{-\delta T} N(d_1)]\}.$$

We are given $S = 85, \quad K = 80, \quad \delta = 0, \quad r = 5.5\%, \quad T = 1.$
By equation (12.2a), d_1 is 0.4812 ≈ 0.48; hence, $N(d_1) \approx 0.6844$.
By equation (12.2b), d_2 is –0.0188 ≈ –0.02; hence, $N(d_2) \approx 0.4920$.
With these values, we obtain

$$Se^{-\delta T} N(d_1) \approx 85 \times e^{0} \times 0.6844 = 58.174,$$
$$Ke^{-rT} N(d_2) \approx 80 \times e^{-0.055} \times 0.4920 = 37.2537.$$

Hence,

$$\Omega = 1/\{1 - [Ke^{-rT}N(d_2)]/[Se^{-\delta T}N(d_1)]\} \approx 2.78,$$

and

$$\sigma_{\text{option}} = \sigma_{\text{stock}} \times |\Omega| \approx 0.50 \times 2.78 = 1.39\,.$$

Remark added in April 2009: See also page 687 of McDonald (2006).

6. Answer = (C)

Because of the identity

$$\text{Maximum}(S_1(3), S_2(3)) = \text{Maximum}(S_1(3) - S_2(3), 0) + S_2(3),$$

the payoff of the claim can be decomposed as the sum of the payoff of the exchange option in statement (v) of the problem and the price of stock 2 at time 3. In a no-arbitrage model, the price of the claim must be equal to the sum of the exchange option price (which is 10) and the prepaid forward price for delivery of stock 2 at time 3 (which is $e^{-\delta_2 \times 3} \times S_2(0)$). So, the answer is

$$10 + e^{-0.1\times3}\times200 = 158.16.$$

Remark: If one buys $e^{-\delta_2 \times 3}$ share of stock 2 at time 0 and re-invests all dividends, one will have exactly one share of stock 2 at time 3.

7. Answer = (E)

By formula (24.32), the call option price is

$$C = P(0, T)[F\,N(d_1) - K\,N(d_2)],$$

where

$T = 1$,

$P(0, T) = P(0, 1) = 0.9434$,

$F = F_{0,1}[P(1, 2)] = P(0, 2)/P(0, 1) = 0.8817/0.9434 = 0.934598$,

$K = 0.9259$.

With $\sigma = 0.05$, we have

$$d_1 = \frac{\ln(F/K) + \frac{1}{2}\sigma^2 T}{\sigma\sqrt{T}} = 0.212011 \approx 0.21,$$

$$d_2 = d_1 - \sigma\sqrt{T} \approx 0.21 - 0.05 = 0.16.$$

Thus,

$N(\text{d}_1) \approx N(0.21) = 0.5832$,

$N(\text{d}_2) \approx N(0.16) = 0.5636$.

Hence,

$$C = P(0, T)[F\,N(d_1) - K\,N(d_2)]$$
$$= 0.9434[0.9346\times0.5832 - 0.9259\times0.5636]$$

$= 0.9434 \times 0.02322 \approx 0.022$

Remarks: (1) The footnote on page 791 points out that the call option price formula can also be expressed as

$$C = P(0, 2)N(d_1) - KP(0, 1)N(d_2).$$

(2) The symbol F in the Black formula (24.32) denotes a *forward* price, but the same symbol in the Black formula (12.7) denotes a *futures* price. There is no contradiction because, in the Black model discussed on page 381, the interest rate is constant. It is stated on page 146 that "if the interest rate were not random, then forward and futures price would be the same."

(3) Consider a forward contract, with delivery date T, for an underlying asset whose price at time T is denoted by $S(T)$. For $t < T$, the time-t prepaid forward price is

$$F_{t,T}^{P}[S(T)] = E_t^*[e^{-R(t,T)} S(T)]$$

by risk-neutral pricing. Here, we use the notation in the last paragraph of page 783; E_t^* means the conditional expectation with respect to the risk-neutral probability measure given the information up to time t, and

$$R(t, T) = \int_t^T r(u)du . \qquad (24.21)$$

Thus, the time-t forward price is

$$F_{t,T}[S(T)] = \frac{1}{P(t,T)} F_{t,T}^{P}[S(T)] = \frac{1}{P(t,T)} E_t^*[e^{-R(t,T)} S(T)].$$

Noting (24.20), we can rewrite this formula as

$$F_{t,T}[S(T)] = \frac{E_t^*[e^{-R(t,T)} S(T)]}{E_t^*[e^{-R(t,T)}]}.$$

If the short-rate, $r(u)$, is not stochastic, then the right-hand side is

$$\frac{E_t^*[e^{-R(t,T)} S(T)]}{E_t^*[e^{-R(t,T)}]} = \frac{e^{-R(t,T)} E_t^*[S(T)]}{e^{-R(t,T)} E_t^*[1]} = E_t^*[S(T)],$$

which is the formula for the time-t futures price of the underlying asset deliverable at time T.

(4) Consider the special case $S(T) = P(T, T + s)$. Then the time-t prepaid forward price of the zero-coupon bond deliverable at time T is

$$\begin{aligned} F_{t,T}^{P}[P(T, T+s)] &= E_t^*[\, e^{-R(t,T)} P(T, T+s)] \\ &= E_t^*[\, e^{-R(t,T)}\; E_T^*[e^{-R(T,T+s)}]] \\ &= E_t^*[\, e^{-R(t,T)} e^{-R(T,T+s)}] \\ &= E_t^*[\, e^{-R(t,T) - R(T,T+s)}] \\ &= E_t^*[e^{-R(t,T+s)}] \\ &= P(t, T+s), \end{aligned}$$

where the third equality is by the law of iterated expectations. Thus, the time-t forward price is

$$F_{t,T}[P(T, T+s)] = \frac{1}{P(t,T)} F_{t,T}^{P}[P(T, T+s)] = \frac{1}{P(t,T)} P(t, T+s),$$

which is equation (24.31).

8. Answer = (C)

By formulas (12.1) and (12.2a, b), with $\delta = 0$, the call option price is

$$S(0)N(\tfrac{1}{2}\sigma\sqrt{T}) - \left[S(0)e^{rT}\right]e^{-rT}N(-\tfrac{1}{2}\sigma\sqrt{T})$$
$$= S(0)\left[N(\tfrac{1}{2}\sigma\sqrt{T}) - N(-\tfrac{1}{2}\sigma\sqrt{T})\right]$$
$$= S(0)\left[2N(\tfrac{1}{2}\sigma\sqrt{T}) - 1\right]$$

where the last equality is due to the identity $N(-x) = 1 - N(x)$.

By (20.12), the random variable $\ln\left[S(t)\right]$ is normally distributed with variance $\sigma^2 t$. Thus, statement (iii) means that $\sigma^2 = 0.4$, and

$$\tfrac{1}{2}\sigma\sqrt{T} = \tfrac{1}{2}\sqrt{0.4\times 10} = \tfrac{1}{2}\times 2 = 1.$$

Therefore, the option price is

$$S(0)\left[2N(1) - 1\right] = 100\left[2\times 0.8413 - 1\right] = 68.26.$$

9. Answer = (A)

This problem is a modification of the example on page 805. Note that the example is about cap payments on a four-year loan, not a three-year loan.

An interest rate cap pays the difference between the realized interest rate in a period and the cap rate, if the difference is positive. Observe that in this problem only r_u and r_{uu} are higher than 7.5%.

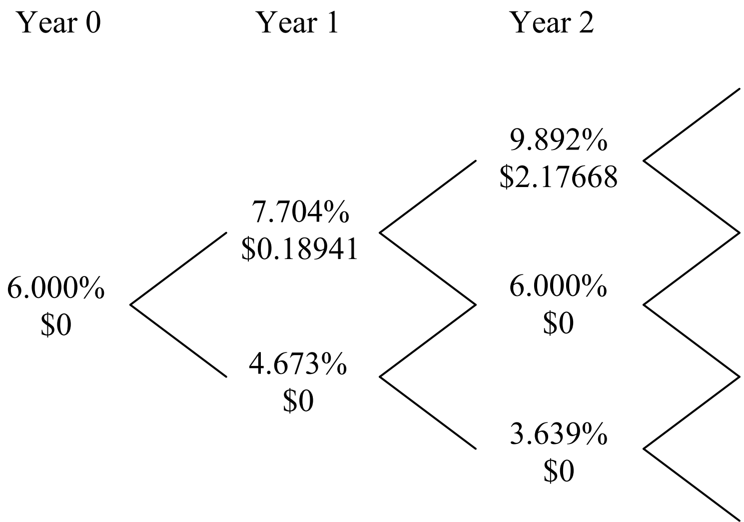
At the "*u*" node, it is expected that a payment of 100×(7.704% − 7.5%) will be made at the end of the year. Thus, the present value of the payment at the node is

$$\frac{100\times(7.704\% - 7.5\%)}{1 + 7.704\%} = 0.18941.$$

At the "*uu*" node, it is expected that a payment of 100×(9.892% − 7.5%) will be made at the end of the year. Thus, the present value of the payment at the node is

$$\frac{100\times(9.892\% - 7.5\%)}{1 + 9.892\%} = 2.17668.$$

The tree below corresponds to Figure 24.5 and Figure 24.9 of McDonald (2006).



By risk-neutral pricing, the time-0 price of the interest rate cap is

$$\frac{1/2}{1+6\%}\times 0.18941 \;+\; \frac{1/2}{1+6\%}\times\frac{1/2}{1+7.704\%}\times 2.17668$$
$$= 0.08934 + 0.47649 = 0.565827 \approx 0.57$$

Remark: The cap payments are not \$0.18941 and \$2.17668. They are \$100×(7.704% – 7.5%) to be paid one year after the *u* node, and \$100×(9.892% – 7.5%) to be paid one year after the *uu* node. One may be tempted to put \$100×(7.704% – 7.5%) at the *uu* node and at the *ud* node, and put \$100×(9.892% – 7.5%) at the *uuu* node and at the *uud* node. Unfortunately, this can be confusing, because these cash flows are not *path-independent*. For example, if one reaches the *ud* node via the *d* node, then there is no cap payment because r_d is less than 7.5%.

10. Answer = (B)

Let y = number of units of the stock you will buy,
z = number of units of the Call-II option you will buy.
If x or y turns out to be negative, this means that you sell.

Delta-neutrality means
$$1000 \times 0.5825 \;=\; y \times 1 \;+\; z \times 0.7773.$$
Gamma-neutrality means
$$1000 \times 0.0651 \;=\; y \times 0 \;+\; z \times 0.0746.$$

From the second equation (the gamma-neutral equation), we obtain

$$z = 65.1/0.0746 = 872.654 \approx 872.7.$$

(This is sufficient to determine that (B) is the correct answer.) Substituting this in the first equation (the delta-neutral equation) yields

$$y = 582.5 - 872.7 \times 0.7773 = -95.8.$$

11. Answer = (D)

With $u = 1.181$, $d = 0.890$, $h = 0.5$, and $\delta = 0$, the risk-neutral probability that the stock price will increase at the end of a period is

$$p^* = \frac{e^{(r-\delta)h} - d}{u - d} = \frac{e^{0.05\times 0.5} - 0.890}{1.181 - 0.890} = 0.465. \qquad (10.5)$$

For the two-period model, the stock prices are

$S_0 = 70$

$S_u = uS_0 = 1.181 \times 70 = 82.67$ $\qquad$ $S_d = dS_0 = 0.890 \times 70 = 62.30$

$S_{uu} = uS_u = 1.181 \times 82.67 = 97.63$ $\qquad$ $S_{ud} = dS_u = 0.890 \times 82.67 = 73.58$

$S_{dd} = dS_d = 0.890 \times 62.30 = 55.45$

Let P_0, P_u, P_d, P_{uu}, P_{ud}, P_{dd} denote the corresponding prices for the American put option. The three prices at the option expiry date are

$$P_{uu} = \max(K - S_{uu}, 0) = \max(80 - 97.63, 0) = 0,$$
$$P_{ud} = \max(K - S_{ud}, 0) = \max(80 - 73.58, 0) = 6.42,$$
$$P_{dd} = \max(K - S_{dd}, 0) = \max(80 - 55.45, 0) = 24.55.$$

By the backward induction formula (10.12), the two prices at time 1 are

$$\begin{aligned} P_u &= \max(K - S_u, e^{-rh}[P_{uu}p^* + P_{ud}(1 - p^*)]) \\ &= \max(80 - 82.67, e^{-0.05/2}[0\times 0.465 + 6.42\times(1 - 0.465)]) \\ &= e^{-0.05/2}\times 6.42\times 0.535 \\ &= 3.35, \end{aligned}$$

$$\begin{aligned} P_d &= \max(K - S_d, e^{-rh}[P_{ud}p^* + P_{dd}(1 - p^*)]) \\ &= \max(80 - 62.30, e^{-0.05/2}[6.42\times 0.465 + 24.55\times(1 - 0.465)]) \\ &= \max(17.70, 15.72) \\ &= 17.70. \end{aligned}$$

Finally, the time-0 price of the American put option is

$$\begin{aligned} P_0 &= \max(K - S_0, e^{-rh}[P_u p^* + P_d(1 - p^*)]) \\ &= \max(80 - 70, e^{-0.05/2}[3.35\times 0.465 + 17.70\times(1 - 0.465)]) \\ &= \max(10, 10.75) \\ &= 10.75. \end{aligned}$$

12. Answer = (A)

Define the function $f(x, t) = xe^{(r-r^*)(T-t)}$. Then,

$$G(t) = f(S(t), t).$$

Obviously, $\frac{\partial}{\partial x} f(x,t) = e^{(r-r^*)(T-t)}$, $\frac{\partial^2}{\partial x^2} f(x,t) = 0$, and $\frac{\partial}{\partial t} f(x,t) = f(x, t)(r-r^*)(-1)$.

By Itô's Lemma, we have

$$\begin{aligned}
\mathrm{d}G(t) &= e^{(r-r^*)(T-t)}\mathrm{d}S(t) + 0 + f(S(t), t)(r^* - r)\mathrm{d}t \\
&= e^{(r-r^*)(T-t)}S(t)[0.1\mathrm{d}t + 0.4\mathrm{d}Z(t)] + G(t)(r^* - r)\mathrm{d}t \\
&= G(t)[0.1\mathrm{d}t + 0.4\mathrm{d}Z(t)] + G(t)(0.10 - 0.08)\mathrm{d}t \\
&= G(t)[(0.1 + 0.02)\mathrm{d}t + 0.4\mathrm{d}Z(t)] \\
&= G(t)[0.12\mathrm{d}t + 0.4\mathrm{d}Z(t)].
\end{aligned}$$

13. Answer = (E)

In a Vasicek model, zero-coupon bond prices are of the form

$$P(t,T,r) = A(t,T)e^{-B(t,T)r}. \qquad (24.26)$$

Furthermore, the functions $A(t,T)$ and $B(t,T)$ are functions of $T-t$. Therefore, we can rewrite formula (24.26) as

$$P(t,T,r) = \exp\left(-\left[\alpha(T-t) + \beta(T-t)r\right]\right).$$

The first two pieces of data tell us:

$$0.9445 = e^{-\alpha(2)-\beta(2)\times 0.04}$$
$$0.9321 = e^{-\alpha(2)-\beta(2)\times 0.05}$$

which, by taking logarithms, are equivalent to

$$0.0571 = \alpha(2) + \beta(2)\times 0.04$$
$$0.0703 = \alpha(2) + \beta(2)\times 0.05$$

The solution of this pair of linear equations is

$$\beta(2) = 1.32$$
$$\alpha(2) = 0.0043$$

The last piece of data says

$$0.8960 = e^{-\alpha(2)-\beta(2)r^*}$$

Taking logarithms yields $0.1098 = \alpha(2) + \beta(2)r^*$,
or

$$r^* = (0.1098 - 0.0043)/1.32 = 0.08.$$

Remark: By comparing (24.29) with (24.26), we see that the word "Vasicek" in this problem can be changed to "CIR."

14. Answer = (E)

This is a one-period binomial model. Let p^* be the risk-neutral probability of an increase in the stock price. (See page 321.) Then,

$$p^* = \frac{60\times e^r - 45}{70-45} = \frac{60\times e^{0.08} - 45}{70-45} = 0.79988896 \approx 0.8.$$

By risk-neutral pricing, the price of the straddle is

$$\begin{aligned} e^{-r}[p^*|70-K| + (1-p^*)|45-K|] &= e^{-0.08}[p^*|70-50| + (1-p^*)|45-50|] \\ &= e^{-0.08}[p^*\times 20 + (1-p^*)\times 5] \\ &= e^{-0.08}[15p^* + 5] \\ &\approx e^{-0.08}[15\times 0.8 + 5] \\ &= e^{-0.08}\times 17 = 15.693 \\ &\approx 15.70. \end{aligned}$$

15. Answer = (C)

This is a variation of Example 12.3 on page 380. Because of the discrete dividend, we are to use the version of the Black-Scholes put option formula that is in terms of prepaid forward prices. The prepaid forward price of the stock is

$$F^P_{0,1/2}(S) = 50 - 1.50e^{-0.05/3} = 50 - 1.5\times 0.983471 = 48.5248.$$

We apply formula (12.2a), with $S = 48.5248$, $K = 50$, $r = 0.05$, $\delta = 0$, $\sigma = 0.3$, and $T = ½$, to obtain

$$\begin{aligned} d_1 &= \{\ln(48.5248/50) + [0.05 - 0 + (0.3)^2/2]\times ½\}/\{0.3\times\sqrt{½}\} \\ &= \{-0.02995 + 0.0475\}/0.212132 \\ &= 0.082740 \\ &\approx 0.08. \end{aligned}$$

(This is the same as applying the formula for d_1 that follows (12.5) on page 380.) Then,

$$d_2 = 0.082740 - 0.212132 = -0.129392 \approx -0.13.$$

It now follows from the prepaid forward price version of (12.3) that the put option price is

$$\begin{aligned} &50e^{-0.05/2}N(+0.13) - 48.5248N(-0.08) \\ &= (50\times 0.975351\times 0.5517) - (48.5248\times 0.4681) \\ &= 26.9039 - 22.7145 \\ &= 4.1894 \\ &\approx 4.19. \end{aligned}$$

Remark added in April 2009: The following sentence can be found on page 381 in new printings of McDonald (2006). "Because the dividend is fixed, the volatility in Example 12.3 is the volatility of the prepaid forward, rather than the volatility of the stock."

16. Answer = (D)

Define $\beta = \frac{r-\delta}{\sigma^2} - \frac{1}{2}$. Then, the formulas on page 403 for h_1 and h_2 are

$$h_1 = -\beta + \sqrt{\beta^2 + \frac{2r}{\sigma^2}}$$

and

$$h_2 = -\beta - \sqrt{\beta^2 + \frac{2r}{\sigma^2}}.$$

Adding these two equations yields

$$h_1 + h_2 = -2\beta.$$

Hence, $-2\beta = 7/9$ or $\beta = -7/18$.

For $r = 5\%$ and $\sigma = 0.3$,

$$h_1 = -\beta + \sqrt{\beta^2 + \frac{2r}{\sigma^2}} = \frac{7}{18} + \sqrt{\left(\frac{-7}{18}\right)^2 + \frac{2\times 0.05}{0.3^2}} \approx 1.51.$$

Alternative solution: The parameters h_1 and h_2 are the positive and negative roots, respectively, of the quadratic equation

$$\frac{\sigma^2}{2}h^2 + (r - \delta - \frac{\sigma^2}{2})h - r = 0; \qquad (*)$$

see the study note "Some Remarks on *Derivatives Markets*." Thus,

$$\frac{\sigma^2}{2}h^2 + (r - \delta - \frac{\sigma^2}{2})h - r = \frac{\sigma^2}{2}(h - h_1)(h - h_2).$$

Consequently,

$$r - \delta - \frac{\sigma^2}{2} = \frac{\sigma^2}{2}(-h_1 - h_2) = \frac{\sigma^2}{2} \times -\frac{7}{9}.$$

Hence, the quadratic equation (*) becomes

$$\frac{0.3^2}{2}h^2 + (\frac{0.3^2}{2} \times -\frac{7}{9})h - 0.05 = 0,$$

the positive root of which is h_1.

Remark: For a positive δ, the positive root h_1 is in fact greater than 1.

Remark added in April 2009: Question 16 is not in the current syllabus of MFE/3F.

17. Answer = (B)

In terms of the notation in Section 14.15, $K_1 = 90$ and $K_2 = 100$.

By (12.1), and (12.2a, b), statement (ii) of the problem is

$$4 = S(0)e^{-\delta T}N(d_1) - K_2 e^{-rT}N(d_2)\,, \qquad (1)$$

where $S(0) = 80$,

$$d_1 = \frac{\ln(S(0)/K_2) + (r - \delta + \frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}}\,,$$

and

$$d_2 = d_1 - \sigma\sqrt{T} = \frac{\ln(S(0)/K_2) + (r - \delta - \frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}}\,.$$

Do note that both d_1 and d_2 depend on K_2, but not on K_1.

From the last paragraph on page 383 and from statement (iii), we have

$$\Delta = e^{-\delta T}N(d_1) = 0.2,$$

and hence equation (1) becomes

$$4 = 80 \times 0.2 - 100e^{-rT}N(d_2)\,,$$

or

$$e^{-rT}N(d_2) = (80 \times 0.2 - 4)/100 = 0.12\,.$$

By (14.15) on page 458, the gap call option price is

$$S(0)e^{-\delta T}N(d_1) - K_1 e^{-rT}N(d_2)$$
$$= 80 \times 0.2 - 90 \times 0.12 = 5.2.$$

Remark: The payoff of the gap call option is

$$[S(T) - K_1] \times I(S(T) > K_2),$$

where $I(S(T) > K_2)$ is the *indicator random variable*, which takes the value 1 if $S(T) > K_2$ and the value 0 otherwise. Because the payoff can be expressed as

$$S(T) \times I(S(T) > K_2) - K_1 \times I(S(T) > K_2),$$

we can obtain the pricing formula (14.15) by showing that the time-0 price for the time-T payoff

$$S(T) \times I(S(T) > K_2)$$

is

$$S(0)e^{-\delta T}N(d_1)\,,$$

and the time-0 price for the time-T payoff

$$I(S(T) > K_2)$$

is

$$e^{-rT}N(d_2).$$

Note that both d_1 and d_2 are calculated using the strike price K_2. We can use risk-neutral pricing to verify these two results:

$$E^*[e^{-rT} S(T)\times I(S(T) > K_2)] = S(0)e^{-\delta T}N(d_1),$$

which is the pricing formula for a European *asset-or-nothing* (or *digital share*) call option, and

$$E^*[e^{-rT} I(S(T) > K_2)] = e^{-rT} N(d_2),$$

which is the pricing formula for a European *cash-or- nothing* (or *digital cash*) call option. Here, we follow the notation on pages 604 and 605 that the asterisk is used to signify that the expectation is taken with respect to the risk-neutral probability measure. Under the risk-neutral probability measure, the random variable $\ln[S(T)/S(0)]$ is normally distributed with mean $(r - \delta - \frac{1}{2}\sigma^2)T$ and variance $\sigma^2 T$.

The second expectation formula, which can be readily simplified as

$$E^*[I(S(T) > K_2)] = N(d_2),$$

is particularly easy to verify: Because an indicator random variable takes the values 1 and 0 only, we have

$$E^*[I(S(T) > K_2)] = \text{Prob}^*[S(T) > K_2],$$

which is the same as

$$\text{Prob}^*(\ln[S(T)/S(0)] > \ln[K_2/S(0)]).$$

To evaluate this probability, we use a standard method, which is also described on pages 590 and 591. We subtract the mean of $\ln[S(T)/S(0)]$ from both sides of the inequality and then divide by the standard deviation of $\ln[S(T)/S(0)]$. The left-hand side of the inequality is now a standard normal random variable, Z, and the right-hand side is

$$\frac{\ln[K_2/S(0)] - (r-\delta-\sigma^2/2)T}{\sqrt{\sigma^2 T}} = -\frac{\ln[S(0)/K_2] + (r-\delta-\sigma^2/2)T}{\sigma\sqrt{T}}$$

$$= -d_2.$$

Thus, we have

$$\begin{aligned} E^*[I(S(T) > K_2)] &= \text{Prob}^*[S(T) > K_2], \\ &= \text{Prob}(Z > -d_2) \\ &= 1 - N(-d_2) \\ &= N(d_2). \end{aligned}$$

The first expectation formula,

$$E^*[e^{-rT} S(T)\times I(S(T) > K_2)] = S(0)e^{-\delta T}N(d_1),$$

is harder to derive. One method is to use formula (18.29). A more elegant way is the actuarial method of *Esscher transforms*, which is not part of the syllabus of any actuarial examination. It shows that the expectation of a product,

$$E^*[e^{-rT} S(T)\times I(S(T) > K_2)],$$

can be factorized as a product of expectations,

$$E^*[e^{-rT} S(T)] \times E^{**}[I(S(T) > K_2)],$$

where ** signifies a changed probability measure. It follows from (20.26) and (20.14) that

$$E^*[e^{-rT} S(T)] = e^{-\delta T} S(0).$$

To evaluate the expectation $E^{**}[I(S(T) > K_2)]$, which is $\text{Prob}^{**}[S(T) > K_2]$, one shows that, under the probability measure **, the random variable $\ln[S(T)/S(0)]$ is normally distributed with mean

$$(r - \delta - \tfrac{1}{2}\sigma^2)T + \sigma^2 T = (r - \delta + \tfrac{1}{2}\sigma^2)T,$$

and variance $\sigma^2 T$. Then, with steps identical to those above, we have

$$\begin{aligned} E^{**}[I(S(T) > K_2)] &= \text{Prob}^{**}[S(T) > K_2], \\ &= \text{Prob}(Z > -d_1) \\ &= 1 - N(-d_1) \\ &= N(d_1). \end{aligned}$$

Alternative solution:

Because the payoff of the gap call option is

$$\begin{aligned} &[S(T) - K_1]\times I(S(T) > K_2) \\ &= [S(T) - K_2]\times I(S(T) > K_2) + (K_2 - K_1)\times I(S(T) > K_2), \end{aligned}$$

the price of the gap call option must be equal to the sum of the price of a European call option with the strike price K_2 and the price of $(K_2 - K_1)$ units of the corresponding cash-or-nothing call option. Thus, with $K_1 = 90$, $K_2 = 100$, and statement (ii), the price of the gap call option is

$$\begin{aligned} &4 + (100 - 90)\times e^{-rT}\text{Prob}^*[S(T) > 100] \\ &= 4 + 10e^{-rT}N(d_2). \end{aligned}$$

On the other hand, from (ii), (iii), and (12.1), it follows that

$$4 = 80(0.2) - 100e^{-rT}N(d_2).$$

Thus, $e^{-rT}N(d_2) = 0.12$, and the price of the gap call option is

$$4 + 10\times 0.12 = 5.2.$$

18. Answer = (A)

In an arbitrage-free model, two assets having the same source of randomness (their prices driven by the same Brownian motion) must have the same Sharpe ratio (which is not necessarily constant with respect to time); see Section 20.4. With $r = 4\%$, we thus have

$$\frac{0.07 - 0.04}{0.12} = \frac{G - 0.04}{H},$$

or

$$G = 0.25H + 0.04. \qquad (1)$$

If $f(x)$ is a twice-differentiable function of x, then Itô's Lemma (page 664) simplifies as

$$df(Y(t)) = f'(Y(t))dY(t) + \tfrac{1}{2}f''(Y(t))[dY(t)]^2,$$

because $\frac{\partial}{\partial t}f(x) = 0$. If $f(x) = \ln x$, then $f'(x) = 1/x$ and $f''(x) = -1/x^2$. Hence,

$$d(\ln[Y(t)]) = \frac{1}{Y(t)}dY(t) + \frac{1}{2}\left(-\frac{1}{[Y(t)]^2}\right)[dY(t)]^2. \qquad (2)$$

We are given that

$$dY(t) = Y(t)[Gdt + HdZ(t)]. \qquad (3)$$

Thus,

$$[dY(t)]^2 = \{Y(t)[Gdt + HdZ(t)]\}^2 = [Y(t)]^2 H^2 dt, \qquad (4)$$

by applying the multiplication rules (20.17) on pages 658 and 659. Substituting (3) and (4) in (2) and simplifying yields

$$d(\ln[Y(t)]) = (G - \tfrac{1}{2}H^2)dt + HdZ(t).$$

Comparing this equation with the one in (i), we have

$$H = \sigma, \qquad (5)$$

$$G - \tfrac{1}{2}H^2 = 0.06. \qquad (6)$$

Applying (1) and (5) to (6) yields a quadratic equation of σ,

$$\tfrac{1}{2}\sigma^2 - 0.25\sigma + 0.02 = 0,$$

whose roots can be found by using the quadratic formula or by factorizing,

$$\tfrac{1}{2}(\sigma - 0.1)(\sigma - 0.4) = 0.$$

By condition (iii), we cannot have $\sigma = 0.4$. Thus, $\sigma = 0.1$. Substituting $H = 0.1$ in (1) yields

$$G = 0.25\times0.1 + 0.04 = 0.065.$$

Remark: Exercise 20.1 on page 675 is to use Itô's Lemma to evaluate $d[\ln(S)]$.

19. Answer = (D)

The delta-gamma approximation is merely the Taylor series approximation with up to the quadratic term. In terms of the Greek symbols, the first derivative is Δ, and the second derivative is Γ. The approximation formula is

$$P(S + \varepsilon) \approx P(S) + \varepsilon\Delta + \tfrac{1}{2}\varepsilon^2\Gamma. \qquad (13.2 \;\&\; 13.5)$$

With $P(30) = 4$, $\Delta = -0.28$, $\Gamma = 0.10$, and $\varepsilon = 1.50$, we have

$$\begin{aligned} P(31.5) &\approx 4 + (1.5)(-0.28) + \tfrac{1}{2}(1.5)^2(0.1) \\ &= 3.6925 \\ &\approx 3.70. \end{aligned}$$