SOCIETY OF ACTUARIES
Quantitative Finance and Investment Core

Exam QFICORE

MORNING SESSION

Date: Wednesday, April 30, 2014
Time: 8:30 a.m. – 11:45 a.m.

INSTRUCTIONS TO CANDIDATES

General Instructions

1. This examination has a total of 100 points. It consists of a morning session (worth 60 points) and an afternoon session (worth 40 points).

    a) The morning session consists of 10 questions numbered 1 through 10.

    b) The afternoon session consists of 6 questions numbered 11 through 16.

    The points for each question are indicated at the beginning of the question.

2. Failure to stop writing after time is called will result in the disqualification of your answers or further disciplinary action.

3. While every attempt is made to avoid defective questions, sometimes they do occur. If you believe a question is defective, the supervisor or proctor cannot give you any guidance beyond the instructions on the exam booklet.

Written-Answer Instructions

1. Write your candidate number at the top of each sheet. Your name must not appear.

2. Write on only one side of a sheet. Start each question on a fresh sheet. On each sheet, write the number of the question that you are answering. Do not answer more than one question on a single sheet.

3. The answer should be confined to the question as set.

4. When you are asked to calculate, show all your work including any applicable formulas. When you are asked to recommend, provide proper justification supporting your recommendation.

5. When you finish, insert all your written-answer sheets into the Essay Answer Envelope. Be sure to hand in all your answer sheets because they cannot be accepted later. Seal the envelope and write your candidate number in the space provided on the outside of the envelope. Check the appropriate box to indicate morning or afternoon session for Exam QFICORE.

6. Be sure your written-answer envelope is signed because if it is not, your examination will not be graded.

Tournez le cahier d'examen pour la version française.

475 N. Martingale Road
Schaumburg, IL 60173-2226

Printed in the U.S.A.

BEGINNING OF EXAMINATION
Morning Session

1. (*5 points*) In a financial market with only two assets, a risk-free account and a risky asset (stock), let $B(t)$ and $S(t)$ be the respective prices of these assets at time t. Assume a single period where at the end of the period, there are only two possible states of the world: ω_1 and ω_2.

You are given:

1. $B(0)=1, S(0)=100$

2. The values of the assets at the end of the period:

State	Risk-free $B(1)$	Stock $S(1)$
ω_1	1.05	a
ω_2	1.05	150

(a) (*0.5 points*) Determine the range of a so that there is no arbitrage opportunity.

Suppose $a=80$.

(b) (*1 point*) Calculate the state prices in this market.

(c) (*0.5 points*) Interpret these state prices.

(d) (*1 point*) Calculate the risk-neutral probabilities of stock price movements in this market.

(e) (*1 point*) Calculate the price of a straddle with payoff equal to $|S(1)-100|$, which expires at the end of the period, using the risk-neutral probabilities.

(f) (*1 point*) Confirm the value in (e) using the state prices calculated in (b).

2. (*4 points*) A binomial lattice is used to model the price of a non-dividend paying stock up to time T. The time interval $(0, T)$ is subdivided into n intervals of length $\Delta t = \frac{T}{n}$.

It is assumed that, at each node in the binomial lattice, the share price will either increase by a factor of:

$$u = e^{\mu \Delta t + \sigma \sqrt{\Delta t}}$$

or decrease by a factor of

$$d = e^{\mu \Delta t - \sigma \sqrt{\Delta t}}$$

where $\sigma > 0$. The movements at each period are assumed to be independent.

(a) (*1.5 points*) Show that the price at time T will be:

$S_T = S_0 \exp\left[\mu T + \sigma\sqrt{T}\left(\frac{2X_n - n}{\sqrt{n}}\right)\right]$, where X_n is the total number of up jumps.

(b) (*1 point*) Specify the distribution of X_n and state how this distribution can be approximated when n is large, assuming the share price will be equally likely to increase or decrease.

(c) (*1.5 points*) Determine the distribution of $\frac{S_T}{S_0}$ when n approaches infinity, using part (b).

3. (*7 points*) You are working on a Ho & Lee interest rate model for the risk-neutral spot rate:

$$dr_t = adt + cdW_t$$

where W_t is a standard Brownian motion.

(a) (*1 point*) Define the technique of calibration as it relates to one-factor interest rate models and explain why the Ho & Lee model facilitates calibration.

(b) (*1 point*) State an argument for and an argument against calibration.

(c) (*1 point*) Show that $r_t = r_0 + at + cW_t$

(d) (*2 points*) Derive a formula for the arbitrage-free price $B(0, T)$ of a default-free zero-coupon bond, as a function of $r_0, a, c,$ and T. Hint: $E(e^{\int_0^T cW_t\,dt}) = e^{c^2T^3/6}$

(e) (*1 point*) Derive a formula for the continuously compounded forward rate $F(0, T, U)$, where $U > T > 0$, as a function of $r_0, a, c, T,$ and U.

(f) (*1 point*) Derive a formula for the instantaneous forward rate $f(0,T)$ as a function of $r_0, a, c,$ and T.

4. (*9 points*) An investment actuary is considering using the following Cox-Ingersoll-Ross (CIR) process to model spot interest rates:

$$dX_t = a(b - X_t)dt + \sigma\sqrt{X_t}dB_t$$

where $a > 0$ and B_t is a standard Brownian motion.

(a) (*1.5 points*) Name the key characteristic of the above CIR process and interpret each of its parameters $a, b,$ and σ.

(b) (*0.5 points*) State the condition(s) on the parameters $a, b,$ and σ so that the spot rate stays positive.

Define $Y_t = X_t^2$, $m_1(t) = E\left[X_t \middle| X_0 = c\right]$ and $m_2(t) = E\left[Y_t \middle| X_0 = c\right]$

(c) (*2 points*) Derive a stochastic differential equation satisfied by Y_t using Ito's Lemma.

(d) (*2 points*) Verify that $m_1(t)$ satisfies the following differential equation:

$$\frac{dm_1(t)}{dt} = a\left(b - m_1(t)\right)$$

(e) (*0.5 points*) Show that $m_1(t) = b + (c - b)e^{-at}$ for all $t \geq 0$.

(f) (*1 point*) Verify that $m_2(t)$ satisfies the following differential equation:

$$\frac{dm_2(t)}{dt} = \left(2ab + \sigma^2\right)m_1(t) - 2a\, m_2(t)$$

(g) (*1.5 points*) Show $\lim_{t\to\infty} Var\left[X_t \middle| X_0 = c\right] = \frac{b\sigma^2}{2a}$ assuming $\lim_{t\to\infty} \frac{dm_2(t)}{dt} = 0$.

5. (*7 points*) Matt is planning to borrow $2,000,000 in 3 years for a 10-year term. Interest rates are currently at historic lows, but Matt worries that rates may increase by the time he is ready to borrow.

(a) (*1.5 points*) Describe the terms of the following interest rate derivatives.

(i) Cap

(ii) Forward Rate Agreements

(iii) Interest Rate Swaps

(b) (*1.5 points*) Explain shortfalls of the Black-Scholes assumptions when applied to interest rate derivatives.

Given the following market prices of risk-free zero-coupon bonds paying $1 at time T:

Maturity T	Bond Price
3	0.95
10	0.78
13	0.73

Matt is contacting dealers to enter into a Forward Rate Agreement (FRA) to hedge his interest rate risk.

(c) (*2 points*)

(i) Calculate the continuously compounded implied forward interest rate of the FRA.

(ii) Describe the arbitrage opportunities if the continuously compounded risk-free spot rate will be constant at 2.42% per annum for the next three years.

(d) (*2 points*) Contrast the Classical and HJM approaches to calculating the arbitrage-free prices of bonds.

6. (*8 points*) Let $S(t)$ be the price of a stock at time t and let it follow the following process:

$$dS = \mu S\,dt + \sigma S\,dB$$

where μ and σ are constants with $\sigma > 0$ and B is a Brownian motion.

Assume the risk-free interest rate, r, is positive.

(a) (*1 point*) Compare and contrast real and risk-neutral random walk.

(b) (*1 point*) Derive, by applying Ito's Lemma, the process that log S follows.

(c) (*3 points*)

(i) Show that

$$v(t,S) = e^{\left(\frac{-\left(4r+\sigma^2\right)(T-t)}{8}\right)}\sqrt{S(t)}, \qquad 0 \le t < T$$

satisfies the Black-Scholes partial differential equation.

(ii) Describe the derivative whose value is given by $v(t, S)$.

An equity option pays 1 if the stock price $S(T) \le K$ and 0 otherwise, where $K > 0$.

(d) (*3 points*) Show that for any time t, $0 \le t < T$, the value of the equity option equals

$$e^{-r(T-t)}\,N\left(\frac{\ln\frac{K}{S(t)} - \left(r - \frac{1}{2}\sigma^2\right)(T-t)}{\sigma\sqrt{T-t}}\right)$$

where $N(\,)$ is the cumulative standard normal distribution.

7. (*7 points*) You are modeling monthly log returns of XYZ stock using historical data from January 1964 to December 2013 (600 observations). You have used AR(1)-GARCH(1,1) model so far.

(a) (*1 point*) Describe the main disadvantage of GARCH(1,1) model.

(b) (*2 points*) Describe how EGARCH(1,1) addresses shortcomings in GARCH(1,1) model.

You have estimated the parameters of the AR(1)-EGARCH(1,1) and the fitted model is

$$r_t = 0.01 + 0.08 r_{t-1} + a_t, \qquad a_t = \sigma_t \varepsilon_t$$

$$\ln\left(\sigma_t^2\right) = -5.5 + \frac{g\left(\varepsilon_{t-1}\right)}{1 - 0.95B}$$

$$g\left(\varepsilon_{t-1}\right) = -0.0825\varepsilon_{t-1} + 0.2856\left[\left|\varepsilon_{t-1}\right| - \sqrt{\frac{2}{\pi}}\right]$$

(c) (*2 points*) Compute the impact of a negative shock of size 2 standard deviations compared to the impact of a positive shock of size 2 standard deviations.

(d) (*2 points*) Compute three step ahead volatility forecast for the fitted model in (c), given that the forecast origin $t = 600$ and $\hat{\sigma}_{600}^2(1) = 5.05 * 10^{-3}$ and $\hat{\sigma}_{600}^2(2) = 5.098 * 10^{-3}$.

8. (*4 points*) You are a portfolio manager for a large bond fund. You are concerned about adverse credit events in the bond holdings of the fund and considering a variety of hedging instruments to mitigate the risks.

Bond Issuer	Credit Events of Concern
X	Rating downgrade
Y	Default
Z	Credit spread widens

(a) (*1.5 points*) Evaluate the appropriateness of using each of the following three hedging instruments to mitigate one or more of the three risks:

(i) A binary credit put option with the credit event specified as a credit rating downgrade.

(ii) A credit spread call option where the underlying is the level of the credit spread.

(iii) A credit spread forward, with the credit derivative dealer firm taking the position that the credit spread will decrease.

One corporate bond in your bond portfolio has the following characteristics:

Rating	BBB
Coupon	5%
Maturity	5-year
Coupon pay frequency	Per annum
Market ask price	$106
Optionality	Callable
Option adjusted spread (in basis points)	55 bps
Nominal spread	90 bps
Nominal spread between an option-free BBB bond and a Treasury bond of similar maturity	60 bps

8. **Continued**

The following table shows present values of cash flows of the bond under different spreads relative to the zero-coupon Treasury curve.

Period	Cash flows	Spot Rates	Present Values of Cash Flows		
			PV at Spot+65 bps	PV at Spot+75 bps	PV at Spot+85 bps
1	$5.00	1.0%	$4.92	$4.91	$4.91
2	$5.00	1.5%	$4.79	$4.78	$4.77
3	$5.00	2.0%	$4.62	$4.61	$4.60
4	$5.00	2.5%	$4.42	$4.40	$4.38
5	$105.00	3.0%	$87.77	$87.35	$86.93

(b) (*1.5 points*) Estimate the zero-spread of this corporate bond.

(c) (*1 point*) Assess whether this bond is cheap or rich. Justify your answer.

9. (*5 points*) Your company has fixed immediate payout annuity liability.

Your CIO would like to use Mortgage Back Security (MBS) to back the payout annuity liability.

You were asked to price the MBS. To estimate prepayment rate, your risk team uses a 100% PSA model, which can be expressed as follows:

$$\begin{cases} \text{if } t \le 30: CPR = 6\% \cdot \dfrac{t}{30} \\ \text{if } t > 30: \qquad CPR = 6\% \end{cases}$$

t is the number of months since the mortgage originated.

The CIO asked you to use higher PSA percentage due to persistent low interest environment.

(a) (*1 point*) Critique your CIO's reasoning for higher PSA percentage.

(b) (*1 point*) Describe advantages and disadvantages of having MBS to back the payout annuity liability.

Some of corporate bonds in your holdings have the following embedded options.

- Make whole call provision
- Sinking fund provision

For each of the above two options,

(c) (*1 point*) Describe the corresponding provision.

(d) (*1 point*) Identify one advantage and one disadvantage to the bondholders.

Your CIO would also like to use bonds with make whole call provisions to back the payout annuity liability.

(e) (*1 point*) Describe advantages and disadvantages of having bonds with make whole call provisions to back the payout annuity liability.

10. (*4 points*) Mary is a US resident and is considering municipal bonds for a personal investment. Her marginal tax rate is 25% and her goal is to maximize after-tax investment return.

(a) (*1 point*) Describe each:

(i) The role that rating agencies play in evaluating municipal bonds.

(ii) How large institutional investors determine the creditworthiness of municipal bonds.

(b) (*1 point*) Critique whether a well-known, high credit quality municipal bond can benefit from using municipal bond insurance.

(c) (*2 points*) Evaluate for each of the following two pairs of bonds separately, which bond is more suitable to Mary given her investment goal.

(i) 5-Year AA corporate bond with a yield of 5% or 5-Year AA municipal bond with a yield of 3.50%.

(ii) 5-Year AA municipal bond selling at par with a yield-to-maturity of 4% or 5-Year AA municipal bond selling below par with a yield-to-maturity of 4%.

****END OF EXAMINATION****
Morning Session

USE THIS PAGE FOR YOUR SCRATCH WORK