

# QFI Core Model Solutions

## Fall 2014

### 1. Learning Objectives:

1. The candidate will understand the fundamentals of mathematics and economics underlying quantitative methods in finance and investments.

### Learning Outcomes:

- (1d) Understand Ito integral and stochastic differential equations.
- (1e) Understand and apply Ito's Lemma.
- (1g) Demonstrate an understanding of the mathematical considerations for analyzing financial time series.

### Sources:

Neftci, Salih An Introduction to the Mathematics of Financial Derivatives, 2nd Edition. Chapters 10-15

Wilmott, Paul Paul Wilmott Introduces Quantitative Finance, Chapter 6

Mild vs. Wild Randomness: Focusing on Those Risks That Matter QFIC-104-13

### Commentary on Question:

*Commentary listed underneath each part.*

### Solution:

- (a) Verify that  $X_t$  satisfies the integral equation:

$$X_t = x_0 e^{-\alpha t} + e^{-\alpha t} \int_0^t e^{\alpha s} dW_s$$

based on the function  $F_t = X_t e^{\alpha t}$  and using Ito's lemma.

### Commentary on Question:

*Good candidates will be able to apply Ito's Lemma correctly to verify the integral equation.*

## 1. Continued

Applying Ito's lemma based on the function  $F_t = X_t e^{\alpha t}$  with intermediate terms

$$\frac{\partial F_t}{\partial X_t} = e^{\alpha t}; \quad \frac{\partial F_t}{\partial t} = \alpha X_t e^{\alpha t}; \quad \frac{\partial^2 F_t}{\partial X_t^2} = 0; \quad dX_t^2 = dt$$

Plug in the intermediate terms

$$dF_t = \alpha X_t e^{\alpha t} dt + e^{\alpha t} dX_t$$

$$dF_t = \alpha X_t e^{\alpha t} dt + e^{\alpha t} (-\alpha X_t dt + dW_t)$$

$$dF_t = e^{\alpha t} dW_t$$

Integrate both sides from 0 to  $t$ , we have

$$X_t e^{\alpha t} - x_0 = \int_0^t e^{\alpha s} dW_s$$

and re-arranging, we get

$$X_t = x_0 e^{-\alpha t} + e^{-\alpha t} \int_0^t e^{\alpha s} dW_s$$

(b) Derive, using part (b), closed-form expressions for:

(i) Expected value,  $E(X_t)$

(ii) Variance,  $Var(X_t)$

### Commentary on Question:

*Most candidates used the integral equation from part (a) and were able to get the expected value correctly. Most candidates received at least partial credit for the variance. Good candidates will have recognized the properties of a standard Wiener process and Ito integral.*

From (a), taking the expectations of both sides and bearing in mind that  $x_0$  is a constant, we find

$$E(X_t) = x_0 e^{-\alpha t} + e^{-\alpha t} E\left(\int_0^t e^{\alpha s} dW_s\right)$$

Since the second term vanishes because of property of standard Wiener process, we have

$$E(X_t) = x_0 e^{-\alpha t}$$

## 1. Continued

For the variance, we have

$$\text{Var}(X_t) = E[X_t - E(X_t)]^2 = E[X_t - x_0 e^{-\alpha t}]^2$$

$$\text{Var}(X_t) = e^{-2\alpha t} E \left[ \int_0^t e^{\alpha s} dW_s \right]^2$$

$$\text{Var}(X_t) = e^{-2\alpha t} E \left[ \int_0^t e^{2\alpha s} (dW_s)^2 \right]$$

$$\text{Var}(X_t) = e^{-2\alpha t} \left[ \int_0^t e^{2\alpha s} ds \right]$$

$$\text{Var}(X_t) = e^{-2\alpha t} \frac{1}{2\alpha} (e^{2\alpha t} - 1) = \frac{1}{2\alpha} (1 - e^{-2\alpha t})$$

- (c) Explain why the process  $\{X_t\}$  is a zero mean reverting process and interpret the parameter  $\alpha$ .

**Commentary on Question:**

*Most candidates did well on this part.*

A zero mean reverting process is one with a zero mean in the long run. This means that  $E(X_t) = 0$  as  $t \rightarrow \infty$ , which holds true for the given process.

The speed at which it reaches zero is controlled by the parameter  $\alpha$ . The larger this value, the faster the process will return to zero.

- (d) Compare the key characteristics of Gaussian (non-scalable) and fractal (scalable) models, and their effectiveness in assessing financial strategies.

**Commentary on Question:**

*Candidates are expected to list key characteristics of both the Gaussian (non-scalable) and fractal (scalable) models. Several candidates mentioned the same characteristic multiple times using different phrasing, but in such a case only one point is awarded.*

Gaussian:

- Easy to handle (closed form formulae)
- Unsuitable where large deviations and stressful events dominate
- Tries to squeeze risk into a single number
- Models may be incompatible with fractal discontinuities
- Suitable for modeling physical quantities
- The most typical member is mediocre
- Total is not determined by a single observation
- Easy to predict from past experience

## 1. Continued

Fractal:

- No precise recipes
- Robust for decision making
- Standard deviation may be infinite, loses its significance
- Fractal memory provides a way of modeling clustering of large events
- Suitable for modeling wealth
- May account for jumps that reflect historical experience
- Total determined by small number of large events
- Hard to predict from past experience

## 2. Learning Objectives:

1. The candidate will understand the fundamentals of mathematics and economics underlying quantitative methods in finance and investments.
2. The candidate will understand how to apply the fundamental theory underlying the standard models for pricing financial derivatives. The candidate will understand the implications for option pricing when markets do not satisfy the common assumptions used in option pricing theory such as market completeness, bounded variation, perfect liquidity, etc.

### Learning Outcomes:

- (1a) Understand and apply concepts of probability and statistics important in mathematical finance.
- (2d) Define and apply the concepts of martingale, market price of risk and measures in single and multiple state variable contexts.
- (2e) Understand and apply Girsanov's theorem in changing measures.

### Sources:

Nefci, Salih: An Introduction to the Mathematics of Financial Derivatives Ch. 5, 14, 15

### Commentary on Question:

*In general, candidates performed well in this question for most parts, except in part (d) where many candidates appeared to lack understanding of the relevant concept.*

### Solution:

- (a) Explain why  $P$  and  $Q$  are equivalent measures.

#### Commentary on Question:

*In order to receive full credits, a candidate needs to mention both the positive probability and the same sample space. Many candidates missed either one or the other.*

$P$  and  $Q$  are equivalent because they operate on the same sample space and an event has positive probability under  $P$  if and only if it has positive probability under  $Q$  (or, equivalently, "An event has zero probability under  $P$  if and only if it has zero probability under  $Q$ ").

- (b) Determine under which measure  $P$  or  $Q$  the process is a martingale

## 2. Continued

Starting at  $t=2$

Top branch

Under  $P$ , the expected value =  $160 * .5 + 90 * .5 = 125$

Under  $Q$ , the expected value =  $160 * .4 + 90 * .6 = 118$

The value of the tree at  $t = 1$  for that branch is 118, so  $Q$  gives that value not  $P$

Lower branch

Under  $P$ , the expected value =  $70 * .5 + 40 * .5 = 55$

Under  $Q$ , the expected value =  $70 * .6 + 40 * .4 = 58$

The value of the tree at  $t = 1$  for that branch is 58, again  $Q$  measure gives that value not  $P$

Initial branch

Under  $P$ , the value =  $118 * .5 + 58 * .5 = 88$

Under  $Q$ , the value =  $118 * .3 + 58 * .7 = 76$

Again,  $Q$  measure gives that value not  $P$  at  $t = 0$

Hence  $Q$  is a martingale measure since  $E[X_j/F_i] = X_i$  for all  $i < j$ , but this does not apply under  $P$

- (c) Calculate the one-period standard deviation from time  $t = 0$  to  $t = 1$  under measure  $P$ .

$$\mu = E[(S_1 - S_0)|F_0] = (118 - 76) * .5 + (58 - 76) * .5 = 12$$

$$E[(S_1 - S_0)^2|F_0] - \mu^2 = (118 - 76)^2 * .5 + (58 - 76)^2 * .5 - 12^2 = 900$$

$$\text{Standard deviation} = \sqrt{900} = 30$$

- (d) Calculate the value of the Radon-Nikodym derivative of  $Q$  with respect to  $P$  for time 1.

The Radon-Nikodym derivative is defined as a random variable on the sample space. In particular, we have, for the finite sample space under consideration:

$$\frac{dQ}{dP}(\omega) = \frac{Q(\omega)}{P(\omega)}$$

for all elementary events  $\omega$  (in the model under consideration, elementary events are paths through the tree).

For time  $t=1$  up

$$dQ/dP = 0.3/.5 = 3/5$$

For time  $t = 1$  down

$$dQ/dP = 0.7/.5 = 7/5$$

### **3. Learning Objectives:**

1. The candidate will understand the fundamentals of mathematics and economics underlying quantitative methods in finance and investments.
2. The candidate will understand how to apply the fundamental theory underlying the standard models for pricing financial derivatives. The candidate will understand the implications for option pricing when markets do not satisfy the common assumptions used in option pricing theory such as market completeness, bounded variation, perfect liquidity, etc.
3. The candidate will understand how to evaluate situations associated with derivatives and hedging activities.

### **Learning Outcomes:**

- (1g) Demonstrate an understanding of the mathematical considerations for analyzing financial time series.
- (1h) Understand and apply various techniques for analyzing conditional heteroscedastic models including ARCH and GARCH.
- (2a) Demonstrate understanding of option pricing techniques and theory for equity and interest rate derivatives.
- (2f) Understand and apply Black Scholes Merton PDE (partial differential equation).
- (2g) Identify limitations of the Black-Scholes pricing formula.
- (2h) Describe and explain some approaches for relaxing the assumptions used in the Black-Scholes formula.
- (3a) Compare and contrast various kinds of volatility (e.g., actual, realized, implied, forward, etc.).

### **Sources:**

Nefci, Salih: An Introduction to the Mathematics of Financial Derivatives Ch. 5, 7, 8, 13

Wilmott, Paul: Paul Wilmott Introduce Quantitative Finance Ch. 6, 9

Tsay, Ruey: Analysis of Financial Time Series, Ch. 3

### **Commentary on Question:**

*Commentary listed underneath each question component.*

### 3. Continued

**Solution:**

(a) Define and explain the following concepts associated with implied volatility.

- (i) Volatility smile
- (ii) Volatility surface

**Commentary on Question:**

*Most candidates were able to define both concepts. However, the volatility smile was not as well explained as the volatility surface.*

- *Being able to explain the relationship of the implied volatilities at different strike prices can earn a full credit for item a(i).*
- *For item a(ii), it was acceptable to say that the volatility surface had 3 dimensions: strike price, implied volatility, and time to maturity*

*Implied volatility*

In actual markets, options are determined by supply and demand. Implied volatility is the volatility that is implied by option prices observed in the market by iteratively solving for the volatility that equals the Black-Scholes value to the market value of the options

*Volatility smile*

It is a description of implied volatilities at different strike prices  
It decreases as strike price increases. Implied distribution has a heavier left tail and a thinner right tail than a lognormal distribution.

*Volatility surface*

It has 2 dimensions: strike price and time to maturity.

(b) Compare the advantages and disadvantages of using stochastic volatility vs. constant volatility in the Black-Scholes model.

**Commentary on Question:**

*In order to receive full credit, a candidate had to explicitly compare AND contrast the use of stochastic volatility vs constant volatility in the Black-Scholes model. Most candidates had difficulty doing this.*

*Advantages using stochastic volatility:*

Exhibit volatility skew and better fit the data than the constant volatility Black Scholes model



### 3. Continued

*Disadvantages using stochastic volatility:*

It is possible that one can find a continuous surface such that the partial derivatives do indeed satisfy the PDE. But it may still be impossible to represent this surface in terms of easy and convenient formula as in the case of the simple Black-Scholes formula.

The volatility factor does not need to be constant to find solutions, it must only be time dependent, volatility can be asset-price dependent but the modified Black-Scholes will have very messy explicit solutions

It is difficult to decide what model to use. What is the volatility of volatility?

When only stock price is random, we have only one source of randomness. Now when volatility is stochastic, we have 2 sources of randomness (stock and volatility) but we still have one traded quantity to hedge with, the stock. We can't easily hedge with volatility to remove volatility risk because that isn't widely traded.

- (c) Calculate the change in  $\sigma_2^2$  when  $\sigma_0^2$  is decreased from 8% to 7%.

**Commentary on Question:** *A good number of candidates did well on this question. Other candidates who calculated the % change instead of arithmetic change were given full credit as long as they demonstrated clear understanding of the topic. For those who did not do so well, it was mainly because they did not follow the GARCH model explicitly provided in the question.*

$$\sigma_0^2 = 0.08$$

$$\sigma_1^2 = 0.003 + 0.1 (.15)^2 + 0.7 * 0.08 = .06125$$

$$\sigma_2^2 = 0.003 + 0.1 (.15)^2 + 0.7 * 0.06125 = .048125$$

if  $\sigma_0^2$  drops by 1%

$$\text{revised } \sigma_1^2 = 0.003 + 0.1 (.15)^2 + 0.7 * 0.07 = .05425$$

$$\text{revised } \sigma_2^2 = 0.003 + 0.1 (.15)^2 + 0.7 * .05425 = 0.043225$$

$$\text{change in } \sigma_2^2 = .043225 - 0.048125 = -0.0049$$

#### 4. Learning Objectives:

1. The candidate will understand the fundamentals of mathematics and economics underlying quantitative methods in finance and investments.

#### Learning Outcomes:

- (1a) Understand and apply concepts of probability and statistics important in mathematical finance.
- (1d) Understand Ito integral and stochastic differential equations.
- (1e) Understand and apply Ito's Lemma.
- (1f) Understand and apply Jensen's Inequality.

#### Sources:

Neftci, Salih An Introduction to the mathematics of Financial Derivatives, 2nd Edition. Chapters 8, 10

#### Commentary on Question:

*Commentary listed underneath each question component.*

#### Solution:

- (a) Calculate  $E(\ln(S_t)/t)$  using Ito's lemma.

#### Commentary on Question:

*Most candidates found this question difficult although it was a simple direct application of Ito's Lemma. Those candidates who were able to demonstrate the correct use of Ito's lemma, and chose the right function to solve, received partial credits.*

Use  $f(S, t) = \ln(S)$

Since

$$\frac{\partial f}{\partial S} = \frac{1}{S}$$

$$\frac{\partial^2 f}{\partial S^2} = -\frac{1}{S^2}$$

$$\frac{\partial f}{\partial t} = 0,$$

from Ito's Lemma we have

#### 4. Continued

$$df = \frac{1}{S} (0.05 S dt + 0.15 S dW) + \frac{(0.15 S)^2}{2} \left(-\frac{1}{S^2}\right) dt.$$

This is,

$$\begin{aligned} d(\ln(S)) &= \left(0.05 - \frac{0.15^2}{2}\right) dt + 0.15 dW \\ &= 0.03875 dt + 0.15 dW. \end{aligned}$$

Integrate both side, we have

$$\ln(S_t) = \ln(S_0) + 0.03875t + 0.15 \int_0^t W_t.$$

Since  $E(\int_0^t W_t) = 0$  by definition and  $\ln(S_0) = 0$ ,

$$E(\ln(S_t)) = 0.03875t.$$

Therefore,

$$E(\ln(S_t)/t) = 0.03875.$$

- (b) Explain why  $\ln(E(S_t))/t$  is greater than the answer from (a).

**Commentary on Question:**

*Some candidates were confused: While they showed the right relationship, they also contradicted their response by stating that it was a convex function, instead of a concave function. However, in general, candidates did fairly well on this question.*

Either show S is lognormal and  $\ln(E(S_t)) = \mu t > \left(\mu - \frac{\sigma^2}{2}\right)t$ .

Or use Jensen's inequality to show ln is concave function therefore  $\ln(E) > E(\ln)$

#### 4. Continued

- (c) Calculate the 95% confidence interval of  $X_5$ .

**Commentary on Question:**

*Most candidates did well on this question. Those who did not receive full credit, either got the mean or variance or the interval wrong.*

For the process  $X$  (which is a generalized Brownian motion):

$$X_t = 1 + 0.15t + 0.25W_t$$

$$X_t \sim N(1 + 0.15t, 0.25\sqrt{t})$$

$$X_5 \sim N(1.75, 0.559)$$

Leading to the confidence interval:

$$(\mu - \sigma \cdot \Phi^{-1}(0.975), \mu + \sigma \cdot \Phi^{-1}(0.975)) = (0.654, 2.846)$$

- (d) Calculate  $S_5$ , given  $X_5 = 3$ .

**Commentary on Question:**

*In general, candidates were able to start with the right solution. However, some did not proceed to complete the solution.*

$$X_5 = X_0 + 0.15T + 0.25 \int_0^5 dW_t = 3$$

$$\text{Solve } \int_0^5 dW_t = 5 \text{ given } X_0 = 1 \text{ and } T = 5.$$

$$S_5 = S_0 \times e^{\left(\mu - \frac{\sigma^2}{2}\right) \times t + \sigma \times \int_0^5 dW_t}$$

$$S_5 = 1 \times e^{\left(0.05 - \frac{0.15^2}{2}\right) \times 5 + 0.15 \times \int_0^5 dW_t} = 2.57$$

## 5. Learning Objectives:

1. The candidate will understand the fundamentals of mathematics and economics underlying quantitative methods in finance and investments.
2. The candidate will understand how to apply the fundamental theory underlying the standard models for pricing financial derivatives. The candidate will understand the implications for option pricing when markets do not satisfy the common assumptions used in option pricing theory such as market completeness, bounded variation, perfect liquidity, etc.

### Learning Outcomes:

- (1a) Understand and apply concepts of probability and statistics important in mathematical finance.
- (1b) Understand the importance of the no-arbitrage condition in asset pricing.
- (1c) Apply the concept of martingale in asset pricing.
- (1d) Understand Ito integral and stochastic differential equations.
- (1e) Understand and apply Ito's Lemma.
- (2a) Demonstrate understanding of option pricing techniques and theory for equity and interest rate derivatives.
- (2d) Define and apply the concepts of martingale, market price of risk and measures in single and multiple state variable contexts.
- (2e) Understand and apply Girsanov's theorem in changing measures.

### Sources:

Neftci Ch. 6, 10, 14, 15, 17, Wilmott Frequently Asked Questions pg. 113-115

Wilmott – Introduces Quantitative Finance Ch. 18

### Commentary on Question:

*This question tests the understanding and application of the martingale pricing method, Ito's lemma for geometric Brownian motion, Girsanov's theorem on changing measures.*

### Solution:

- (a) Derive the SDE of  $\ln(G(t))$ .

### Commentary on Question:

*Candidates rarely succeeded in getting the correct answer by applying Ito's Lemma.*

## 5. Continued

$$\begin{aligned}\frac{dP(t, m)}{P(t, m)} &= r(t)dt + \sigma(t, m)dX_t \\ d(\log[P(t, T)]) &= (r(t) - \frac{\sigma^2(t, T)}{2})dt + \sigma(t, T)dX_t \\ d(\log[P(t, U)]) &= (r(t) - \frac{\sigma^2(t, U)}{2})dt + \sigma(t, U)dX_t \\ d(\log(G(t))) &= -\frac{1}{2}(\sigma^2(t, U) - \sigma^2(t, T))dt + (\sigma(t, U) - \sigma(t, T))dX_t\end{aligned}$$

- (b) Explain how you can use Girsanov theorem to claim that there exists a martingale measure under which the drift of  $G(t)$  vanishes.

### Commentary on Question:

*In addition, most candidates got full credits for writing down Novikov condition (77) of p.329 of Neftci Ch 14.*

*(See equation (92) on p. 360 of Neftci Ch 15.5.1)*

*(See equations (93) & (94) on p. 360 of Neftci Ch 15.5.1)*

*(See equation (97) on p. 361 of Neftci Ch. 15.5.1.1)*

*(p. 363 of Neftci Ch 15.5.2)*

*(See equation (86) on p. 331 of Neftci Ch 14.4)*

Given that  $P(t, U)$  and  $P(t, T)$  are both tradeable bonds, according to the Girsanov theorem, we can find an  $\mathcal{I}_t$ -adapted process  $Y_t$  and a new Wiener process  $\tilde{X}_t$  such that  $d\tilde{X}_t = dX_t + dY_t$ .

The probability measure associated with  $\tilde{X}_t$  is given by

$$\begin{aligned}dH &= \zeta d\tilde{H}_t, \text{ where} \\ \zeta_t &= e^{\int_0^t Y_u dX_u - \frac{1}{2} \int_0^t Y_u^2 du}\end{aligned}$$

If we define the SDE in Part (a) under the new probability  $\tilde{H}$  with the new Wiener process  $\tilde{X}_t$ , then the discounted asset price (i.e.,  $G(t)$ ) becomes a martingale and  $\tilde{H}$  will be a martingale measure if we equate the drift term to zero i.e., the drift of

$G(t) = \frac{P(t, U)}{P(t, T)}$  is zero. In addition, the probability measure  $\tilde{H}$  is called an equivalent martingale measure. This operation gives the drift adjustment term  $\gamma$ , required by the Girsanov theorem, such that

$$d\tilde{X}_t = dX_t - \gamma dt.$$

## 5. Continued

(c)

- (i) Derive the SDE of  $G(t)$  under the martingale measure in (b).
- (ii) Determine the drift-shift needed to create the martingale measure in (b).

**Commentary on Question:**

*Many candidates couldn't provide the martingale representation of  $G$ , which follows geometric Brownian motion and got only partial credits. Some candidates got right answer for c(ii) by directly making drift zero.*

- (i) By Girsanov Theorem we can find a drift-shift for  $dX_t$  such that the drift of  $G$  vanishes, that is, for some  $\gamma$  and  $\xi$  :

$$d\tilde{X}_t = dX_t - \gamma dt \quad (1)$$

$$dG(t) = \xi G(t) d\tilde{X}_t \quad (2)$$

Applying Ito's Lemma and then substituting (1) we find that

$$d \ln(G(t)) = -\frac{\xi^2}{2} dt + \xi d\tilde{X}_t = -(\gamma\xi + \frac{\xi^2}{2})dt + \xi dX_t \quad (3)$$

Comparing (3) with the SDE derived in Part (a) for the coefficient of  $dX_t$  we have

$$\xi = \sigma(t, U) - \sigma(t, T). \quad (4)$$

Therefore,

$$dG(t) = (\sigma(t, U) - \sigma(t, T))G(t)d\tilde{X}_t \quad (5)$$

- (ii) Comparing (3) with the SDE derived in Part (a) for the coefficient of  $dt$  we have

$$\gamma\xi + \frac{\xi^2}{2} = \frac{1}{2}(\sigma^2(t, U) - \sigma^2(t, T))$$

Using (4) we solve that

$$\gamma = \sigma(t, T).$$

(d)

- (i) Express  $F$  in terms of  $P(0, T)$  and  $P(0, U)$ .
- (ii) Express  $V$  as an integral involving  $\sigma(t, T)$  and  $\sigma(t, U)$ .

## 5. Continued

### Commentary on Question:

Some candidates got full credits by simply explaining that  $G$  is a martingale for  $d(i)$ . (p. 240-241, Neftci Ch. 10.4)

- (i) By using Ito's Lemma on (5), we get

$$d(\ln(G(t))) = -\frac{1}{2}\xi^2 dt + \xi dX_t^T,$$

where  $\xi = \sigma(t, U) - \sigma(t, T)$  as in (4).

Thus

$$\ln(G(T)) - \ln(G(0)) \sim \text{Normal}\left(-\frac{1}{2}\int_0^T \xi^2 ds, \int_0^T \xi^2 ds\right). \quad (6)$$

Therefore

$$F = E[G(T)] = G(0)e^{-\frac{1}{2}\int_0^T \xi^2 ds + \frac{1}{2}\int_0^T \xi^2 ds} = G(0) = \frac{P(0, U)}{P(0, T)}$$

Alternatively:  $E[G(T)] = G(0)$  from the property of a martingale measure

- (ii) From (6), the variance is

$$V = \int_0^T \xi^2 ds = \int_0^T (\sigma(s, U) - \sigma(s, T))^2 ds.$$



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### Learning Outcomes:

- (1a) Understand and apply concepts of probability and statistics important in mathematical finance.
- (1b) Understand the importance of the no-arbitrage condition in asset pricing.
- (1c) Apply the concept of martingale in asset pricing.
- (1d) Understand Ito integral and stochastic differential equations.
- (2a) Demonstrate understanding of option pricing techniques and theory for equity and interest rate derivatives.
- (2b) Apply the basic concepts of currency markets (purchase price parity, law of one price, etc.).

### Sources:

Neftci, Salih An Introduction to the Mathematics of Financial Derivatives, 2<sup>nd</sup> Edition. Chapters 2, 3, 6, 8

### Commentary on Question:

*Candidates found this question to be a very challenging question. Most candidates were unable to move beyond part(a)-(i). It appears that candidates may have misinterpreted the volatility in the table*

### Solution:

- (a) Let  $Z$  represent the number of annual up-movements of asset  $S$  for  $t \leq 6$ :
  - (i) Derive the price of  $S$  at  $t = 6$  as a function of  $Z$ .
  - (ii) Calculate the expected payout of  $C$ .

## 6. Continued

- (i)  $S_6 = 1.2^Z * 0.95^{(6-Z)}$
- (ii) Let  $X_j, j = 1, \dots, 6$  be indicator variables taking on the value 1 if there is an up move at time  $j$ , and the value 0 if there is a down move at time  $j$   
let  $p_j = E[X_j]$  be the probability of an up move at time  $j$

The payoff of the option is  $\ln(S_6) \cdot X_1$ .

The expected payoff is:

$$\begin{aligned} E[\ln(S_6)X_1] &= E[\ln(S_6)X_1 | X_1 = 1]p_1 \\ &= p_1(E[6 \cdot \ln(0.95) + (\ln(1.2) - \ln(0.95))Z | X_1 = 1]) \\ &= p_1(6 \cdot \ln(0.95) + (\ln(1.2) - \ln(0.95))\sum_{j=1}^6 E[X_j | X_1 = 1]) \\ &= p_1(6 \cdot \ln(0.95) + (\ln(1.2) - \ln(0.95))(1 + \sum_{j=2}^6 p_j)) \\ &\approx 0.4079 \end{aligned}$$

(b)

- (i) Derive a formula for  $E(A_3)$  in terms of  $i_{euro,2}$  and/or  $i_{euro,3}$ .
- (ii) Calculate  $i_{euro,2}$  and  $i_{euro,3}$  such that the process  $A_t$  is a martingale.

- (i) Assuming dollar return in year  $t$ :  $A_t = S_t/S_{t-1} * X_t/X_{t-1} - 1$   
Since  $S$  and  $X$  are independent,  
Expectation of dollar return  $E(A_t) = E(S_t/S_{t-1}) * E(X_t/X_{t-1}) - 1$

$$\text{At } t=3: E(S_t/S_{t-1}) = 0.6 * 1.2 + 0.4 * .95 = 1.1$$

$$E(X_t/X_{t-1}) = \exp(i_{us,t} - i_{euro,t}) = \exp(3\% - i_{euro,3})$$

$$E(A_t) = E(S_t/S_{t-1}) * E(X_t/X_{t-1}) - 1 = 1.1 * \exp(3\% - i_{euro,3}) - 1$$

- (ii) Since  $A_t$  follows martingale, that means  $E(A_{t+n}|I_t) = E(A_t)$

$$E(A_1) = (0.8 * 1.2 + 0.2 * .95) * \exp(3\% - 4\%) - 1 = 0.13856$$

$$E(A_2) = (0.7 * 1.2 + 0.3 * .95) * \exp(3\% - i_{euro,2}) - 1 = 1.125 * \exp(3\% - i_{euro,2}) - 1$$

$$E(A_3) = (0.6 * 1.2 + 0.4 * .95) * \exp(3\% - i_{euro,3}) - 1 = 1.1 * \exp(3\% - i_{euro,3}) - 1$$

## 6. Continued

Since  $A_1$ ,  $A_2$  and  $A_3$  are independent, that means  $E(A_1) = E(A_2) = E(A_3)$   
Solve for  $i_{\text{euro},2}$  and  $i_{\text{euro},3}$  using  $E(A_1) = E(A_2) = E(A_3)$

$$3\% - i_{\text{euro},2} = 0.01198, \quad \Rightarrow i_{\text{euro},2} = 0.01802 \text{ or } 1.802\%$$

$$3\% - i_{\text{euro},3} = 0.03445, \quad \Rightarrow i_{\text{euro},3} = -0.00445 \text{ or } -0.445\%$$

## 7. Learning Objectives:

7. The candidate will understand the theory and techniques of portfolio asset allocation.

### Learning Outcomes:

- (7b) Propose and critique asset allocation strategies.
- (7c) Evaluate the significance of liabilities in the allocation of assets.
- (7d) Incorporate risk management principles in investment policy and strategy, including asset allocation.

### Sources:

Managing Investment Portfolios: A Dynamic Process, Maginn & Tuttle, 3<sup>rd</sup> Edition (Ch. 7, Equity Portfolio Management) Pages : 414-416; 420-422; 425-427; 433-435; 440-441

### Commentary on Question:

*The question was not difficult, but parts a) and b) were not very well answered. This contributed to a reduction of the mean score on the whole question.*

### Solution:

- (a) Determine, given the guidelines, which of the following indices is the most appropriate to serve as the equity benchmark to evaluate

#### Commentary on Question:

*Surprisingly, the qualification of each index as to its concentration risk and as to its representation of a global market was not well answered. For example, many candidates said that the Dow-Jones was a global market index and with no concentration risk, while it is the opposite. Few have selected Value Line as being the right answer.*

Dow-Jones index is price weighted and has only 30 stocks in it. It is not a broad equity market; so it cannot be chosen as the benchmark.

S&P Midcap 400 is a market capitalization weighted index. It is subject to concentration risk because it could be concentrated on few large and mature stocks, or overvalued stocks whose values have already risen the most. This is by the mandate to be avoided.

Value Line is the only one that can fit this mandate, because it is a broad equity market index, including about 1,700 stocks, and is equal weighted, which means no concentration risk to a few stocks.

## 7. Continued

- (b) Recommend the most appropriate method. Justify your answer

**Commentary on Question:**

*Here again and surprisingly, the candidates have in great numbers selected the optimization over the replication method, reducing the average score. The choice candidates made in part (a) might have affected their thinking for part (b).*

Full replication is the most appropriate method.

When the index (S&P 500 index) contains highly liquid stocks, full replication is usually the preferred index construction method.

Apart from minimizing tracking risk, a full replication portfolio has the advantage of being self rebalancing, given that it is based on a value- or float- weighted index, which S&P 500 is.

Optimization is preferred if a portfolio manager wishes to track an index containing a large number of stocks, particularly for stocks that are more difficult and costly to trade.

- (c) Describe the advantages and disadvantages of the following choices:
- (i) Hiring a single manager in *either* a growth *or* value style (but not both).
  - (ii) Hiring one growth manager and one value manager.
  - (iii) Hiring one manager with a market-oriented style.

**Commentary on Question:**

*Good results in general.*

- (i) **Value or growth manager (but not both)**

**Advantage:**

If the investor has a position on the desirability of these equity investment styles, this choice would lead to a portfolio expressing the clear conviction of the investor.

Such a portfolio has the potential for strong gains if the investor's style is favored by the market.

## 7. Continued

**Disadvantage:**

The choice creates tracking risk relative to the equity benchmark. Substantial underperformance may occur if the manager's style is not in favor.

(ii) **Both value and growth manager**

**Advantage:**

Lower tracking risk relative to the benchmark than investing in a single growth or value-oriented portfolio, because it does not make an overall style bet.

It is a kind of barbell approach to achieving an overall market orientation, which may have the advantage of combining the expertise of both value and growth managers.

**Disadvantage:**

Higher overall management fees than investing in a single portfolio.

Rely on security selection alone to overcome the transaction costs and higher fees associated with the active management.

(iii) **One manager with a market-oriented style**

**Advantage:**

Simplest way to invest consistently with the equity benchmark.

**Disadvantage:**

Need to confirm that the market-oriented style reflects an appropriate and consistent process that promises to add value, as opposed to an unfocused process that has averaged to a market orientation.

No obvious disadvantage to this approach exist.

(d) Critique your assistant's selection.

**Commentary on Question:**

*Good results.*

The selection has the following issues:

- A large number of stocks that are too small to be selected by S&P 500 and also too large to be selected by Russell 2000 are omitted.
- These omitted stocks could be characterized as mid-cap stocks.

## 7. Continued

- The selection should be mutually exclusive and at least approximately exhaustive with respect to your investment universe.
  - Using a faulty set of indices could lead to misleading results.
- (e) Recommend a more appropriate selection of indices.

The following selection of indices would be best:

Russell Top 200 Growth and Value

Russell Midcap Growth and Value

Russell 2000 Growth and Value

- The selection is exhaustive.
- The selection is adequate for determining a distinct style weight for mid-cap issues because it breaks out mid-cap issues via the Russell Midcap indices.

Russell 1000 Growth and Value

Russell 2000 Growth and Value

The selection is also exhaustive but would be inferior to the one recommended because it does not suffice to give a specific weight for mid-cap because the Russell 1000-based indices include both large- and mid-cap stocks.

## 8. Learning Objectives:

4. The candidate will understand and identify the variety of fixed instruments available for portfolio management. This section deals with fixed income securities. As the name implies the cash flow is often predictable, however there are various risks that affect cash flows of these instruments. In general candidates should be able to identify the cash flow pattern and the factors affecting cash flow for commonly available fixed income securities. Candidates should also be comfortable using various interest rate risk quantification measures in the valuation and managing of investment portfolios.
7. The candidate will understand the theory and techniques of portfolio asset allocation.

### Learning Outcomes:

- (4h) Construct and manage portfolios of fixed income securities using the following broad categories:
  - (i) Managing funds against a target return
  - (ii) Managing funds against liabilities.
- (7a) Explain the impact of asset allocation, relative to various investor goals and constraints.
- (7b) Propose and critique asset allocation strategies.

### Sources:

Managing Investment Portfolios: A Dynamic Process, Maginn & Tuttle, 3rd Edition (Ch. 5 & Ch. 6, Fixed Income Portfolio Management)

V-C 194-12, Revisiting the Role of Insurance Company ALM within a Risk Management Framework

Wilmott Chapter 18

### Commentary on Question:

*Commentary listed underneath each question component.*

### Solution:

- (a)
  - (i) Explain what is tracking risk and the meaning of the target tracking risk specified in the investment objectives, assuming normality of excess portfolio returns.
  - (ii) Suggest ways to achieve a lower tracking risk.



## 8. Continued

### **Commentary on Question:**

*Question 8(a) was generally well done. Maximum points earned for the second part was 4.*

- (i) Tracking risk (aka tracking error) is a measure of the variability with which a portfolio's return tracks the return of the benchmark. Specifically, Tracking Risk = Std Dev. of the Active Returns, where Active Return = Portfolio's return - Benchmark return

The target tracking risk of 1% means that in at least two thirds of the time periods, the return of the fixed income portfolio is within plus or minus 1 percent of the return of the liability benchmark.

The smaller the tracking risks, the more closely the portfolio's return matches the liability required earn rate.

- (ii) The target tracking risks could be reduced by choosing the fixed income assets such that the chosen assets
- Share same duration as the liability's duration
  - Share same key rate duration as liability's key rate duration
  - Share same convexity as the liability's convexity
  - Have the same country mix as liability's (i.e. if liabilities are sold in foreign countries with non US denominated currencies)
  - Inherit embedded put options that are consistent with cash surrender value feature
  - Replicate or passively track the benchmark index
  - Including replicating allocations by sectors, etc

(b)

- (i) Recommend one of the four strategic asset allocations solely based on the investment objectives and constraints listed above.
- (ii) Critique your recommendation in part (i) from the interest rate risk perspective.

### **Commentary on Question:**

*Consistent with the example in Maggin and Tuttle MIP Pages 241-245, including Footnote 16 at the bottom of Page 242, the intent was to use percentage points rather than the decimal versions for the input returns and standard deviations in the calculation of the risk adjusted returns.*

## 8. Continued

*However we also accepted the consistent use of the decimal version that significantly minimized by a factor of 1/100, the impact of the risk adjustment, thus, e.g. 8.97%, 7.98%, 6.99%, 6.00% instead of the intended 6.0%, 5.8%, 6.0%, 5.8%*

*Interestingly, the majority of candidates used this alternate approach. The grading allowed for the CONSISTENT follow through on this alternate approach, e.g. SAA # 1, 2 and 3 would then all satisfy the requirement of a minimum 6% risk-adjusted return, and must all be then reviewed under the other criteria.*

*Note that for this question #8, the intent was to use percentage points, whereas in Question #10 on delta-hedging, the decimal format was required for interest rates and volatility.*

*Many candidates overlooked the RBC C-1 Risk Charge for Equities.*

*Very few candidates scored any points on the RBC Capital C-3 ALM Risk Charge.*

Given the utility function, we can determine the risk-adjusted return  $E(R_m) - 0.01 * \sigma_m^2$

Asset Allocation /Portfolio	Expected Return	Stand. Deviation of Return	Risk-Adjusted Return
1	9%	17.3%	$9 - (0.01) (17.3^2) = 6.0\%$
2	8%	15%	$8 - (0.01) (15^2) = 5.8\%$
3	7%	10%	$7 - (0.01) (10^2) = 6.0\%$
4	6%	5%	$6 - (0.01) (5^2) = 5.8\%$

Asset Allocations 1 (SAA1) and 3 (SAA 3) meet the minimum 6% risk-adjusted return.

Next, we calculate the RBC requirement for SAA1 and SAA3 to decide on the optimal choice.

The portfolio is subject to C-1 risk (default risk) and C-3 risk (interest rate risk due to duration mismatch).

## 8. Continued

### C-1 Risk:

C-1 risk	SAA 1			SAA 3		
	Allocation	Exposure	C-1 Capital	Allocation	Exposure	C-1 Capital
Equities	30%	2.9	0.86	20%	1.9	0.58
Corporate Bonds (NAIC 2)	40%	3.8	0.05	55%	5.3	0.07
Structured Assets (NAIC 5)	30%	2.9	0.66	25%	2.4	0.55
Total	100%	9.6	1.58	100%	9.6	1.20

SAA 1 requires more C-1 capital (\$1.58Bn) than SAA 3 (\$1.2Bn) due to higher allocations of equities and structured assets to meet the higher return target.

### C-3 Risk:

C-3 capital charge is the greater of the prescribed factor by the NAIC and the factor computed by running the cash flow models under the prescribed set of stochastic scenarios. In this case, \$0.6Bn for both SAAs

### Recommendation

While both SAAs have the same C-3 capital charge, SAA 3 has a lower C-1 capital charge so it is more efficient from a statutory capital perspective

SAA 3 has a significantly longer duration profile than the liabilities (8 vs 5). This would expose the company to a significant interest rate risk.

Rising interest rates can:

- 1) Reduce economic surplus as the reduction in asset value would be larger than the reduction in liability value.
  - 2) Increase policyholder withdrawals and surrenders, potentially resulting in selling longer duration assets at a loss to meet the liquidity needs (disintermediation risk).
- (c) Recommend two market instruments for hedging the interest rate risk inherent in the recommendation under part (b).

### **Commentary on Question:**

*For each of the two instruments, points were awarded for defining the instrument and for explaining how it hedges the portfolio, consistent with the recommendation from (b)*

*Other instruments for hedging against increases in interest rates include "puts on bonds", "call on interest rates", "forward rate agreements", ....*

## 8. Continued

*Partial credit for recommending a switch of part of the portfolio to "floating rate assets"*

The company needs to mitigate rising interest rate risk due to the duration of assets being longer than the duration of liabilities.

This can be done through:

1. Purchase 5-year interest rate caps

Definition: a cap is a contract that guarantees its holder that otherwise floating rates will not exceed a specified amount. The variable rate is capped. A cap's cash flow is  $\max(r_f - r_c, 0)$ , multiplied by the principal.  $r_f$  is the basic floating rate, and  $r_c$  is the fixed cap rate.

Caps can help mitigate rising IR risk as increasing likelihood of the payoff from the caps when interest rate goes above the cap rate can be used to offset the reduction in asset value and the additional liability payments due to increased withdrawals and surrenders.

2. Purchase 5-year payer swaptions - option to pay fixed and receive floating

Definition: a swaption has a strike rate,  $R_E$ , that is the fixed rate that will be swapped against floating rate if the option is exercised.

In a payer swaption, the buyer has the right to become the fixed rate payer.

Payer swaptions can help mitigate rising IR risk as when interest rate goes above the strike rate and the option is exercised, the payoff from the swaptions can be used to offset the reduction in asset value and the additional liability payments due to increased withdrawals and surrenders.

## 9. Learning Objectives:

6. The candidate will understand how to develop an investment policy including governance for institutional investors and financial intermediaries.

### Learning Outcomes:

- (6c) Determine how a client's objectives, needs and constraints affect investment strategy and portfolio construction. Include capital, funding objectives, risk appetite, and risk-return trade off, tax, accounting considerations and constraints such as regulators, rating agencies, and liquidity.
- (6d) Incorporate financial and non-financial risks into an investment policy, including currency, credit, spread, liquidity, interest rate, equity, insurance product, operational, legal and political risks.

### Sources:

Maginn, Tuttle, et al Managing Investment Portfolios, A Dynamic Process, Chapter 3

Lietterman and GSAM, Modern Investment Management, Equilibrium Approach, Chapter 10

V-C194-12 Revisiting the Role of Insurance Company ALM within a Risk Management Framework

### Commentary on Question:

*Commentary listed underneath each question component.*

### Solution:

- (a) Develop a suitable investment policy for the contingency fund. List all elements.

### Commentary on Question:

*There are seven elements to be listed with an explanation. Correct and complete justification would have earned full points. Candidates did relatively well on this part of the question.*

The purpose of the contingency fund is to stabilize the revenue of the members.

**Risk objectives:** The contingency fund will experience highly variable and unpredictable contributions and payments, which argues for a low risk investment policy. However, the contingency fund, which has no stated maturity date and long term nature has some time to make up for investment losses.

**Return objectives:** The main goal is to hedge the production of the member, providing revenue from the fund when their revenues are low and collecting contributions when their revenue is high. All cash flow, inflows and outflows, are unpredictable.

## 9. Continued

**Liquidity:** The unpredictable nature of the cash flows requires highly liquid assets.

**Time horizon:** There is no defined maturity (i.e. unlimited), as the fund is for the benefit of moral entity not human being.

**Tax concern:** The Association will not be taxed as long as it is not-for-profit and administered solely for the benefits of its members. The Association can invest in otherwise taxable investments; tax deductible investments are not attractive to them.

**Unique circumstances:** The Association promotes healthy environment. The Fund is against investing in non-renewable sources of energy and is against investing in polluting industries.

- (b) Discuss the appropriateness of each asset for the contingency fund.

**Commentary on Question:**

*Candidates were expected to recognize and explain the importance of correlation of assets with liabilities, and whether the investment was or was not against the Association's core value. Partial credits were given for comments about the correlation and correct relationship of each asset to OCGAGV, and whether the asset was good or not good environmental investment. Partial credits were also given for providing additional information on each asset expected return, variability, and credit risk.*

The contingency fund compensate for low production. Thus assets that are negatively correlated with cauliflower production are preferred.

DDP: This stock has a high expected return of 10% and Sharpe ratio that is higher than the other two stocks, ST and AIR. Its negative correlation with cauliflower production (OCGAGV) makes it a good fit. But, it is not an ecologically oriented company which does not fit the environmental values of OCGAGV.

ST: This stock has a high expected return and a moderate standard deviation. It is positively correlated with OCGAGV, thus not a good fit.

AIR: This stock has a good expected return and low standard deviation. Its high negative correlation with OCGAGV makes it a good fit.

The three bonds GP, UST and HT are all positively correlated with OCGAGV. The bonds have predictable income and elements of safety. But the fund liabilities cash flow are highly unpredictable.

## 9. Continued

GP: Fair expected return, a good credit rating and moderate credit risk.

UST: Low expected return, minimal credit risk, and a long duration with low reinvestment risk.

HT: Higher expected return than the other two bonds, GP and UST. Also, it has more credit risk and a reinvestment risk, and a shorter duration.

- (c) Discuss the appropriateness of overweighting bonds relative to equities to support the liabilities.
- (i) In the presence of a deficit
  - (ii) In the presence of a surplus

**Commentary on Question:**

*Candidates did relatively well on this part of the question.*

- (i) In the presence of a deficit (underfunded):
  - In the context of maximizing RACS, weighting more towards equities increases the likelihood of returning to a surplus.
  - Bonds are acceptable for hedging, but are unlikely to be able to fully hedge liabilities in a deficit scenario.
- (ii) In the presence of a surplus (overfunded):
  - It is appropriate to use bonds to hedge liabilities. Beyond the level required to hedge liabilities, it is appropriate to use equities.

## 10. Learning Objectives:

3. The candidate will understand how to evaluate situations associated with derivatives and hedging activities.

### Learning Outcomes:

- (3d) Understand how to delta hedge and the interplay between hedging assumptions and hedging outcomes.

### Sources:

Introduces Quantitative Finance Chapter 10

### Commentary on Question:

*This question was designed to test a candidate's understanding of dynamic hedging of an option under a Black-Scholes framework.*

### Solution:

- (a) Complete the table below. Show your calculations.

### Commentary on Question:

*Credit was primarily awarded for demonstrating correct methodology rather than numerical correctness. Most candidates understood how to calculate "Option Delta," "Value of Shares," and "Portfolio Value". Many candidates had trouble calculating "Cash / (Borrowing)" and improperly solved for the post-rebalancing value by setting "Portfolio Value" to be 0 and back-solving.*

*There was some confusion for the last value of "Number of Shares", possibly due to ambiguity in the question text stating that for the 2-year option, delta hedging was done "once a year". The intention was for one rebalance over the two-year period, but a position rebalanced again at maturity was not penalized as long as all other columns were adjusted accordingly.*

Time (Year)	Underlying Stock Value	Option Value / Payoff (A)	Option Delta	Number of Shares	Value of Shares (B)	Cash / (Borrowing) (C)	Portfolio Value (A+B+C)
0	100	14.9	A	D	H	L	0
1 prior to rebalancing	115	20.5	B	E	I	M	P
1 post rebalancing	115	20.5	C	F	J	N	Q
2	105	5	1	G	K	O	R



## 10. Continued

Option Delta is equal to  $N(d1)$  where  $d1 = \frac{1}{\sigma\sqrt{T-t}} \left[ \ln\left(\frac{S}{K}\right) + \left(r + \frac{\sigma^2}{2}\right)(T-t) \right]$

$$\sigma = 0.25, \quad T - t = 2, \quad \frac{S}{K} = 1, \quad r = 0.01 \quad \text{for Time 0}$$

$$\sigma = 0.25, \quad T - t = 1, \quad \frac{S}{K} = 1.15, \quad r = 0.01 \quad \text{for Time 1}$$

$d1 = 0.233345$ ,  $N(d1) = 0.592253$  [A] for time 0

$d1 = 0.724048$ ,  $N(d1) = 0.765482$  [B] & [C] for time 1

Number of Shares is equal to Option Delta (post rebalancing):

Short (0.59) shares [D] & [E]

Short (0.77) shares [F] & [G]

Value of Shares is equal to Underlying Stock Value \* Number of Shares:

$$100 * (0.59) = (59) \text{ [H]}$$

$$115 * (0.59) = (67.85) \text{ [I]}$$

$$115 * (0.77) = (88.55) \text{ [J]}$$

$$105 * (0.77) = (80.85) \text{ [K]}$$

Cash/(Borrowing) is equal to (Option Premium) + Stock (Purchased)/Shorted:

$$\text{Cash @ Time 0} = (14.9) + 100 * 0.59 = 44.10 \text{ [L]}$$

$$\text{Cash @ Time 1 pre-rebalancing} = 44.10 * 1.01 = 44.54 \text{ [M]}$$

$$\text{Cash @ Time 1} = 44.54 + 115 * (0.77 - 0.59) = 65.24 \text{ [N]}$$

$$\text{Cash @ Time 2} = 65.24 * 1.01 = 65.89 \text{ [O]}$$

Portfolio Value is Option Value + Value of Shares + Cash/(Borrowing)

$$20.5 - 67.85 + 44.54 = (2.81) \text{ [P]}$$

$$20.5 - 88.55 + 65.24 = (2.81) \text{ [Q]}$$

$$5.00 - 80.85 + 65.89 = (9.96) \text{ [R]}$$

- (b) Explain why Bob's final portfolio value did not turn out to be close to 0.

### Commentary on Question:

*Most candidates did not do well on this section by providing only one reason.*

Bob's final portfolio value did not turn out to be close to 0 because:

- The hedging frequency is too low / Continuous rebalancing is not possible
- The option has convexity (Gamma)
- The option has time decay (Theta)
- Implied volatility (25%) differs from actual/realized volatility
- Black-Scholes framework is incompatible with fractal/jump discontinuities

## 11. Learning Objectives:

3. The candidate will understand how to evaluate situations associated with derivatives and hedging activities.

### Learning Outcomes:

- (3c) Understand the different approaches to hedging.
- (3d) Understand how to delta hedge and the interplay between hedging assumptions and hedging outcomes.

### Sources:

Paul Wilmott Introduction to Quantitative Finance Chapter 8

### Commentary on Question:

*Commentary listed underneath each question component.*

### Solution:

- (a) Define and estimate the following Greeks for the EIUL benefit:
  - (i) Delta
  - (ii) Gamma
  - (iii) Vega

### Commentary on Question:

*Most candidates were able to give definitions of the greeks and list the formulae. About 1/3 of candidates were able to calculate correctly. A few candidates missed the calculation but recalled the definition correctly and they received half of the points. Note that the greeks can be defined either as change per \$1 move or change per 1% move. Although only the latter approach is presented below, full credits were given to both answers.*

- (i) Delta is the rate of price change of the insurance benefit with respect to the price change of the underlying asset (ie the S&P 500 index). Delta = change in liability / change in S&P index

$$\text{Delta} = (\$71.7 - \$64.8) / (1.01 - 0.99) / 100 = \$3.45 \text{ million}$$

For 1% increase in the underlying asset (S&P 500 index), the liability increases \$3.45 million.

- (ii) Gamma is the rate of the change of the portfolio's delta with respect to the price change of the underlying asset (ie the S&P 500 index). Gamma is the 2nd partial derivative of the portfolio with respect to asset price.

$$\text{Gamma} = ((\$71.7 + \$64.8) - 2 * \$68.2) / 0.01^2 / 100 = \$10 \text{ million.}$$

For 1% increase in the underlying asset (S&P index), the dollar delta increases about \$10 million.

- (iii) Vega is the rate of change of the insurance benefit with respect to the volatility of the underlying asset (i.e. the S&P 500 index).

$$\text{Vega} = (\$68.7 - \$67.7)/2\%/100 = \$0.5 \text{ million.}$$

For 1% increase in volatility of underlying asset, the liability increases 0.5 million.

- (b) Explain how you can statistically hedge the crediting strategy of the EIUL block, assuming zero-coupon bond investment will provide the minimum crediting interest, using a bull call spread.

**Commentary on Question:**

*Candidates did poorly on this question. Most candidates didn't show that they understood what a bull call spread is. Although the question specifies to use bull call spread, most candidates tended to choose their own hedging strategies. Some candidates mentioned using interest rate options which was not correct. The crediting rate is guaranteed by the investment gains from investing in the S&P index. Very few candidates were able to correctly articulate the bull call spread strategy and choose the appropriate strike prices.*

Bull call spread consists of the simultaneous buying and selling call options on the same underlying asset with the same time to maturity. To construct a long position (bull) call spread, we need to buy a call option with lower strike price  $K_1$  and sell a call option at a higher strike price  $K_2$ . ( $K_2 > K_1$ ). In order to hedge the 0%-15% crediting rates to policyholders, we choose to:

- Buy call option with strike price = 1700 (current price, ATM)
- Sell call option with strike price = 1955 ( $1700 * 115\%$ )

- (c) Calculate using Black-Scholes-Merton model, the number(s) of SP call options needed to implement the hedge in part (b).

**Commentary on Question:**

*Candidates did poorly on this question. Most candidates did not understand how to delta hedge the liability portfolio using options. Some candidates attempted to calculate option prices in this part, which did not help to determine the deltas of the hedging strategy. Very few candidates were able to calculate the call spread delta correctly (even though the formula is given on the formula sheet) and then derive the correct numbers of spreads needed. Partial credits were given to candidates who understood that they needed to calculate the delta but failed to calculate correctly.*

## 11. Continued

$$d1 = \frac{LN\left(\frac{S}{K}\right) + \left(r - d + \frac{\sigma^2}{2}\right)(T - t)}{\sigma\sqrt{(T - t)}}$$

Buy long ATM call option,  $S=K=1700$ ,  $r = 5\%$ ,  $d = 3\%$ ,  $t = 1$ ,  $\sigma = 20\%$

$$d1 = \frac{LN\left(\frac{1700}{1700}\right) + \left(5\% - 3\% + \frac{0.2^2}{2}\right)(1)}{0.2\sqrt{1}} = 0.2$$

$$d2 = 0.2 - 0.2\sqrt{1} = 0$$

$N(d1) = 0.5793$ ,  $N(d2) = 0.5$

$$\begin{aligned} \text{Delta of call} &= e^{-dt}N(d1) \\ \text{Delta of long call} &= e^{-0.03*1}N(0.2) \\ &= e^{-0.03*1} * 0.5793 = 0.5622 \end{aligned}$$

Sell short Out of the money option,  $S = 1700$ ,  $K = 115\% * 1700 = 1955$ ,  $r = 5\%$ ,  $d = 3\%$ ,  $t = 1$ ,  $\sigma = 20\%$

$$d1 = \frac{LN\left(\frac{1700}{1955}\right) + \left(5\% - 3\% + \frac{0.2^2}{2}\right)(1)}{0.2\sqrt{1}} = -0.4988$$

$$d2 = -0.4988 - 0.2\sqrt{1} = -0.6988$$

$N(d1) = 0.3090$ ,  $N(d2) = 0.2423$

$$\text{Delta of short call} = e^{-0.03*1}N(-0.4988) = e^{-0.03*1} * 0.3090 = 0.2999$$

Delta of bull call spread =  $0.5622 - 0.2999 = 0.2623$ , and Dollar Delta of bull call spread =  $0.2623 * 100 * 1700 = \$44,593$

$$\text{Dollar Delta of Liab} = \frac{71.7 - (64.8)}{(1.01 - 0.99)} * 10^6 = 345 \text{ million}$$

Therefore to neutralize liability delta, the number of bull call spread that needs to be bought is  $345 \text{ million} / \$44,593 = 7,737$

Therefore we need to buy 7,737 ATM call option and sell 7,737 out of the money call option ( $K = 115\%$  times  $S$ ).

- (d) Calculate the initial cost of setting up the hedge portfolio in (b).

## 11. Continued

### Commentary on Question:

Candidates did poorly on this question. Most candidates failed to identify that the correct initial cost of hedging = number of options needed \* cost per bull call spread. The price of the call is given on the formula sheet.

$$\begin{aligned} \text{Long ATM call} &= Se^{-dt}N(d1) - Ke^{-rt}N(d2) \\ &= 1700(e^{-0.03*1} * 0.5793 - e^{-0.05*1} * 0.5) \\ &= 147.15 \end{aligned}$$

$$\begin{aligned} \text{OTM call} &= Se^{-dt}N(d1) - Ke^{-rt}N(d2) \\ &= 1700(e^{-0.03*1} * 0.3090) - 1955(e^{-0.05*1} * 0.2423) \\ &= 59.18 \end{aligned}$$

One spread cost (long ATM, sell OTM call) = 147.15 - 59.18 = \$87.97

Initial cost of setting up the hedge = \$87.97 \* 100 \* 7,737 = \$68.12 million

- (e) Calculate the Gamma and Theta of the 1,000 long ATM call options on the day they were established.

### Commentary on Question:

Most candidates were able to identify the formulae needed for estimating Gamma and Theta (appears on formula sheet); however, only a few candidates were able to use the given variables and correctly calculate the two greeks. Partial credits were given for having the correct formulae.

$$N'(d1) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}d1^2} = 0.391$$

$$\text{Portfolio Gamma} = \frac{e^{-D(T-t)}N'(d1)}{\sigma S\sqrt{T-t}} * 1000 = 1.116$$

$$\text{Portfolio Theta} = \left[ -\frac{\sigma Se^{-D(T-t)}N'(d1)}{2\sqrt{T-t}} + DSN(d1)e^{-D(T-t)} - rKe^{-r(T-t)}N(d2) \right] * 1000 = -76270.7 \text{ (per year)}$$

- (f) Estimate, using the Gamma and Theta, the gain or loss on the 1,000 long ATM calls on the next trading day.

### Commentary on Question:

Candidates did poorly on this question. Few candidates were able to identify the change in underlying (S&P) = \$2 and time passage = 1 day = 1/250 year, and even fewer candidates were able to identify the correct formula to use to estimate the loss.

## 11. Continued

Assume the same risk free rate, dividend yield, and constant volatility.

Change in S&P 500 index =  $1700 - 1698 = 2$

Change in time = 1 day =  $1/250$  year

For a Delta-Neutral portfolio, the G/L of portfolio can be approximated by

$$\Delta\Pi = \theta\Delta t + \frac{1}{2}\Gamma\Delta S^2$$

$$\Delta\Pi = -76270.7/250 + 0.5 * 1.116 * 2^2 = -\$302.85$$

## 12. Learning Objectives:

4. The candidate will understand and identify the variety of fixed instruments available for portfolio management. This section deals with fixed income securities. As the name implies the cash flow is often predictable, however there are various risks that affect cash flows of these instruments. In general candidates should be able to identify the cash flow pattern and the factors affecting cash flow for commonly available fixed income securities. Candidates should also be comfortable using various interest rate risk quantification measures in the valuation and managing of investment portfolios.

### Learning Outcomes:

- (4b) Demonstrate an understanding of par yield curves, spot curves, and forward curves and their relationship to traded security prices.
- (4c) Demonstrate understanding of the different characteristics of securities issued by government agencies.
- (4e) Describe the cash flow of various corporate bonds considering underlying risks such as interest rate, credit and event risks.
- (4f) Evaluate different private money market instruments.
- (4g) Demonstrate understanding of cash flow pattern and underlying drivers and risks of mortgage-backed securities and collateralized mortgage obligations.

### Sources:

Managing Investment Portfolios: A Dynamic Process, Maginn & Tuttle, 3rd Edition (Ch. 6, Fixed Income Portfolio Management)

An Overview of Mortgages and Mortgage Market, Fabozzi Handbook, Ch. 24

Agency Mortgage Backed Securities, Fabozzi Handbook, Ch. 25

Quantitative Finance, Wilmott, 2nd Edition, Ch. 14 (Yield, Duration, Convexity)

Quantitative Finance, Wilmott, 2nd Edition, Ch. 26 (Orange County used Inverse floater)

Quantitative Finance, Wilmott, 2nd Edition, Ch. 10 (How to Delta hedge)

Floating-Rate Securities, Fabozzi Handbook, Ch. 17, pages 354-356; pages 360-361

### Commentary on Question:

*Commentary listed underneath each question component.*

## 12. Continued

### Solution:

- (a) Calculate cash flows of the inverse floater given the following projected 6-month LIBOR.

#### Commentary on Question:

*Most candidates were able to provide the formula for inverse floater coupon and did well. Candidates also received full points for calculating semi-annual cash flows, and for calculating actual cash flows instead of per \$100 notional. Some candidates did not notice that there was no cap or floor on the inverse floater.*

Inverse floater coupon =  $K - L \times (\text{reference rate})$

$L = 1.0$

$K = 10\%$

Reference rate = 6-month LIBOR

Coupon =  $10\% - 1 \times (\text{6-month LIBOR})$

For 6-month LIBOR = 2%,

Coupon =  $10\% - (1 \times 2\%) = 8\%$

Cash flow per \$100 notional =  $8\% \times \$100 = \$8$

For 6-month LIBOR = 12%,

Coupon =  $10\% - (1 \times 12\%) = -2\%$

Cash flow per \$100 notional =  $-12\% \times \$100 = -\$2$

- (b)
- (i) Explain how the cash flows of the inverse floater may be replicated using the trading strategy involving a bond and a floater.
- (ii) Calculate the duration of the inverse floater, given the following estimates.

#### Commentary on Question:

*Most candidates were able to answer part (i) and gave a general replication strategy. Candidates who recognized that two Treasury bonds were needed in this specific case showed understanding of application of the strategy and were rewarded partial points.*

- (i) An inverse floater can be created by acquiring a fixed rate security and splitting it into a floater and an inverse floater. The cash flows an inverse floater can be replicated by buying a fixed rate bonds and shorting a floater.



## 12. Continued

- (ii) Since the reference rate is 10%, we can replicate the inverse floater by going long two 10-year 5% coupon Treasury bonds and shorting a floater.

The durations are additive. Thus,

Duration(floater) + Duration(inverse floater) = 2 x Duration(10-year 5% Treasury bond)

Duration(floater) = 0.5

Duration(10-year 5% Treasury bond) = 6.0

Duration(inverse floater) = (2 x 6) – 0.5 = 11.5

- (c)
- (i) Determine whether you need to buy or sell futures contracts.
- (ii) Estimate how many futures contracts would be needed to achieve your objective. Justify your answer.

### Commentary on Question:

*More than a half of the candidates had the correct hedge strategy. Candidates who were able to identify the correct formula for number of contracts generally did well. Candidates still received full points even if they used an incorrect duration from the previous part. Some candidates used the notional value of the inverse floater instead of the market value and received partial points.*

- (i) As rate rises, the value of the inverse floater declines. As rate rises, the value of Treasury futures also declines. So, to delta hedge the interest rate risk, short Treasury futures.

- (ii)

$D_T$  = target duration for the portfolio

$D_I$  = initial duration for the portfolio

$P_I$  = initial market value of the portfolio

$D_{CTD}$  = the duration of the cheapest-to-deliver bond

$P_{CTD}$  = the price of the cheapest-to-deliver bond

Conversion factor for the cheapest-to-deliver bond

$$\text{Approximate number of contracts} = \frac{(D_T - D_I)P_I}{D_{CTD}P_{CTD}} \times \text{Conversion factor for the CTD bond}$$

## 12. Continued

DT = 0; DI = 11.5; PI = \$9.5 million; DCTD = 8.5; PCTD = \$0.1 million

$$\begin{aligned} &\text{Futures contracts needed} \\ &= (-1.2 * 11.5 * 9.5) / (8.5 * 0.1) \\ &= -154 \end{aligned}$$

Therefore, sell 154 Treasury futures contracts. (Selling 155 contracts is also fine.)

- (d) Calculate the repo rate.

### **Commentary on Question:**

*The formula for repo rate is in the Floating-Rate Securities chapter of Fabozzi's Handbook of Fixed Income Securities. Candidates who were able to recall the correct formula did well. Some candidates did not understand the term repo rate. Some could not recall the formula.*

$$\text{Dollar interest} = \text{dollar principal} \times \text{repo rate} \times \left( \frac{\text{repo term}}{360} \right)$$

$$\begin{aligned} \text{Repo rate} \\ &= (\$1,008,214.15 - \$1,007,500.50) / \$1,007,500.50 \times 360/30 \\ &= 0.85\% \end{aligned}$$

- (e) List two benefits for each party to enter into this repurchase agreement.

Benefits to Evergreen (investor):

- Higher return than Treasury bills.
- Repo agreement is collateralized with Treasury notes, therefore there is less counterparty risk.
- Able to receive a good return for an investment that is highly liquid, a good alternative to holding cash.

Benefits to Goldman (AA-rated investment bank)

- Repo rate is less than the cost of bank financing.
- Collateral to a repo is not limited to government securities. Money market instruments, federal agency securities, and mortgage-backed securities can also be used.

### 13. Learning Objectives:

5. The candidate will understand the variety of equity investments and strategies available for portfolio management.

#### Learning Outcomes:

- (5e) Identify methods for establishing passive exposure to an equity market.
- (5g) Recommend and justify, in a risk-return framework, the optimal portfolio allocations to a group of investment managers.
- (5h) Describe the core-satellite approach to portfolio construction with a completeness fund to control overall risk exposures.
- (5i) Explain alpha and beta separation as an approach to active management and demonstrate the use of portable alpha.
- (5j) Describe the process of identifying, selecting, and contracting with equity managers.

#### Sources:

Managing Investment Portfolios: A dynamic process, Maginn & Tuttle, 3rd Edition (Ch. 5, and 7, Equity Portfolio Management)

QFIC-106-13: Modern Investment Management: An Equilibrium Approach, Litterman (Ch. 10)

#### Commentary on Question:

*Commentary listed underneath each question component.*

#### Solution:

- (a) Assess the correctness of each of the three comments in the above statement.

#### Commentary on Question:

*This question tests the candidate's understanding of risk measures such as the Sharpe ratio and the Information Ratio and how they may be used.*

The statement on how the Sharpe ratio measures risk is correct, while it does not distinguish the trade-off between investing in risky assets and investing in risk-free assets.

The statement that the Sharpe ratio uses a passive benchmark is incorrect. The information ratio is used for measuring active managers against a passive benchmark.

### 13. Continued

The statement on how to best use the Sharpe ratio is incorrect. Investors using the Sharpe ratio can rank or compare funds within their risk tolerance to see which funds have the best risk-adjusted returns. The information ratio, on the other hand, is best used by investors to measure or rank manager performance. It can be used as an absolute measure of portfolio manager skill.

- (b) Draft a corrected version of the above statement.

**Commentary on Question:**

*This question tests the candidate's understanding of risk measures such as the Sharpe ratio and the Information Ratio and how they may be used.*

Recommended statement:

The Sharpe ratio measures a portfolio's excess return to its total risk; it answers the question of how much an investor was compensated for investing in a risky asset versus a risk-free asset.

The information ratio measures a portfolio's excess return relative to its benchmark tracking error. It answers the question of how much reward a manager generated in relation to the risks he or she took deviating from the benchmark.

All portfolios measured with the Sharpe ratio, then, have the same benchmark: the risk free asset. The higher the Sharpe ratio, the better the risk-adjusted return is. Investors using the Sharpe ratio can rank or compare funds within their risk tolerance to see which funds have the best risk-adjusted returns.

The information ratio is used for measuring active managers against a passive benchmark. The information ratio is best used by investors to measure or rank manager performance. It can be used as an absolute measure of manager skill.

- (c) Design a portable alpha strategy that can maintain the current strategic asset allocation and enhance performance by adding small cap alpha. List your implementation steps.

**Commentary on Question:**

*This question tests the candidate's understanding of a portable alpha strategy and how it can be used to add enhance performance. Some candidates showed understanding of the basic concepts. Some missed the strategy entirely.*

The following steps implement a portable alpha strategy that can maintain the 4% alpha under both up market and down market conditions.

## 13. Continued

Step 1: Buy S&P 500 index futures to establish market exposure equal to \$10 million (10% of \$100 million AUM) to bring the large cap asset class allocation back to the original 35%.

Step 2: Buy \$10 million in small cap stocks through the small cap manager. The manager's Long-Only portfolio is designed to beat the Russell 2000 Index, but it has a beta of 1.0 relative to the index.

Step 3: Short/sell \$10 million worth of Russell 2000 index futures to eliminate market exposures or beta.

- (d) Illustrate the portable alpha strategy that you proposed above by showing its annual returns under the following market scenarios, assuming that the small cap manager generates the expected alpha:
- Up market: S&P 500 up 10% and Russell 2000 up 15%.
  - Down market: S&P 500 down 10% and Russell 2000 down 15%.

### **Commentary on Question:**

*This question tests the candidate's understanding of how a portable alpha strategy enhances performance in up and down markets. Once candidates figured out part (a), they generally did well on this part.*

### **Up market:**

S&P up 10%, the \$10 million long position in the S&P 500 futures gains \$1 million ( $\$10 \text{ million} * 10\%$ ).

Russell 2000 up 15%, the \$10 million short position loses \$1.5 million. ( $\$10 \text{ million} * 15\%$ ).

The small cap Long-only manager should return 20% (15%+5%), giving a return of \$2 million.

So overall, the strategy returns \$1.5 million ( $\$1 - \$1.5 + \$2$ ), generating additional 5%, which is the portable alpha.

### **Down market:**

S&P down 10%, the \$10 million long position in the S&P 500 futures loses \$1 million ( $\$10 \text{ million} * 10\%$ ).

Russell 2000 down 15%, the \$10 million short position gains \$1.5 million. ( $\$10 \text{ million} * 15\%$ ).

### 13. Continued

The small cap Long-only manager should return -10% (-15%+5%), losing \$1million.

So overall, the strategy loses \$0.5 million (-\$1+\$1.5-\$1), beating the allocation with this strategy by 5% (w/o this strategy, portfolio would lose \$1 million), which is the portable alpha.

- (e) Identify the asset allocation strategy.

**Commentary on Question:**

*This question tests the candidate's understanding of asset allocation types. Most candidates did well on this part.*

The asset allocation strategy is the core-satellite strategy, with the S&P 500 as the core to passively track the broad market. The satellites, the actively managed funds in high-yields and commodities, are to provide an opportunity to earn greater returns than the passive portion of the portfolio.

- (f) Compare and contrast this asset allocation strategy to:

- Strategy I: 100% allocated to S&P 500 Index Fund.
- Strategy A: 100% allocated to US Growth Fund.

**Commentary on Question:**

*This question tests the candidate's understanding of the characteristics of various asset allocation types. Most candidates did well on this part.*

Strategy I – Index approach:

- Seeks market return where Core-Satellite (C/S) seeks to outperform broad market
- Lower cost
- Lower manager risk
- Higher potential of tax efficiency
- However, cannot outperform the broad market

Strategy A – Active approach:

- Seek to outperform
- Higher cost
- Higher manager risk
- Lower potential of tax efficiency

## 13. Continued

- (g) Assess this suggestion.

**Commentary on Question:**

*This question tests the candidate's understanding of ETFs. Most candidates did well on this part.*

This is a novel idea (innovative).

Traditionally, core/satellite investing is used exclusively with actively managed products. The ETF might push the core-satellite approach even further in both efficiency and cost:

- Lower cost: ETFs have significantly lower management fees than actively managed funds.
- More accessible and flexible: ETFs offer exposure to markets that may be expensive to invest in or difficult for fund managers to outperform.

## 14. Learning Objectives:

6. The candidate will understand how to develop an investment policy including governance for institutional investors and financial intermediaries.
7. The candidate will understand the theory and techniques of portfolio asset allocation.

### Learning Outcomes:

- (6c) Determine how a client's objectives, needs and constraints affect investment strategy and portfolio construction. Include capital, funding objectives, risk appetite, and risk-return trade off, tax, accounting considerations and constraints such as regulators, rating agencies, and liquidity.
- (7a) Explain the impact of asset allocation, relative to various investor goals and constraints.
- (7b) Propose and critique asset allocation strategies.
- (7c) Evaluate the significance of liabilities in the allocation of assets.

### Sources:

Maginn, Tuttle, et al Managing Investment Portfolios, A Dynamic Process, Chapter 5

Litterman and GSAM, Modern Investment Management, Equilibrium Approach, Chapter 10

### Commentary on Question:

*Commentary listed underneath each question component.*

### Solution:

- (a) Describe the ALM approaches under each of the above listed dimensions.

#### Commentary on Question:

*Candidates generally did well on this part of the question.*

ALM approaches under Low Risk Option

Cash flow matching (exact matching)

- Investments are structured using bonds to match future asset and liability cash flows.



## 14. Continued

### Immunization

- Structures investments to match/offset the weighted average duration/convexity of liabilities
- Hedges against shifts in the yield curve/interest rate risk

ALM approaches permitting higher risk levels include those specifying the satisfaction of liabilities as constraints under which the asset allocation is chosen as well as those incorporating an objective function that includes a penalty for failing to satisfy liabilities.

### Time Horizon approaches under ALM

- Dynamic – ALM approach recognizes that the asset allocation and actual asset returns and liabilities in a given period affect the optimal decision that will be available in the next period
- Static – ALM approach that does not consider links between optimal decisions at different time periods.

- (b) Recommend the ALM approach for the pension fund by considering each dimension.

### **Commentary on Question:**

*Partial credits were given for comments and recommendations on an ALM approach to a risk level dimension and to a time period dimension. Full credit would be given for responses that included (proper) recommendation and provided support for the candidate's recommendations.*

### Risk level

#### Recommendation: Immunization strategy

- Having long duration liabilities means cash flow matching may be impractical as there may be no assets available for longest liability durations
- Due to the penalties for not meeting liabilities when due (legal/regulatory), the upside immunization provides is preferable
- Employees are risk-tolerant so immunization is more preferable since it involves more risk/upside
- Perfect cash flow matching eliminates upside potential since equities are eliminated.

-OR-

#### Recommendation: Higher risk approach

- Having long duration liabilities means an all-fixed-income strategy may be impractical as there may be no assets available for longest liability durations (also liquidity issues)

## 14. Continued

- Employees are risk-tolerant so more risk/upside is appropriate
- High growth - more risk/upside a appropriate
- Plan underfunded – more growth needed

Time Horizon

Recommendation: Dynamic strategy

- Pension liabilities are multi-period
- Has an accumulation and a payout phase
- Company is in growth phase and will have significant future liabilities, so dynamic strategy may be more appropriate.

-OR-

Recommendation: Static strategy

- Dynamic approach is complex, requires a lot of computing power, and is costly to implement
- High growth, so future liabilities less predictable, dynamic approach may be imprecise.

(c) Assess the validity of CFO's statements.

### **Commentary on Question:**

*Candidates were expected to provide valid responses to the CFO's statement on the pension fund investments. Partial credits were given for correct points of agreement and points of disagreement with the CFO's statement.*

Points of agreement:

- Asset-only approach is used in practice for many defined benefit pension plans.
- Younger, more risk-tolerant clientele may be able to accommodate a riskier retirement portfolio at this stage.
- The current underfunded status of the fund will require a higher equity allocation in order to increase the likelihood of reverting back to adequate funding status. An asset-only approach that tries to maximize the Sharpe ratio would typically gravitate more towards equities, especially in the current low interest rate environment

Points of disagreement

- In the long term, the penalties for not meeting the pension liabilities are high. Risks have to be mitigated.
- Pension plans need fixed income assets to emulate liability payouts in duration-matching/immunization strategies in order to minimize exposure to interest rate risk

## 14. Continued

- Argument makes better sense in the context of a defined contribution plan, not a defined benefit plan. Defined contribution plans allow pensioners to influence their pension by directly allocating assets based on their expectations. Defined benefit plans are tied to salary/work performance.
- Sharpe ratio is not a good risk-return metric for the pension in the long run because it does not incorporate the liability stream.
- Sharpe ratio is not a good risk-return metric because it only considers a one-period model where the clients do not require intermediate consumption in addition to final wealth protection.

## 15. Learning Objectives:

7. The candidate will understand the theory and techniques of portfolio asset allocation.

### Learning Outcomes:

- (7b) Propose and critique asset allocation strategies.
- (7c) Evaluate the significance of liabilities in the allocation of assets.
- (7d) Incorporate risk management principles in investment policy and strategy, including asset allocation

### Sources:

Maginn, Tuttle, et al *Managing Investment Portfolios, a dynamic process*, Chapter 5

### Commentary on Question:

*Parts a) and b) were well answered. In particular, the interpolation between corner portfolios to select the appropriate asset mix was well done. However, the results for parts c) and d) were not as good and not homogenous so the dispersion of the distribution of final results was considerable.*

### Solution:

- (a) Explain the characteristics and concerns for strategic asset allocation, as they relate to life insurance companies.

Investment portfolio policy for insurers seeks to achieve the most appropriate mix of assets (1) to **counterbalance the risks inherent in the mix of insurance products involved** and (2) to achieve the stated return objectives.

#### The most important considerations are:

##### **Asset/liability management concerns**

Yield, duration, convexity, key rate sensitivity, value at risk, and the effects of asset risk on capital requirements given the spread of risk-based capital regulation

##### **Regulatory influences**

Public policy frequently views insurance portfolios as quasi-trust funds. Restrictions on junk bond holdings

##### **Time Horizon**

Selection of bond maturities is substantially dictated by its need to manage the interest rate risk exposure arising from asset/liability duration mismatch. The portfolio's maturity schedule is usually structured around the short and medium term estimated liability cash outflows.

## 15. Continued

### Tax Considerations

Because life insurers are taxable entities, they must consider potential investment taxability, (i.e. state and municipal bonds).

### Additional Considerations:

**Portfolio segmentation** is used to break the company's portfolio into subportfolios in order to allocate the most appropriate securities to fund each sub portfolio. Fixed income investments constitute the majority holdings of the most life insurers.

Insurance companies are sensitive to **cash flow volatility** and **reinvestment risk**. Private placement bonds are a large part of a life insurer's portfolio, so **credit analysis** is important. Life insurers hold equity in **separate accounts** for products like variable annuities and variable life insurance.

Common stocks, investments in real estate, and venture capital are the most widely used investment to achieve **surplus growth**.

**Surplus adequacy:** companies are looking for more aggressive investments and leverage to offset the narrowing contributions to surplus from newer product lines.

**Valuation risk** has led life insurers to limit common stock holdings (at market value) as a percentage of surpluses rather than a percentage of assets.

The need to maintain limited **liquidity reserves** to ensure that surrenders and/or policy loans can be funded with little or no loss of principle income.

- (b) Propose an asset allocation strategy for ABC Life's universal life insurance portfolio. Show your work.

Corner portfolios mark the edge of the minimum-variance frontier. We want to find the portfolio with the maximum return subject to the risk objective of an overall standard deviation of 8.0%.

We identify the adjacent corner portfolios as Corner Portfolio 4 (with a 8.99% expected standard deviation) and Corner Portfolio 5 (with a 7.82% expected standard deviation). It then follows that  $8.0 = 7.82w + 8.99(1-w)$ . **Note that this will find an upper bound on the standard deviation of the portfolio with these weights. The actual standard deviation of the portfolio with these weights would be less.**

We find that  $w = 0.1538$  and  $(1-w) = 0.8462$ . The detailed arithmetic follows:

## 15. Continued

Weight of Real Estate  $0.1538(32.53\%) + 0.8462(19.93\%) = \mathbf{21.87\%}$   
Weight of Private Placement  $0.1538(14.30\%) + 0.8462(21.09\%) = \mathbf{20.05\%}$   
Weight of Public Short Term Bonds  $0.1538(0.00\%) + 0.8462(16.85\%) = \mathbf{14.26\%}$   
Weight of Public Medium Term Bonds  $0.1538(8.74\%) + 0.8462(0.00\%) = \mathbf{1.34\%}$   
Weight of Public Long Term Bonds  $0.1538(44.44\%) + 0.8462(42.13\%) = \mathbf{42.49\%}$

- (c) Describe your concerns with the portfolio you proposed in part (b).

The asset portfolio should be considered in conjunction with the liabilities that it supports. The asset portfolio from part (b) has a duration of 8.72 based on the weights from the table. The liabilities have duration equal to 6.5. This mismatch means that the asset portfolio is more sensitive to changes in interest rates than the liability portfolio. If interest rates were to spike, the value of the surplus (assets less liabilities) would decrease.

- (d) Propose changes to your allocation strategy in part (b) to address the CRO's concerns. Justify your answer.

An **immunization** approach is recommended. We should attempt to match the durations of the assets and liabilities. Corner portfolio 6 has a liability duration of 6.48, which is very close to the duration of the universal life liabilities.

This portfolio has weights of 26.61% in private placement bonds, 37.81% in short term public bonds, 35.58% in long term public bonds. This portfolio has an expected return of 6.13%, which is above the minimum required. The expected standard deviation is 5.94%, which is below the maximum required.

This portfolio meets the risk and return objectives, and addresses the interest rate risk in the universal life portfolio.

**Convexity** matching is also recommended.