QFI Core Model Solutions
Fall 2013

1. **Learning Objectives:**

3. The candidate will understand how to evaluate situations associated with derivatives and hedging activities.

Learning Outcomes:

(3a) Compare and contrast various kinds of volatility (e.g., actual, realized, implied, forward, etc.).

Sources:

Quantitative Finance, Wilmott, Paul, 2nd Edition, Chapter 9

Frequently Asked Questions in Quantitative Finance, Wilmott, Paul, 2nd Edition

Commentary on Question:

Overall, candidates did well on this question. For part (a), full credits were given to complete discussion of the 3 volatilities including definition, time period, associated timescales, and the ability to be measured. Partial credits were given to a partial discussion. For part (b), full credits were given to candidates who defined the formula for ARCH and EWMA models, and demonstrated understanding of at least one advantage and one disadvantage. Partial credits were given to a partial discussion. For part (c), full credits were given to candidates who named the GARCH model, and recalled its formula. Partial credits were given to either naming the model or recalling the formula.

Solution:

(a) Describe each of the following: actual volatility, implied volatility and realized volatility. Identify how they differ in terms of time period, number of associated time scales, and ability to be measured.

Actual volatility

- measure of the amount of randomness in an asset return at any particular time
- no associated timescale
- very difficult to measure

Realized volatility

- measure of the amount of randomness over some period in past
- There are two time scales associated with a realized volatility, short and long.
- Easy to measure with specified period and calculation method.

1. **Continued**

Implied volatility

- Volatility which when put into Black-Sholes option pricing formula give the market price of the option
- Some interpret as market view of future actual volatility over lifetime of option. It's influenced by supply and demand as well as other effects
- Timescale is expiration of option
- Easy to measure

(b) You wish to estimate 60-day future volatility using historical data. Describe ARCH and EWMA including their advantages and disadvantages.

<u>ARCH Model (Autoregressive Conditional Heteroscedasticity)</u>

Model specified by:

$$\sigma_n^2 = \alpha\bar{\sigma}^2 + (1-\alpha)\frac{1}{n}\sum_{i=1}^{n} R_i^2$$

- Volatility tends to vary about a long-term mean, $\bar{\sigma}$.
- Weighting assigned to the long-run volatility estimate and the current estimate based on the last *n* returns.

Pluses:
Time varying volatility
Captures variance around an assumed long term mean

Minuses:
Each term equally weighted, can lead to plateau effect

<u>Exponential Weighted Moving Average (EWMA)</u>

- The more recent the return the more weight is attached to value

Model specified by

$$\sigma_n^2 = \lambda\sigma_{n-1}^2 + (1-\lambda)R_n^2 \quad or \quad \sigma_n^2 = (1-\lambda)\sum_{i=1}^{\infty}\lambda^{(i-1)}R_{n-i+1}^2$$

$0 < \lambda < 1$

Pluses:
The more recent the return the more weight attached to it
Avoids plateaus
Time varying volatility

1. Continued

Minuses:
Not able to capture information regarding a long term average

(c) Identify and specify a common model that combines the features of ARCH and EWMA.

Common model is GARCH, Specified by:

$$\sigma_n^2 = \alpha\bar{\sigma}^2 + (1-\alpha)(\lambda\sigma_{n-1}^2 + (1-\lambda)R_n^2)$$

2. Learning Objectives:

1. The candidate will understand the fundamentals of mathematics and economics underlying quantitative methods in finance and investments.

2. The candidate will understand how to apply the fundamental theory underlying the standard models for pricing financial derivatives. The candidate will understand the implications for option pricing when markets do not satisfy the common assumptions used in option pricing theory such as market completeness, bounded variation, perfect liquidity, etc.

Learning Outcomes:

(1b) Understand the importance of the no-arbitrage condition in assert pricing.

(1c) Apply the concept of martingale in asset pricing.

(1d) Understand Ito integral and stochastic differential equations.

(1e) Understand and apply Ito's Lemma.

(2d) Define and apply the concepts of martingale, market price of risk and measures in single and multiple state variable contexts.

(2e) Understand and apply Girsanov's theorem in changing measures.

Sources:

An Introduction to the Mathematics of Financial derivatives, Neftci, Salih, 2nd Edition, Chapters 2, 3, 6, 10, 12 14 and 15

Frequently Asked Questions in Quantitative Finance, Wilmott, Paul, 2nd Edition, page 113

Quantitative Finance, Wilmott, Paul 2nd Edition, Chapter 6

Commentary on Question:

Commentary listed underneath question component.

Solution:

(a) State the Girsanov's Theorem.

Commentary on Question:

Candidates generally answered this question well. The question was seeking a definition of Girsanov's Theorem, so candidates answering what the Theorem could be used for – while important to know – did not receive credits.

Let Wt be a Brownian motion with probability measure P and sample space Ω. If γt is a pre-visible process satisfying the constraint

2. Continued

$E^p[\exp(1/2\int_0^T \delta^2{}_t dt)] < \infty$

Then there exist an equivalent measure Q such that

$\hat{W}t = Wt + Kt$

$K_t = \int_0^t \delta_s ds$

Where $\hat{W}t$ is a Brownian motion

(b) Show that D_t is not a martingale.

Commentary on Question:

Candidates generally answered this question well, the most straightforward approach was to apply Ito's lemma directly which is the answer shown below, other approaches were accepted too. Because the answer for this question as given upfront, it was particularly important for candidates to show their calculation steps clearly to earn exam points.

Apply Ito's Lemma to $D_t = S_t / (e^{rt})$.

$dD_t = \partial D_t / \partial S_t \, dS_t + \partial D_t / \partial t \, dt + ½ (\delta S_t)^2 \partial^2 D_t / \partial^2 S_t \, dt$

$= 1/e^{rt} [(\mu + ½ \delta^2) S_t \, dt + \delta S_t \, dW_t] - rD_t \, dt + 0$

$= \delta D_t \, dW_t + (\mu - r + ½ \delta^2) D_t \, dt$

Drift term $= (\mu - r + ½ \delta^2) > 0$ since $\mu > r$ and $\delta > 0$

So D_t is not a martingale

(c) Demonstrate how to use Girsanov's Theorem to convert D_t into a martingale.

Commentary on Question:

Candidates who performed well on this question generally understood the progression from parts (a) and (b) of this question, and used that information to quickly identify how Girsanov's Theorem applies and the change of measure needed.

Set $\gamma = (\mu - r + ½ \delta^2) / \delta$

$d\hat{W}_t = dW_t + \gamma dt$

$dW_t = d\hat{W}_t - \gamma dt$

$dD_t = \delta D_t (d\hat{W}_t - \gamma dt) + (\mu - r + ½ \delta^2) D_t \, dt$

2. Continued

$dD_t = \delta D_t (d\hat{W}_t - (\mu - r + ½ \delta^2) / \delta dt) + (\mu - r + ½ \delta^2) D_t dt$

$dD_t = \delta D_t (d\hat{W}_t)$

This has 0 drift under martingale measure Q

(d) Explain how you can construct a replicating strategy for the derivative.

Commentary on Question:
A common error was for candidates to buy units of the derivative itself as part of the replicating strategy – it's important to recognize that while a replicating strategy can use derivatives, purchasing units of the original derivative in question does not constitute a replicating strategy. Similarly, we did not consider answers that invoke the put-call parity to be valid answers as derivative in question was never described as a put or call option.

Because the derivative and underlying asset follow the same stochastic process, a replicating strategy for the derivative can be constructed by entering into $\theta = \partial F / \partial S$ units of the underlying asset S and and $\varphi t = Et - \theta t\ Dt$ of risk-free bond such that the value of the replicating portfolio equals the value of the derivative upfront. At each stage adjust the portfolio to match the delta of the derivative.

3. **Learning Objectives:**

1. The candidate will understand the fundamentals of mathematics and economics underlying quantitative methods in finance and investments.

Learning Outcomes:

(1a) Understand and apply concepts of probability and statistics important in mathematical finance.

(1c) Apply the concept of martingale in asset pricing.

(1d) Understand Ito integral and stochastic differential equations.

(1e) Understand and apply Ito's Lemma.

Sources:

An Introduction to the Mathematics of Financial Derivatives, Neftci, Salih, 2nd Edition, Chapters 5, 6, 8, 9, and 10

Commentary on Question:

Commentary listed underneath question component.

Solution:

(a) State the conditions on the process $\sigma\left(S_t, t\right)$ for which $I\left(T\right)$ is well defined.

Commentary on Question:

Many candidates forgot to mention that $\sigma(S_t, t)$ must be non-anticipative. Some candidates did not give the mathematical definition for a non-explosive process in this context. In general when asking for conditions we seek both descriptions and mathematical expressions where a precise definition is appropriate.

The random variables $\sigma(S_t, t)$ is non-anticipative (they are independent of the future) and they are non-explosive.

$$E[\int_0^T \sigma^2 (S_t, t) dt] < \infty$$

(b) Determine the mean and variance of $I\left(T\right)$.

Commentary on Question:

Many candidates forgot to mention the reason why the mean is 0 – it's important to go beyond simply stating the answer but also explaining the underlying reasoning. Some candidates incorrectly assumed that $\sigma(S_t, t)$ is a constant and gave $\sigma^2 T$ as the variance.

3. Continued

Mean of I(T)= 0 because I(T) is a martingale.

Variance of I(T)= $E[I(T)^2] = E[\int_0^T \sigma^2(S_t, t)dt]$

(c) Determine whether or not $\int_0^T W_{2t} dW_t$ is an Ito-integral.

Commentary on Question:
Most candidates did poorly on this question. Many candidates went through the conditions given in part (a) of this question, but did not have a good understanding of the conditions and how they apply.

The integral is not well-defined since W_{2t} depends on Brownian motion after time t (that is not non-anticipative)

OR

The integral is not well-defined since W_{2t} is not I_t- measurable.

(d) Show that Y_t is a martingale if and only if $V(F_t, t)$ satisfies the partial differential equation.

Commentary on Question:
This is an easy question and most of the candidates got it right. It's important for candidates to realize that in questions where the answer is given, it is particularly important for the candidate to show their working steps in order to get credit for their answers.

By Ito's lemma we have the following.

$$dY = \frac{\partial Y}{\partial F} dF + \frac{\partial Y}{\partial t} dt + \frac{1}{2}\sigma^2 F^2 \frac{\partial^2 Y}{\partial F^2} dt$$

$$\frac{\partial Y}{\partial F} = e^{-rt} \frac{\partial V}{\partial F}$$

$$\frac{\partial Y}{\partial t} = e^{-rt} \frac{\partial V}{\partial t} - re^{-rt} V$$

$$\frac{\partial^2 Y}{\partial^2 F} = e^{-rt} \frac{\partial^2 V}{\partial^2 F}$$

3. **Continued**

$$dY = e^{-rt}\frac{\partial V}{\partial F}(\sigma F\, dW) + \left(e^{-rt}\frac{\partial V}{\partial t} - re^{-rt}V\right)dt + \; e^{-rt}\frac{\partial^2 V}{\partial^2 F}\frac{1}{2}\sigma^2 F^2 dt$$

$$dY = e^{-rt}\left(\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 F^2\frac{\partial^2 V}{\partial^2 F} - rV\right)dt + \; e^{-rt}\sigma F\frac{\partial V}{\partial F}dW$$

Y is a martingale if and only if the drift term vanishes. This means we must have

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 F^2\frac{\partial^2 V}{\partial^2 F} - rV = 0$$

(e) Show that $V(F_t, t) = e^{-r(T-t)}E[g(F_T) \mid I_t]$.

Commentary on Question:
In general, candidates did not do well on this question. Candidates who did well generally understood and demonstrated the application of martingale property in pricing of options.

It is known that Y *is a martingale. This gives us the following.*

$\mathrm{E}[\mathrm{Y_T}|\mathrm{I_t}] = \mathrm{Y_t}$

This means
$\mathrm{E}[\mathrm{e}^{-\mathrm{rT}}\mathrm{V}(\mathrm{F_T},\mathrm{T})|\mathrm{I_t}] = \mathrm{e}^{-\mathrm{rt}}\mathrm{V}(\mathrm{F_t},\mathrm{t})$

We set $\mathrm{V}(\mathrm{F_T},\mathrm{T}) = \mathrm{g}(\mathrm{F})$ to obtain the following.

$\mathrm{E}[\mathrm{e}^{-\mathrm{rT}}\mathrm{g}(\mathrm{F})|\mathrm{I_t}] = \mathrm{e}^{-\mathrm{rt}}\mathrm{V}(\mathrm{F_t},\mathrm{t})$
which gives
$\mathrm{e}^{-\mathrm{rT}}\mathrm{E}[\mathrm{g}(\mathrm{F})|\mathrm{I_t}] = \mathrm{e}^{-\mathrm{rt}}\mathrm{V}(\mathrm{F_t},\mathrm{t})$

which gives
$\mathrm{V}(\mathrm{F_t},\mathrm{t}) = \mathrm{e}^{-\mathrm{r(T-t)}}\mathrm{E}[\mathrm{g}(\mathrm{F})|\mathrm{I_t}]$

4. **Learning Objectives:**

1. The candidate will understand the fundamentals of mathematics and economics underlying quantitative methods in finance and investments.

Learning Outcomes:

(1g) Demonstrate an understanding of the mathematical considerations for analyzing financial time series.

(1h) Understand and apply various techniques for analyzing linear time series.

Sources:

QFIC-100-13: Chapters 1 and 2 of Analysis of Financial Time Series, Tsay, 3rd Edition, pages 19, 37-29, 50 and 56

Commentary on Question:

Commentary listed underneath question component.

Solution:

(a) Determine the mean and variance of the unconditional distribution of r_t.

Commentary on Question:

Many candidates did not do well on this question. Most did not do what was asked and just wrote the formula down from the formula sheet without any derivation.

Normal distribution implies stationary.

Mean = $E[r_t] = E[\phi_0 + \phi_1 r_{t-1} + \varepsilon_t] = \phi_0 + \phi_1 E[r_t] = \dfrac{\phi_0}{1-\phi_1}$

Variance $= V[r_t] = E[r_t^2] - E^2[r_t\ ]$, where

$$E[r_t^2] = E[\phi_0^2 + 2\phi_0\phi_1 r_{t-1} + \phi_1^2 r_{t-1}^2 + \phi_0\varepsilon_t + \phi_1 r_{t-1}\varepsilon_t + \varepsilon_t^2]$$

$$= \phi_0^2 + 2\phi_0\phi_1 \frac{\phi_0}{1-\phi_1} + \phi_1^2 E[r_t^2] + \sigma^2$$

$$= \frac{\phi_0^2}{(1-\phi_1)^2} + \frac{\sigma^2}{1-\phi_1^2}$$

Hence $V[r_t] = \dfrac{\phi_0^2}{(1-\phi_1)^2} + \dfrac{\sigma^2}{1-\phi_1^2} - \left(\dfrac{\phi_0}{1-\phi_1}\right)^2 = \dfrac{\sigma^2}{1-\phi_1^2}$

(b) Develop the log-likelihood function, $\ln f\left(r_2,\ldots,r_T \mid r_1,\theta\right)$, that you will need to maximize to obtain the MLE parameters from all the data provided.

4. Continued

Commentary on Question:
Very few candidates were able to develop the log-likelihood function. More than half understood the concept of translating the function into a summation or multiplication but then were challenged to develop it to the final steps.

$$\ln f(r_2,\ldots,r_T \mid r_1,\theta) = \ln\left(\prod_{j=2}^{T} f(r_j \mid r_{j-1},\theta)\right) = \ln\left(\prod_{j=2}^{T} \frac{1}{\sqrt{2\Pi\sigma^2}} e^{-\frac{(r_j-(\phi_0+\phi_1 r_{j-1}))^2}{2\sigma^2}}\right)$$

$$= \ln(2\Pi\sigma^2)^{-\frac{T-1}{2}} - \sum_{j=2}^{T} \frac{(r_j-(\phi_0+\phi_1 r_{j-1}))^2}{2\sigma^2}$$

$$= -\frac{T-1}{2}\ln 2\Pi - (T-1)\ln\sigma - \frac{1}{2\sigma^2}\sum_{j=2}^{T}(r_j-\phi_0-\phi_1 r_{j-1})^2$$

(c) Determine the system of algebraic equations that needs to be solved to get the MLE parameters for the AR(1) model.

Commentary on Question:
Very few candidates answered this question accurately since they could not answer part (b). About two-thirds of the candidates understood that finding MLEs involved differentiation of the log-likelihood function and solving for the estimator when the differential equation was set to 0. Only a few candidates successfully followed through with the mathematics to get the proper equations for the MLEs.

Differentiate log-likelihood and set to 0 to find MLE

For $\hat{\phi}_0$, the estimator of ϕ_0,

$$\frac{\partial \ln f}{\partial \phi_0} = -\frac{1}{2\sigma^2}\times 2\times\sum_{j=2}^{T}(r_j-\phi_0-\phi_1 r_{j-1})(-1) = 0$$

MLE = $\hat{\phi}_0 = \frac{1}{T-1}\times\sum_{j=2}^{T}(r_j-\phi_1 r_{j-1})$

For $\hat{\phi}_1$, the estimator of ϕ_1,

$$\frac{\partial \ln f}{\partial \phi_1} = -\frac{1}{2\sigma^2}\times 2\times\sum_{j=2}^{T}(r_j-\phi_0-\phi_1 r_{j-1})(-r_{j-1}) = 0$$

4. Continued

$$\text{MLE} = \hat{\phi}_1 = \frac{\sum_{j=2}^{T}(r_j - \hat{\phi}_0)}{\sum_{j=2}^{T} r_{j-1}}$$

For $\hat{\sigma}^2$, the estimator of σ^2,

$$\frac{\partial \ln f}{\partial \sigma^2} = -\frac{(T-1)}{2\sigma^2} + \frac{1}{2\sigma^4}\sum_{j=2}^{T}(r_j - \phi_0 - \phi_1 r_{j-1})^2 = 0$$

$$\text{MLE} = \hat{\sigma}^2 = \frac{\sum_{j=2}^{T}(r_j - \hat{\phi}_0 - \hat{\phi}_1 r_{j-1})^2}{T-1}$$

(d) Calculate the half-life measure of the speed of mean-reversion if the estimated parameters are $\theta = \{0.015,\ 0.7,\ 0.0025\}$.

Commentary on Question:
Half the candidates answered this question or knew the formula but substituted incorrect parameters.

$$\text{half-life} = \ln\frac{0.5}{|\phi_1|} = \ln\frac{0.5}{0.7} = 1.94$$

5. Learning Objectives:

1. The candidate will understand the fundamentals of mathematics and economics underlying quantitative methods in finance and investments.

Learning Outcomes:

(1g) Demonstrate an understanding of the mathematical considerations for analyzing financial time series.

(1h) Understand and apply various techniques for analyzing linear time series.

Sources:

QFIC-100-13: Chapters 1 and 2 of Analysis of Financial Time Series, Tsay, 3rd Edition

Commentary on Question:

Commentary listed underneath question component.

Solution:

(a) Identify the periodicity and the parameters of the time series.

Commentary on Question:
Most candidates successfully answered this part.

Periodicity = 4
$\theta = 0.58$
$\Theta = 0.17$

(b) Consider the forecast origin is 50. Calculate the 1-step ahead prediction of the log earnings.

Commentary on Question:
Less than 10% of the candidates were able to answer this successfully. Most candidates did not write down the proper formula for the 1-step ahead prediction.

1-step ahead prediction of log-earnings is:

$w_{t+1} = (1 – B)(1 – B^4)y_{t+1} = (1 – 0.58B)(1 – 0.17B^4)a_{t+1}$, or

$y_{t+1} - y_t - y_{t-3} + y_{t-4} = a_{t+1} – 0.58a_t – 0.17a_{t-3} + 0.0986a_{t-4}$, so

$y_{51} – y_{50} – y_{47} + y_{46} = a_{51} – 0.58a_{50} – 0.17a_{47} + 0.0986a_{46}$

$y_{51} – 1.19 – 1.40 + 1.06 = a_{51} – 0.58(0.0115) – 0.17(-0.0316) + 0.0986(0.0223)$

$E(y_{51}) – 1.53 = E(a_{51}) + 0.0009$, and $E(a_{51}) = 0$ since $a_t \sim N(0,1)$

$E(y_{51}) = 1.531$

5. Continued

(c) Determine the Autocorrelation Function (ACF) of w_t.

Commentary on Question:
Majority of candidates answered this part. It was a matter of knowing the formulas and substitution.

$$\rho_1 = -\frac{\theta}{1+\theta^2} = -\frac{0.58}{1+0.58^2} = -0.434$$

$$\rho_4 = -\frac{\Theta}{1+\Theta^2} = -\frac{0.17}{1+0.17^2} = -0.165$$

$$\rho_3 = \rho_5 = \frac{\theta \cdot \Theta}{(1+\theta^2)(1+\Theta^2)} = 0.0717$$

$\rho_i = 0$, for $i \neq 1,3,4,5$

6. **Learning Objectives:**

2. The candidate will understand how to apply the fundamental theory underlying the standard models for pricing financial derivatives. The candidate will understand the implications for option pricing when markets do not satisfy the common assumptions used in option pricing theory such as market completeness, bounded variation, perfect liquidity, etc.

Learning Outcomes:

(2j) Demonstrate understanding of interest rate models.

Sources:

An Introduction to the Mathematics of Financial Derivatives, Neftci, Salih, 2nd Edition, Chapter 18

Commentary on Question:

Within the context of modeling the term structure of rates, the question asks candidates to derive formulas related to the conditional expectation and conditional variance of Vasicek spot rate model with a given set of parameters.

Candidates generally did well on part (b) but very poorly on part (a). Candidates who worked on the questions at the end of the chapter would have been better prepared than those who did not.

Solution:

(a) Show for $t < s$ that

- $E\left[r_s \mid r_t\right] = \mu + \left(r_t - \mu\right)e^{-\alpha(s-t)}$
- $Var\left[r_s \mid r_t\right] = \dfrac{\sigma^2}{2\alpha}\left(1 - e^{-2\alpha(s-t)}\right)$

Commentary on Question:

To achieve full credit candidates needed to provide a derivation leading to the conditional expected value and the conditional variance. Identification of the following steps was necessary to achieve full credit. Proportional partial credit was given for identifying some but not all of the steps.

$dr_t = \alpha(\mu - r_t)dt + \sigma dW_t$

$dr_t + \alpha r_t dt = \alpha\mu dt + \sigma dW_t$ rearrange terms

$e^{\alpha t}[dr_t + \alpha r_t dt] = e^{\alpha t}[\alpha\mu dt + \sigma dW_t]$ multiply both sides by $e^{\alpha t}$

$\dfrac{\partial[e^{\alpha t} r_t]}{\partial t} = e^{\alpha t}[\alpha\mu dt + \sigma dW_t]$ take partial derivative

6. Continued

$\int_t^s \frac{\partial[e^{\alpha x} r_x]}{\partial x} dx \quad = \quad \alpha\mu \int_t^s e^{\alpha x}\, dx \quad + \sigma \int_t^s e^{\alpha x}\, dW_x$ integrating from t to s

$$e^{\alpha x} r_x |_{x=t}^{x=s} \quad = \mu e^{\alpha x} |_{x=t}^{x=s} + \sigma \int_t^s e^{\alpha x}\, dW_x$$

$$e^{\alpha s} r_s - e^{\alpha t} r_t \;= \mu e^{\alpha s} - \mu e^{\alpha t} + \sigma \int_t^s e^{\alpha x}\, dW_x$$

$$r_s \;= e^{-\alpha(s-t)} r_t + \mu - \mu e^{-\alpha(s-t)} + \sigma e^{-\alpha s} \int_t^s e^{\alpha x}\, dW_x$$

$$r_s \;= \mu + (r_t - \mu) e^{-\alpha(s-t)} + \sigma e^{-\alpha s} \int_t^s e^{\alpha x}\, dW_x$$

From the above equation, the conditional mean and variance are obtained

$E\,[r_s|r_t] = \mu + (r_t - \mu)e^{-\alpha(s-t)}$ as $E\left[\sigma e^{-\alpha s} \int_t^s e^{\alpha x}\, dW_x\right] = 0$

$$Var\,[r_s|r_t] = Var\,\left[\sigma e^{-\alpha s} \int_t^s e^{\alpha x}\, dW_x\right]$$

$$= \sigma^2 e^{-2\alpha s} \langle \int_t e^{\alpha x}\, dW_x \rangle s$$

$$= \sigma^2 e^{-2\alpha s} \int_t^s e^{2\alpha x}\, dx$$

(b) Describe what these two equations imply for the conditional mean and variance of spot rates as $s \to \infty$.

Commentary on Question:
Utilizing the formulas identified in part (a) candidates needed to identify the boundary effects of $s \to \infty$.

As $s \to \infty$. for a fixed t, the terms $e^{-\alpha s}$ and $e^{-\alpha(s-t)}$ converge to zero since α is a positive constant.

Therefore the conditional mean approaches μ and the conditional variance approaches $\frac{\sigma^2}{2\alpha}$

7. Learning Objectives:

2. The candidate will understand how to apply the fundamental theory underlying the standard models for pricing financial derivatives. The candidate will understand the implications for option pricing when markets do not satisfy the common assumptions used in option pricing theory such as market completeness, bounded variation, perfect liquidity, etc.

Learning Outcomes:

(2k) Understand the concept of calibration and describe the issues related to calibration.

Sources:

Quantitative Finance, Wilmott, Paul, 2nd Edition, Chapter 17

Commentary on Question:

Commentary listed underneath question components.

Solution:

(a) Describe the reasons for and against yield curve fitting using a one-factor interest rate model.

<u>**For**</u>

- Building blocks for bond pricing equation are delta hedging and no arbitrage
  - To use 1-factor model correctly must abide by delta hedging assumptions
- Buying and selling must be done at market prices not theoretical prices
  - Hedging with model output prices markedly different from market prices is useless
  - Model collapses and cannot be used to prices other instruments
- Once model fitted with prices of traded products can dynamically or statically hedge with these products
  - Working theory: Even if model is “wrong” and lose money on contracts will make money back on hedging instruments

<u>**Against**</u>

- Constant refitting required
  - Even over time periods as short as one week
- Value of $\eta^*(t)$ depends on slope of the market yield curve at the short end. $\eta^*(t^*)$ is large and positive.
  - Slope of $\eta^*(t)$ depends on curvature of yield curve at the short end. Derivative of $\eta^*(t)$at t = t* is large and negative.
- No 1-factor model can capture high slope and curvature that is usual for yield curves

7. **Continued**

(b) Show that the functional form for $\eta(t)$ must be

$$\eta^*(t) = c^2\left(t - t^*\right) - \frac{\partial^2}{\partial t^2}\log Z_M\left(t^*; t\right)$$

in order for zero coupon bonds for all maturities to have the correct value.

where

- t^*, to be fitted, is today's date, and
- $Z_M(t^*;t)$ are the discount factors in the market.

$$Z_M(t^*;T) = e^{A(t^*;T) - r^*(T-t^*)}$$

Taking log of both sides in the equation above

$$A(t^*;T) - r^*(T - t^*) = \log\left(Z_M(t^*;T)\right)$$

Substituting terms for $A(t^*;T)$

$$-\int_{t*}^{T} \eta^*(s)\ (T-s)ds + \frac{1}{6}c^2(T-t^*)^3\ - r^*(T-t^*) = \log\left(Z_M(t^*;T)\right)$$

Rearranging terms

$$\int_{t*}^{T} \eta^*(s)\ (T-s)\,ds \quad = -\log\left(Z_M(t^*;T)\right) - r^*(T-t^*) + \frac{1}{6}c^2(T-t^*)^3$$

Differentiating once with respect to T

$$\eta^*(T)\ (T-T)\ + \int_{t*}^{T} \eta^*(s)\ ds \quad = -\frac{\partial}{\partial T}\log\left(Z_M(t^*;T)\right) - r^* + \frac{1}{2}c^2(T-t^*)^2$$

Differentiating again with respect to T

$$\eta^*(T)\ = -\frac{\partial^2}{\partial T^2}\log\left(Z_M(t^*;T)\right) + \ c^2(T-t^*)$$

(c) Express $A(t;T)$ in terms of $Z_M(t^*;T)$, $Z_M(t^*;t)$, t, T and c.

Commentary on Question:
Many candidates responded to "Express" by simply copying the formula from the Formula packet with little or no attempt to derive.

7. Continued

$$A(t;T) = -\int_t^T [c^2(s-t^*) - \frac{\partial^2}{\partial s^2}\log Z_M(t^*;s)](T-s)ds + \frac{1}{6}c^2(T-t)^3$$

Expressing $A(t;T)$ in two terms:

$A(t;T) = I + II$

where

$I = -\int_t^T c^2(s-t^*)\,(T-s)\,ds + \frac{1}{6}c^2(T-t)^3$

$II = \int_t^T \frac{\partial^2}{\partial s^2}\log Z_M(t^*;s)\,(T-s)\,ds$

For the first term I, we let $u = T-s$ in calculating the integral

$$I = -\int_{T-t}^{T-T} c^2(T-u-t^*)\,u\,d(-u) + \frac{1}{6}c^2(T-t)^3$$

$$= c^2\int_0^{T-t}[u^2 - (T-t^*)u]\,du + \frac{1}{6}c^2(T-t)^3$$

$$= c^2[\frac{1}{3}(T-t)^3 - (T-t^*)\,\frac{1}{2}(T-t)^2] + \frac{1}{6}c^2(T-t)^3$$

$$= -c^2(T-t^*)\,\frac{1}{2}(T-t)^2 + \frac{1}{2}c^2(T-t)^3$$

$$= -\frac{1}{2}c^2(t-t^*)(T-t)^2$$

For the second term II, we integrate by parts to get

$$II = \int_t^T (T-s)\,d\frac{\partial}{\partial s}\log Z_M(t^*;s)$$

$$= [(T-s)\frac{\partial}{\partial s}\log Z_M(t^*;s)]_{s=t}^{s=T} - \int_t^T \frac{\partial}{\partial s}\log Z_M(t^*;s)\,(-ds)$$

Thus

$$II = -(T-t)\frac{\partial}{\partial t}\log Z_M(t^*;t) + \log Z_M(t^*;T) - \log Z_M(t^*;t)$$

It follows that

$$A(t;T) = \log\left(\frac{Z_M(t^*;T)}{Z_M(t^*;t)}\right) - (T-t)\frac{\partial}{\partial t}\log Z_M(t^*;t) - \frac{1}{2}c^2(t-t^*)(T-t)^2$$

8. **Learning Objectives:**

3. The candidate will understand how to evaluate situations associated with derivatives and hedging activities.

Learning Outcomes:

Sources:
Managing Investment Portfolios: A dynamic Process, Maginn & Tuttle, 3rd Edition

- Chapter 6 Fixed Income Portfolio Management, pages 373 - 384

Commentary on Question:
Candidates did relatively well on this question.

Solution:
(a) Describe how the following derivatives can be used to manage the duration of a bond portfolio and comment on their advantages and disadvantages.

(i) Interest Rate Futures

(ii) Interest Rate Swaps

(iii) Interest Rate Options

Futures Contracts
Prices of interest rate futures contracts are negatively correlated with changes to interest rates
Buying a futures contract will increase exposure to interest rate risk and the portfolio's duration will increase.
On the other hand selling futures will lower interest rate risk and lower duration.
Advantages to using futures contracts to manage duration include:

- For duration reduction, shorting the contract is very effective
- Portfolio Duration can be adjusted by buying (to increase duration) or selling (to increase duration) futures contracts.

They are very useful in interest rate anticipation strategies where the portfolio's duration is adjusted in anticipation of future rate changes
Risk to the value of a cash position can be hedged by selling futures contracts.
As rates drop Futures contracts will change in the same direction as bonds
The hedge ratio determines the correct number of futures to sell so that changes to portfolio value are offset by changes in the short position.
Cross hedging occurs when the bonds in the portfolio differ from those used for the futures contract.
Cross hedging introduces basis risk

8. **Continued**

<u>**Swaps**</u>
An interest rate swap is a contract between two parties to exchange periodic interest payments on a specified dollar amount of principal

Alternatively candidates may say swap involves paying fixed while receiving floating on a notional principle.
Swaps can be used to more closely match an insurer's asset and liability cash flows
Swaps are a very efficient method from a transactions cost perspective

<u>**Interest Rate Options**</u>
Duration interest rate options depends on the duration of the underlying instrument and the option's delta (the price responsiveness of the option to changes in the underlying instrument and the leverage of the option)
Protective put buying and covered call writing both protect against interest rate increases
Put buying establishes a floor value for the portfolio while allowing the manager to benefit from rate declines

Call writing does not fully protect against rate increases but provides some income to offset losses
Call writing also eliminates potential for gains from falling rates.

(b) Calculate the number of futures contracts the portfolio manager should buy or sell to achieve the target duration.

Initial market value of the portfolio = 1+2+1,5=4.5 million

Initial portfolio dollar duration =(A duration * A MV + B duration * B MV + C duration * C MV)
= (5*1 + 7*2 + 10*1.5) = 34 million
Target dollar duration of the portfolio
= dollar duration of the bond portfolio without futures + dollar duration of the futures

Dollar duration of the futures = dollar duration per future contract * number of future contract * price of the cheapest to deliver bond /conversion factor
=6*X*100,000/1.1

24.3= 34 +0.6x/1.1
X = (24.3-34)*1.1/.6=-17.7

The fund would need to sell 18 future contracts.

9. **Learning Objectives:**

2. The candidate will understand how to apply the fundamental theory underlying the standard models for pricing financial derivatives. The candidate will understand the implications for option pricing when markets do not satisfy the common assumptions used in option pricing theory such as market completeness, bounded variation, perfect liquidity, etc.

3. The candidate will understand how to evaluate situations associated with derivatives and hedging activities.

Learning Outcomes:

(2a) Demonstrate understanding of option pricing techniques and theory for equity and interest rate derivatives.

(2j) Demonstrate understanding of interest rate models.

(3c) Understand the different approaches to hedging.

(3d) Understand how to delta hedge and the interplay between hedging assumptions and hedging outcomes.

Sources:
Quantitative Finance, Wilmott, Paul, 2nd Edition, Chapter 8, The Black-Scholes formulae and the "Greeks"

Commentary on Question:
This question tests the understanding of the Black-Scholes model and how to use the delta. Unfortunately, almost ½ of candidates skipped this question or got zero. Of the others – most only answered part (d).

Solution:

(a) Calculate the static hedging budget for the first year to achieve the target profit margin and expenses.

Commentary on Question:
Common errors were the use of 3% cap rather than 2% interest rate, and forgetting to include the minimum guarantee of 0.5%

Earned Rate – minimum guaranteed rate – profit margin+expenses
= 2.0% - 0.5% - 0.9% = 0.6%

(b) Recommend a static hedging strategy for this crediting rate policy.

9. Continued

Commentary on Question:
Most students did not provide the name of the strategy – just the actions – we felt this was sufficient even though question writer originally required it. There were a number of different ways of achieving the strategy – the following were the two most popular.

Call-spread strategy
buy 1-year atm call option (strike 100.5)
sell 1-year otm call option where otm strike = crediting rate cap

Put-spread strategy
buy1-year atm put option (strike 100.5)
sell 1-year otm put option where otm strike = crediting rate cap

(c) Determine the first year cap of the crediting rate (ACRC) given the profit margin and expenses.

Commentary on Question:
There were two alternatives to this as well – two different formulas for the DELTA of the Black-Scholes model – both appeared on the Formula sheet. Many students did not read correctly and calculated the annual crediting rate rather than the cap (ACRC). Some calculated the option price.

cost of atm call option = stike K= S0*1.005

DELTA= N(d1)
d1 = {ln(S/K) + [(r-D) +1/2 □^□□□* t } / □□sqrt(t)
={ln(1/1.005) + [(0.02-0.03) + 0.04/2]* 1 } / 0.2
= {-0.005-0.01 + 0.02} / 0.2 = 0.025
DELTA = N(d1) = N(0.0025) = 0.510
0.6% = change in strike * Delta * 1% = change in strike * 0.510 * 1%
        = change in strike * 0.510% = 0.6%/0.510% = 1.176
Short Strike = 101.68
Cap = 1.68%

DELTA = e^(-d(T-t) * N(d1)
DELTA = e^-.03 *N(d1) =.495
…. =.6/.495 = 1.212
Short Strike = 101.72
Cap = 1.72%

(d) Assess the disintermediation risk and recommend a risk mitigation strategy.

9. Continued

Commentary on Question:
If the candidate read the full question – they almost always got the full answer. Only one risk mitigation strategy was required although there were many options – the two below were most popular.

Disintermediation = as interest rate increase, policyholders will surrender or borrow to invest at higher rates elsewhere.
Company will be forced to sell bonds at depressed prices and lose money.

Recommendations

- Add mva feature
- Hedge with payer swaption, cap, etc

10. Learning Objectives:

Learning Outcomes:
"Asset Allocation," by Sharpe, Chen, Pinto, and McLeavey, Chapter 5

Sources:

Commentary on Question:
Although this was a relatively easy question, many candidates struggled to provide the correct answers.

Solution:
(a) Describe the portfolio segmentation approach.

Portfolio segmentation is:
- Creation of subportfolios within the general account portfolio according to the company product mix offerings;
- Group liabilities in accordance to product line of business;
- Or based on certain characteristics, such as duration;
- Portfolios are then constructed by segment and each has its own objectives;
- Based on expected return, interest rate risk, credit risk characteristics, liquidity etc...

(b) Recommend and justify your recommendation to Mr. Yao to adopt the portfolio segmentation approach.

Segmentation offers the following benefits to JHN:
- Company offers a diversity portfolio of products from traditional whole life, UL, VUL to fixed and variable annuities;
- Segmentation provides a focus for meeting return objectives by product line;
- Provides a simple way to allocate investment income by line of business;
- Provides more accurate measurement of profitability by line of business;
- Aids in managing interest rate risk and/or duration mismatch by product line;
- Assists regulations and senior management in assessing the suitability of investments.

(c) Recommend a tactical asset allocation for the participating insurance segment that incorporates Mr. Yao's plan and the segment's objectives.

10. Continued

- The portfolio has an implicit annual guarantee of 5%, so the new asset mix has to still earn over 5%;
- Current portfolio return of 5.6% and 5% return gives 60bps differential;
- Maximum reallocation from private to public allowed:
- 60bps = 8bps yield reduction * X% + 20bps cost of tactical adjustments;
- X% = 5%;
- So based purely on return and cost, we can reallocate 5% from private to public;
- That is, 25% in private bond and 35% in public bond;
- However, public and private bond has a permissible range of 26% to 34%;
- Hence, the maximum reallocation is only 4%;
- Resulted tactical asset allocation:
- 2% Cash and Short Equivalents; 34% public; 26% private; 20% mortgage; 18% equities.

(d) Evaluate what additional information should be considered before implementing a tactical asset allocation for JNH.

Additional considerations to implement tactical asset allocation:

- The increase in tracking risk and the change in expected absolute risk in relation to the expected benefits;
- The logic of the tactical assessment, if it's still hold true over time;
- Cost of adjustment versus the benefits;
- Diversification impacted after reallocation.

(e) Describe three factors that affect the price at which an ETF trades.

The value of the underlying securities in the fixed income ETF portfolio;
The flow factor: the balance of ETF flows in the market drives how much of the total creation cost is priced into the ETF as well as the ETF bid/ask spread;
Market Liquidity;
Market volatility.

11. **Learning Objectives:**

4. The candidate will understand and identify the variety of fixed instruments available for portfolio management.

Learning Outcomes:

(4h) Construct and manage portfolios of fixed income securities using the following broad categories:
   (i) Managing funds against a target return
   (ii) Managing funds against liabilities.

Sources:

Managing Investment Portfolios: A Dynamic Process, Maginn & Tuttle, 3rd Edition

- Chapter 6 fixed Income Portfolio Management, pgs 331 - 339

Commentary on Question:

Commentary listed underneath question component.

Solution:

(a) Describe the considerations that come in play when selecting a benchmark.

Commentary on Question:
The scores are low on this part. Many candidates did refer to the general qualities of a good benchmark (unambiguous, measurable, owned…) but a few did replicate the expected answer for a bond index.

The following considerations are important when selecting a benchmark:

- The market value risk of the portfolio and the benchmark should be comparable;
- The portfolio and benchmark should provide comparable assured income streams;
- The liability framework risk should be minimized;
- However the bond market index portfolio is not a real portfolio.

(b) List and describe the various strategies that can be used when creating a fixed income portfolio based on an index.

Commentary on Question:
Credit was given for describing all strategies that can be used to build a fixed income portfolio with an emphasis on indexing techniques.

11. Continued

1. **Pure bond Indexing (or full replication approach):** owing all bonds in the index in the same percentage as the index;
2. **Enhanced indexing by matching primary risk factors:** such as level of interest rates, twists in yield curve, change in spread between treasuries and non-treasuries;
3. **Enhanced indexing by small risk factor mismatches:** tilt the portfolio in favor of any other risk factors while matching duration;
4. **Active management by larger risk factor mismatches:** actively pursuing opportunities in the market to increase return;
5. **Full-blown active management:** aggressive mismatches on duration, sector weights and other factors.

In order to match the portfolio's risk exposures to those of the benchmark index we can use a cell-matching technique (stratified sampling) which divides the benchmark index into cells representing risk factors, or a multifactor technique that makes use of a set of factors that drive bond returns.

(c) Identify the advantages and disadvantages of active management of fixed-income portfolios.

Advantages of active management:
Expectation of a higher return than the benchmark: ["The portfolio's return should increase if the manager's forecast of the future path of the factors that influence fixed-income returns are more accurate than those reflected in the current prices of fixed-income securities"]

Disadvantages of active management:
Investor must be willing to accept a large tracking risk: ["Outperforming an index on a consistent basis is a difficult task"].
Active managers have a set of activities that they must implement that passive managers are not faced with (implies costs) such as:

- Identify which index mismatches are to be exploited;
- Extrapolate the market's expectations (or inputs) from the market data;
- Independently forecast the necessary inputs and compare these with the market's expectations and;
- Estimate the relative values of securities in order to identify areas of under- or overvaluation.

12. **Learning Objectives:**

3. The candidate will understand how to evaluate situations associated with derivatives and hedging activities.

Learning Outcomes:

(3c) Understand the different approaches to hedging.

Sources:

Quantitative Finance, Wilmott, Paul, 2nd Edition. Chapter 18, Interest Rate Derivatives

Commentary on Question:

This question tests the understanding of interest rates derivatives and how they can be used to hedge the risks inherent in an annuity product with a variety of guarantees. For full marks - part c) it was necessary to understand the implication of the product changes on the risk and address those rather than just repeat was said in (b)

Solution:

(a) Define and describe the payment structure of the following derivative instruments:

(i) Cap & Floor
(ii) Receiver Swaption & Payer Swaption
(iii) Index amortization swap

Commentary on Question:

Many candidates chose to define OR describe the payment structure, rather than both. Some candidates chose to define the payment structure for the impact of owning an investment and a cap, rather than the cap alone – we gave full points for this. For (ii) it was necessary to specify "the right to" enter the swap

(i) A cap is a contract that guarantees its holder that otherwise floating rates will not exceed a specified amount. The variable rate is capped.

A cap's cash flow is max (rl – rc, 0), multiplied by the principal. rl is the basic floating rate, and rc is the fixed cap rate. These payments continue for the lifetime of the cap. The rate rl to be paid at time t is set at time t-1.

A floor is similar to a cap except that the floor ensures that the interest rate is bounded below.
A floor's cash flow is max(rf – rl, 0), multiplied by the principal, where rf is the fixed floor.

12. Continued

(ii) Swaption has a strike rate, R_E, that is the fixed rate that will be swapped against floating rate if the option is exercised

Call swaption / (fixed) payer swap where the buy has the RIGHT to become the fixed payer.

Put swaption / (fixed) receiver swap where the buy has the RIGHT to become the fixed reciever.

(iii) In an Index amortizating rate swap the amount of principal (notional) decreases, or amortizes according to the behavior of an 'index' over the life of swap; typically, that index is a short-term interest rate.

(b) Evaluate and recommend trades for ABC using interest rate derivatives listed above if:

(i) ABC portfolio consists of 10-year floating rate bond
(ii) ABC portfolio consists of 10-year fixed rate bond
(iii) ABC portfolio consists of 10-year residential mortgage pass-throughs

Commentary on Question:
Many candidates did not realize that they needed to identify the specific Company ABC annuity product risks to be addressed and then recommend how they may be hedged.

(i) Minimum guarantee at issue – purchase floor
Minimum guarantee rate at the renewal - enter additional 5 year reciever swap at the time of renewal
For disintermediation risk, - buy some amount of 5-year cap to protect from upside

(ii) minimum guarantee rate at issue – no requirement (assume fixed rate = minimum rate)
Minium guarantee at the renewal → enter 5 year payer swap at the time of renewal
For disintermediation risk, buy some amount of 5-year cap to protect from upside

(iii) Need additional protection for the prepayment risk – use Index amortizing rate swap
Also buying floor or selling cap will be good hedging for the prepayment risk
Similar interest rate risk management for (i) or (ii)

12. Continued

(c) Evaluate and recommend trades for ABC using interest rate derivatives listed above if:

(i) ABC portfolio consists of 10-year floating rate bond
(ii) ABC portfolio consists of 10-year fixed rate bond

Commentary on Question:
Many students were confused as to how this question was different from (b) and mostly just repeated their answer for (b).

Market value adjustment with principal guarantee does not eliminiate disintermediation risk
For the principal guarantee, buy additional interest rate that would pay if the rate increases (payer swaption, or cap)

(i) Same issue re: minimum interest guarantee and disintermediation as per (i) above.
(ii) Same issues re: minimum interest guarantee and disintermediation as per (ii) above.

13. Learning Objectives:

5. The candidate will understand the variety of equity investments and strategies available for portfolio management.

Learning Outcomes:

(5b) Demonstrate and understanding of the basic concepts surrounding passive, active, and semi-active (enhanced index) equity investing, including managing exposures.

(5d) Demonstrate an understanding of equity indices and their construction, including distinguishing among the weighting schemes and their biases.

Sources:

Managing Investment Portfolios: A dynamic Process, Maginn & Tuttle, 3rd Edition

- Chapter 7 Equity Portfolio Management

Commentary on Question:

Although the question was addressing relatively easy topics, not many candidates were able to provide satisfactory answers.

Solution:

(a) Identify and describe the structure of the portfolio of investment managers used by this pension fund.

- The portfolio of managers represents a core-satellite portfolio.
- An indexed investment manager (Manager A) representing more than half of the portfolio's value and functions as the core manager.
- Actively managed portfolio managers representing the satellite portfolios surrounding the core.

(b) Evaluate whether this portfolio of managers is expected to meet the trustees' investment objectives.

- Expected Alpha = 75/500 * 0.50% + 75/500 * 1.5% + 50/500 * 4% = .075 + .225 + 0.4 = 0.7%
- Tracking Risk = [(75/500 * 1%)^2 + (75/500 * 5%)^2 + (50/500 * 9%)^2]^(1/2) = 1.18%
- Information Ratio = 0.7% / 1.18% = 0.593
- Tracking risk of 1.18% satisfies the trustee's objective of tracking risk of no more than 2% annually. Information ratio falls just short of the trustee's target.

13. **Continued**

(c)

(i) Describe the concepts of "true" and "misfit" active return and risk for a manager and the circumstances when these concepts are useful.
(ii) Evaluate Manager C's performance based on his "true" information ratio.
(iii) Assess Lawrence's plan to minimize "misfit" risk of each manager.

Section (i)

- Manager's "true" active return = Manager's return – Manager's normal benchmark return.
- Manager's "misfit" active return = Manager's normal benchmark return – Investor's benchmark return.
- Manager's true active risk = Standard deviation of true active return.
- Manager's misfit active risk = Standard deviation of misfit active return.
- These concepts are useful when we consider managers that have different investment styles.
- True/misfit distinction may be used in performance appraisal and/or in optimization of a portfolio of managers.

Section (ii)

- Manager's C true active return is (7% + 1.5% - 5%) = 3.5%
- Manager's C true active risk is obtained by solving the following equation:
- 5% = sqrt(X^2 + 4%^2). X = 3%
- Manager's C true information ratio is 3.5% / 3% = 1.17
- Since S&P 500 Growth index represents Manager C's selection universe, the above calculated information ratio shows that his performance during this period has been better than if we had measured Manager's C information ratio using portfolio benchmark.

Section (iii)

- This may not be the most optimal approach.
- By disaggregating the active risk and return in two component, it is possible to create optimal solutions that maximize total active return at every level of total active risk and that also allow for the optimal level of "misfit" risk .
- A high level of true active return may more than compensate for a given level of "misfit" risk.

(d) Recommend an approach that would allow Park to keep broad sector weightings in line with the benchmark while retaining the alpha from active management.

- Park should consider establishing a completeness fund for the equity portfolio.
- A completeness fund when added to active managers' positions establishes an overall portfolio with approximately the same sector weightings as the investor's overall equity benchmark.

13. **Continued**

- The completeness fund may be constructed with an objective of making the overall portfolio sector/style neutral with respect to the benchmark while attempting to retain the value added from the active manager's stock-selection ability.

(e) Propose a fee structure so that the managers' interests are aligned with those of the pension fund while their incentive to take a high level of risk is limited.

- The fee schedule should be performance-based. This will align the manager's interests to the fund performance.
- The structure can be set up as a combination of a base fee based on the size of asset under management plus an additional fee based on realized positive alpha.
- A fee cap and high water mark should be included to limit managers' incentive to pursue high risk strategies.
- A fee cap will limit the total fee paid regardless of performance.
- A high water mark would require the portfolio manager to have cumulatively generated a minimal level of performance before he/she is paid any fee.

14. Learning Objectives:

4. The candidate will understand and identify the variety of fixed instruments available for portfolio management.

7. The candidate will understand the theory and techniques of portfolio asset allocation.

Learning Outcomes:

(4h) Construct and manage portfolios of fixed income securities using the following broad categories:
  (i) Managing funds against a target return
  (ii) Managing funds against liabilities.

(7a) Explain the impact of asset allocation, relative to various investor goals and constraints.

(7b) Propose and critique asset allocation strategies.

(7d) Incorporate risk management principles in investment policy and strategy, including asset allocation.

Sources:
Chapter 5 "Asset Allocation," by Sharp, Chen, Pinto and McLeavey

Commentary on Question:
Commentary listed underneath question component.

Solution:
(a)

(i) State the return and risk objectives of EDF endowment fund.

(ii) Identify the relevant considerations and constraints for EDF endowment fund.

Commentary on Question:
For (i) many candidates used additive method, but this had little impact on scoring. For (ii) credit was also given for mentioning taxes, legal and regulatory, unique circumstances).

(i) Return objectives: earn (1.03)*(1.03)*(1.004)-1=6.51% annual return
Risk objectives: standard deviation of portfolio <= 9%
Minimize probability of falling under 4.5% return

(ii) Considerations and constraints: preserve capital and purchasing power, minimize liquidity, minimize downside risk

14. Continued

(b) Using the results of the mean-variance analysis summarized above:

(i) Determine the composition of the efficient portfolio with the expected return equal to the EDF objective, assuming that only the four asset classes considered by the trustees are permissible.

(ii) Determine and justify the overall most appropriate strategic asset allocation for EDF, assuming that T-bills are also a permissible asset class.

Commentary on Question:
Many candidates stopped after finding the tangency portfolio, and suggested a combination with T-bills. This would have received partial credit.

Portfolio 2 expected return = 8.0% ((.58*.06)+(.27*.10)+(.15*.12))=.0798
Portfolio 3 standard deviation = 16.0%
Efficient portfolio is a combination of corner portfolios 1 and 2
Solve for w, where 6.5=w*5 + (1-w)*8
W=0.5
Bonds: 0.5*82+0.5*0=41
Mortgages: 0.5*5+0.5*58=31.5
Large Cap Equities: 0.5*7+0.5*27=17
Small Cap Equities: 0.5*6+0.5*15=10.5
Consider the Sharpe Ratio of the three portfolios:
Portfolio 1: (5%-2%)/5.9% = .508
Portfolio 2: (8%-2%)/9.5% = 0.632
Portfolio 3: (10.5%-2%)/16% = 0.531
Portfolio 2 has the highest Sharpe Ratio, so it is the tangency portfolio
The optimal portfolio is a combination of the tangency portfolio and T-bills
First find portfolio A with st. dev. = 9%
9 = w*9.5, so w = 0.947
Exp. return for portfolio A is 8*0.947 + 2*(1-0.947) = 7.682
Also consider portfolio B that minimizes st. dev. subject to meeting return objective
Solve for w, where 6.5 = w*8 + (1-w)*2
w = 4.5/6 = 0.75
St. dev. for portfolio B is 0.75*9.5 = 7.13
Consider the Roy's Safety criterion of portfolios A & B:

- Portfolio A: (7.682%-4.5%)/9% = 0.35
- Portfolio B: (6.5%-4.5%)/7.13% = 0.28

Both portfolios have the same Sharpe Ratio, but for portfolio A Roy's Safety Criterion Ratio is higher than for portfolio B
The higher Roy's ratio is the smaller the prob. that return will fall below the threshold

14. Continued

So portfolio A is the optimal portfolio.
Allocation:
Bond: 94.7% * 0% = 0%
Mortgage: 94.7% * 58 = 54.9%
LC Equities: 94.7% * 27% = 25.6%
SC Equities: 94.7% * 15% = 14.2%
T-bills: 5.3%

(c) Critique the mean-variance optimization approach in setting asset allocation.

Commentary on Question:
Credits were also given with the second set of following answers.

- Composition of efficient portfolios is very sensitive to small changes in inputs
- Forecasting returns, volatilities, and correlations is difficult subject to substantial estimation error

OR

- Covers only single period, no re-balancing (static, not dynamic)
- No liability is considered
- Does not consider changes in future volatility and correlations
- Does not factor in investor's view
- Does not allow short sales
- May concentrate on certain asset classes (not diversified)
- Assumes returns are normal distribution which usually is not the case

(d) Describe briefly the process of the following alternative approaches to setting optimal asset allocation:

(i) Black-Litterman

(ii) Monte Carlo Simulation

(iii) Experience Based

Commentary on Question:
Most candidates do fairly well on 14 (d).

14. **Continued**

(i) Black-Litterman

- Define equilibrium market weights and covariance matrix for all asset classes
- Back-solve equilibrium expected returns
- Express views and confidence for each view
- Calculate the view adjusted market equilibrium return
- Run mean-variance optimization

(ii) Monte Carol Simulation

- Involves the calculation and statistical description of the outcomes resulting in a particular strategic asset allocation
- Under random scenarios for investment returns, inflation, and other relevant variables
- Evaluate the possible range of investment outcomes
- As well as the likelihood that each will occur
- Choose the most appropriate scenarios to fit the purpose

(iii) Experience Based (will receive full credit with any 4 of the answers below)

- Rely on traditions, experience, and rules of thumb in making strategic asset allocation recommendation
- A 60/40 stock/bond asset allocation is appropriate or at least a starting point for an average investor's asset allocation
- The allocation to bonds should increase with increasing risk aversion
- Investors with longer time horizons should increase their allocation to stocks
- A rule-of-thumb for the percentage allocation to equities is 100 minus the age of investor
- Yong investors, an aggressive portfolio is more appropriate due to longer investment time horizon

15. Learning Objectives:

2. The candidate will understand how to apply the fundamental theory underlying the standard models for pricing financial derivatives. The candidate will understand the implications for option pricing when markets do not satisfy the common assumptions used in option pricing theory such as market completeness, bounded variation, perfect liquidity, etc.

Learning Outcomes:

(2a) Demonstrate understanding of option pricing techniques and theory for equity and interest rate derivatives.

Sources:

The Handbook of Fixed Income Securities, Fabozzi, Frank, 8th Edition, Chapter 16, 18, and 21

Commentary on Question:

The candidates' scores were low on this question, especially on part (a). Many could not correctly derive the asset portfolio cash flows, as they did not realize that first they need to determine the amount of at-issue ILS face value purchased.

Solution:

(a) Derive the following cash flows assuming that Bill's predictions will materialize. Show all calculations:

(i) Liability cash flows
(ii) Expected asset portfolio cash flows

(i) The GIC (Guaranteed Investment Contract) has a 2.5% annual credited interest.
The annual cash flow = 2.5% * 100 million = $2.5 million
The principal is paid back at the end of the term = $100 million

Year	20x3	20x4	20x5	20x6
GIC $CFs	$100m	- $2.5m	- $2.5m	- $102.5m

(ii)

- The solution of this section consists of the following four steps.
- Step 1. Calculate the Price, P, to be paid to by "$1 at-issue ILS face value" on January 1, 2013. P = a x b.

  "a" is the change – increase - in value due to real interest rate decreasing from 2% in 20x1 to 1% in 20x3 and is calculated as follows:

15. Continued

i = 1%
c = 2%
N = 8

a = (c/i)* (1-1/(1+i)^N)+1/(1+i)^N = 1.0765.

"b" is the change – increase – in value due to changes in prices from 20x1 to 20x3:

b = 230/220 = 1.0455.

P = a x b = 1.0765 x 1.0455 = $1.1254.

Therefore, with $100 million, we can buy (100/P) = 100/1.1254 = $88.853 millions of "20x1 at-issue ILS face values".

- Step 2: What will be the sale price of $88.853 mln at-issue ILS in 20x6:
  = $88.853 * (5 * 0.02 + 1) *(239/220) = $106.18 mln
- Step 3: Coupon payments we will receive by holding ILS for three years:
  - Coupon received in 2004: $88.853 * 0.02 * (225/220) = $1.8174 m
  - Coupon received in 2005: $88.853 * 0.02 * (232/220) = $1.8742 m
  - Coupon received in 2006: $88.853 * 0.02 * (239/220) = $1.9305 m

- Step 4: We conclude that the cash flow from 2003 to 2004 will be:

Year	20x3	20x4	20x5	20x6
ILS $CFs	- $100m	$1.82m	$1.87m	$1.93m + $106.18m = 108.11m

(b) Explain the reasons why pension plans, endowments and individuals, respectively, use Inflation-Linked Bonds (ILBs).

Reasons of using ILBs for Pension Plans:
ILBs may match the active-life liability well
ILBs can protect a surplus
ILBs can help offset substantial equity risk exposure
ILBs can help reduce variability of annual funding requirements
Reasons of using ILBs for Endowments:
ILBs can help set up return objectives in real terms
ILBs can help truncate downside risk
Reasons of using ILBs for Individuals:
Individuals whose fixed incomes that are vulerable to inflationary erosion can use ILBs to hedge inflation risk and keep retirement living standards

15. Continued

(c) Critique Bill's decision to invest entirely in ILBs to back the GIC.

This decision is based on speculation only
There is no cashflow matching
GIC payments are independent of inflation and level of real rates

Low yield will not generate enough to cover GIC interest payments
If inflation is lower than predicted by Bill and/or real rates do not decline, this investment strategy will lead to losses

(d) Calculate the break-even inflation rate implied by the ILB for the 8-year period beginning on Jan. 15, 20X3, if the nominal yield of the conventional 8-year bond with the same credit rating as the ILB was 2.8% at that date.

nominal yield = 2.8%
TIPS_real_yield = 1.0%

Breakeven inflation rate = $\frac{1+nominal\_yield}{1+TIPS\_real\_yield}-1=1.8\%$

16. **Learning Objectives:**

6. The candidate will understand how to develop an investment policy including governance for institutional investors and financial intermediaries.

Learning Outcomes:

(6c) Determine how a client's objectives, needs and constraints affect investment strategy and portfolio construction. Include capital, funding objectives, risk appetite, and risk-return trade off, tax, accounting considerations and constraints such as regulators, rating agencies, and liquidity.

(6d) Incorporate financial and non-financial risks into an investment policy, including currency, credit, spread, liquidity, interest rate, equity, insurance product, operational, legal and political risks.

Sources:

Managing Investment Portfolios: A Dynamic Process, Maginn & Tuttle, 3rd Edition

- Chapter 3 Managing Institutional Investment Portfolios

Commentary on Question:

This was one of the easiest questions on the exam. Many candidates performed well on this question.

Solution:

(a) Explain how the following factors affect the risk tolerance for a DB pension plan.

(i) Plan status

- -Expressed as the funded status of the plan, e.g., surplus or deficit
- -The greater the funded status, the greater the risk tolerance

(ii) Plan features

- Examples are early retirement option or lump sum distributions
- These options will affect the duration of the plan liabilities
- The shorter the duration of the liabilities, the lower the risk tolerance

(iii) Sponsor financial status

- Expressed as debt to total assets
- Lower debt ratios imply higher risk tolerance

(iv) Sponsor profitability

- Both current and future/expected profitability
- Higher profitability implies a higher risk tolerance

16. **Continued**

(b) Assess the risk tolerance of Buddy Inc.'s DB plan.

Commentary on Question:
The answer for part (b) requires restatement of the concepts from (a) but in this particular context.

The plan funded status is calculated as the assets less the liabilities, i.e. $45million - $40 million = $5 million. This is a surplus.
Avg age 55, avg retirement 65, this implies about 10 years until liabilities due, which is an intermediate time horizon for the investments
Buddy's earnings are stable or increasing, which is positive for risk tolerance.
Overall, the risk tolerance is good/positive/slightly above average due to:

- Surplus of the fund
- Positive profitability outlook
- No early retirement and lump sums

(c) Calculate the maximum portion of the plan assets to be invested in bonds in order to minimize the surplus risk.

α is the fraction to be invested in equities and β reflects the duration of liabilities relative to the bond index.
As such, reflects the uncertainties in the value of liabilities due to changes in interest rates.

Denoting the returns on equity and fixed income at time t as $R_{E,t}$ and $R_{B,t}$, respectively, and the fraction of the surplus invested in equity as ∝, we write the surplus as:

$$S_{t+1} = A_t\left[\alpha\left(1+R_{E,t+1}\right)+\left(1-\alpha\right)\left(1+R_{B,t+1}\right)\right]-L_t\left[1+R_f+\beta\left(R_{B,t+1}-R_f\right)+\varepsilon_{t+1}\right]\left(10A.10\right)$$

$$\beta = \frac{LiabilityDuration}{BondINdexDuration}$$

$$=15/5=3$$

Funding ration $= Lt/At = 40/45$
Based on assumptions that equities track closely to the benchmark Barclay's Global Equity Index and bonds track closely to the benchmark Barclay's Global Bond Index, we have:

$\sigma_B = 6.0\%$

$\sigma_E = 15\%,\ \rho = 0.50$

16. Continued

To minimize surplus risk, the fraction of asset to be invested in equity:

$$\frac{\left(1-\beta\frac{L_t}{A_t}\right)\left(\sigma_B{}^2-\rho\sigma_E\sigma_E\right)}{\sigma_E{}^2+\sigma_E{}^2-2\rho\sigma_E\sigma_E}$$

$=8.80\%$

The portion of assets to be invested in bond, thus, should be at most 91.2%.

(d) Assess the impact on the plan's risk tolerance for each of the following new provisions that Buddy Inc. considers to introduce:

(i) Early retirement at age 55

(ii) Lump sum settlements

Early retirement option

- Increases volatility in terms of timing of payments
- As the average age of the employees is also 55, this shorts the time horizon which reduces the risk tolerance
- Buddy has greater liquidity risk as assets may need to fund retirement benefits immediately
- Buddy has greater shortfall risk as the surplus may be lower since more may be needed to fund liabilities

Lump sum settlements also decrease duration and increase liquidity needs

- Overall, Buddy's risk tolerance is lower and no longer above average

17. **Learning Objectives:**

3. The candidate will understand how to evaluate situations associated with derivatives and hedging activities.

Learning Outcomes:

(3c) Understand the different approaches to hedging.

(3d) Understand how to delta hedge and the interplay between hedging assumptions and hedging outcomes.

Sources:

Quantitative Finance, Wilmott, Paul, 2nd Edition, Chapter 10

Commentary on Question:

For part (b), candidates needed to realize that the profit was deterministic, so no integration was needed. For part (c), the question asked for advantages so more than one distinct advantage was needed to get full credit.

Solution:

(a) Calculate the volatility implied by the market.

Candidate is not given strike price. Using the Put-Call parity, the strike price can be solved for as -(7.227-12.350-90)/(e^-.05)=100

Given the strike price, the candidate must recognize that d1 and d2 can be solved for in terms of theta, or implied volatiliy.

Using the given ratio between d1 and d2, the candidate can solve for the implied volatiliy to be 26%

d1=(theta^2/2 + ln(.9) +.05)/theta
d2=(- theta^2/2 + ln(.9) +.05)/theta

d1/d2=.2418=(theta^2/2 + ln(.9) +.05)/ (- theta^2/2 + ln(.9) +.05)
theta^2/2 = .0338, theta = .26

(b) You decide to sell the European Call option and to delta hedge it using implied volatility. On the first day, the actual volatility is 37%.

Estimate your mark-to-market profit for that day.

The candidate must recognize that by hedging with implied volatility, the resulting profit is deterministic. Further, the profit can be shown to be ½(theta(actual)^2-theta(implied)^2)*S^2*gamma*dt (formula 10.1)

Plugging in the given numbers gives : ½(.37^2-.26^2)*90^2*.017*1/250 = 0.091

17. Continued

(c) Explain the advantages of hedging with implied volatility.

- Markets are not perfectly efficient
- Implied volatility impounds expected volatility and everything else that affects option supply and demand but is not modeled
- Options are modelled differently by different people
- Volatility in actuality is not constant over time (as some models suggest)
- Implied vol can be easily calculated, actual vol is unknown (don't need to know actual vol)
- Guaranteed profit if actual vol > implied vol (must say actual vol > implied vol, not just guaranteed profit)
- There will be no day to day profit/loss (hedging with implied vol results in a deterministic profit)