QFI CORE Model Solutions
Fall 2018

1. **Learning Objectives:**

(1a)   Understand and apply concepts of probability and statistics important in mathematical finance.

Learning Outcomes:

(1c)   Understand Ito integral and stochastic differential equations.

(1d)   Understand and apply Ito's Lemma.

Sources:
Chin et al (Pg 116-117 #13)

Neftci Ch 8, 9, 10

Commentary on Question:
This question tested properties of the Brownian motion and the Ito's integral. Most candidates did well on this question.

Solution:

(a)   Show that $E(W_s W_t W_u) = 0$ for $s \leq t \leq u$.

Commentary on Question:
This part tests the independency property of the Brownian motion. Most candidates did well on this part by writing $W_u = W_u - W_t + W_t - W_s + W_s$ and $W_t = W_t - W_s + W_s$. However, some candidates split $W_s = W_s - W_t + W_t$ and $W_t = W_t - W_u + W_u$. This will not lead to independent components because, for example, $W_s - W_t$ and W_t are not independent.

1. **Continued**

A standard approach is as follows:

Let $X = W_s$ and $Y = W_t - W_s$ and $Z = W_u - W_t$
By the independence increment property of Wiener process, X, Y and Z are independent random variables

$$E[W_s W_t W_u] = E[X\,(X + Y)(X + Y + Z)]$$
$$= E[X^3 + 2X^2 Y + XY^2 + X^2 Z + XYZ]$$
$$= E[X^3] + 2E[X^2 Z]E[Y] + E[X]E[Y^2] + E[X^2]E[Z] + E[X]E[Y]E[Z]$$

Since the third raw moment of the normal variable is zero and

$$E[X] = E[Y] = E[Z] = 0$$

we have $E[W_s W_t W_u] = 0$ as desired.

The above proof can be simplified using the tower property of conditional expectations:

$$E[W_s W_t W_u] = E[E[W_s W_t W_u | I_t]]$$
$$= E[E[W_s W_t W_u | I_t]]$$
$$= E[W_s W_t E[W_u | I_t]]$$
$$= E[W_s W_t^2]$$
$$= E[W_s (W_t - W_s + W_s)^2]$$

The remaining steps are similar to the standard approach.

(b)   Show that $Cov(M_s, M_t) = \dfrac{s^3}{3}$ for $s \le t$.

Commentary on Question:
Most candidates did well on this part by recognizing that M_s and $M_t - M_s$ are independent and used the Ito's isometry. Some candidates wrote $M_t - M_s$ as M_{t-s} but these two terms are not the same.

For $s \le t$,

$$Cov\left(\int_0^s u\,dW_u, \int_0^t v\,dW_v\right)$$
$$= Cov\left(\int_0^s u\,dW_u, \int_0^s v\,dW_v + \int_s^t v\,dW_v\right)$$
$$= Cov\left(\int_0^s u\,dW_u, \int_0^s v\,dW_v\right) + Cov\left(\int_0^s u\,dW_u, \int_s^t v\,dW_v\right)$$

Now from independence we have

$$Cov\left(\int_0^s u\,dW_u, \int_s^t v\,dW_v\right) = 0$$

1. Continued

And from Ito Isometry we find

$$Cov\left(\int_0^s u\,dW_u, \int_0^s v\,dW_v\right) = E\left[\left(\int_0^s u\,dW_u\right)^2\right] = \int_0^s u^2\,du = \frac{s^3}{3}$$

Thus we have

$$Cov\left(\int_0^s u\,dW_u, \int_0^t v\,dW_v\right) = \frac{s^3}{3}$$

(c)   Show that $E\left(e^{\theta M_t}\right) = e^{\frac{\theta^2}{2}\left(\frac{t^3}{3}\right)}$ for any given constant θ.

Commentary on Question:
Most candidates recalled that M_t is normally distributed (from pages 126-127 of Chin et al.) and used the property of the moment generating function of the normal distribution to prove this part. However, some candidates tried to prove that M_t is normally distributed but failed to do so. Very few candidates used the approach given below.

Let $M_t = \int_0^t v\,dW_v$. Then $dM_t = t\,dW_t$

Let $Z_t = e^{\theta M_t - \frac{\theta^2}{2}\left(\frac{t^3}{3}\right)}$ for any given parameter θ.

Since

$$\frac{\partial Z_t}{\partial M_t} = \theta Z_t, \qquad \frac{\partial Z_t}{\partial t} = -\frac{\theta^2}{2}t^2 Z_t, \qquad \frac{\partial^2 Z_t}{\partial M_t^2} = \theta^2 Z_t$$

from Ito's Lemma we have

$$dZ_t = \theta Z_t dM_t + \left(-\frac{\theta^2}{2}t^2 Z_t\right)dt + \frac{t^2}{2}\theta^2 Z_t dt = \theta Z_t\, t\,dW_t$$

By taking integral and noting $Z_0 = 1$ we have

$$Z_t = 1 + \int_0^t \theta Z_s\, s\,dW_s$$

Since the Ito integral is a martingale we have
$$E[Z_t] = 1$$

Thus

$$E\left[e^{\theta M_t}\right] = e^{\frac{\theta^2}{2}\left(\frac{t^3}{3}\right)}$$

1.   Continued

(d)   Show that X_t is actually not a Brownian motion.

Hint: Derive $Var(X_{t+u} - X_t)$ using the result of part (b).

Commentary on Question:
Most candidates did well on this part. Some candidates did not calculate the variance of X_{t+u} correctly.

For $u > 0$ and $t > 0$

$$Var(X_{t+u} - X_t) = Var(X_{t+u}) + Var(X_t) - 2Cov(X_{t+u}, X_t)$$

$$= Var\left(\frac{\sqrt{3}}{t+u} M_{t+u}\right) + Var\left(\frac{\sqrt{3}}{t} M_t\right) - 2Cov\left(\frac{\sqrt{3}}{t+u} M_{t+u}, \frac{\sqrt{3}}{t} M_t\right)$$

$$= t + u + t - \frac{6}{t(t+u)}\left[\frac{1}{3} t^3\right] = u + \frac{2tu}{t+u} \neq u.$$

Thus X_t is not a Brownian motion

2. **Learning Objectives:**
   1.   The candidate will understand the fundamentals of stochastic calculus as they apply to option pricing.

Learning Outcomes:
(1c)   Understand Ito integral and stochastic differential equations.

(1d)   Understand and apply Ito's Lemma.

(1e)   Understand and apply Jensen's Inequality.

(1h)   Define and apply the concepts of martingale, market price of risk and measures in single and multiple state variable contexts.

Sources:
An Introduction to the Mathematics of Financial Derivative, Neftci, Hirsa, 3rd Edition, 2014, Ch6

Commentary on Question:
Candidates generally did well on this question but very few received full credits. Most candidates missed the Jensen's Inequality in proving non-explosiveness of Martingale.

Solution:
(a)   State the conditions for a process to be a martingale.

   Commentary on Question:
   This part is memory retrieving. Most candidates can answer this part.

   A process $\{S_t, t \in [0, \infty)\}$ is a martingale with respect to the family of information sets I_t and with respect to the probablity P if for all $t > 0$,
   1. S_t is known, given I_t
   2. Unconditional "forecasts" are finite: $E[|S_t|] < \infty$
   3. $E_t[S_T] = S_t$ for all $t < T$

(b)   Show that W_t^2 is not a martingale by identifying a condition in part (a) that is not satisfied.

   Commentary on Question:
   Most candidates could successfully invalidate condition 3. However, some candidates incorrectly had $E[W_{t+s}^2|I_t] = t + s$, leading to partial credits.

   $E[W_{t+s}^2|I_t] = E[(W_{t+s} - W_t)^2 + 2W_{t+s}W_t - W_t^2|I_t] = s + 2W_t^2 - W_t^2 = W_t^2 + s \neq W_t^2$
   Therefore, W_t^2 is not a martingale since Condition 3 fails.

2. **Continued**

Alternative solution for partial credit:
From Ito's lemma
$$d(W_t^2) = 2W_t dW_t + dt$$
W_t^2 is not a martingale since the drift term is not vanishing.

(c) Determine a deterministic function $f(t)$ such that $W_t^2 - f(t)$ is a martingale.

Commentary on Question:
Most candidates were successfully in proving condition 3, however, some candidates did not show conditions (1) and (2)

It can be converted to a martingale by subtracting the time trend, and $W_t^2 - t$ is a martingale.
This can be shown below:
   (1) It's obvious that X_t is known, given I_t
   (2) $E[|W_t^2 - t|] \le E[|W_t^2|] + E[|t|] = 2t < \infty$
   (3) $E[W_{t+s}^2 - (t+s)|I_t] = E[W_{t+s}^2|I_t] - (t+s) = W_t^2 + s - (t+s) = W_t^2 - t.$

Alternate solution for partial credit:
$$d(W_t^2 - t) = 2W_t dW_t + dt - dt = 2W_t dW_t$$
$W_t^2 - t$ is a martingale since the drift term is vanishing.

(d) Show that X_t is a martingale by verifying that all the conditions in part (a) hold.

Commentary on Question:
Candidates did not do well for this question, most candidates had trouble to validate condition 2, some candidates did not know $2E\left[\int_t^{t+s} uW_u du|I_t\right] = 2\int_t^{t+s} uE[W_u|I_t]du = 2\int_t^{t+s} uW_t du$, and some candidates did not show the process of $2E\left[\int_t^{t+s} uW_u du|I_t\right] = 2\int_t^{t+s} uE[W_u|I_t]du = 2\int_t^{t+s} uW_t du = 2W_t \int_t^{t+s} udu = W_t[(t+s)^2 - t^2],$

Some candidates used the alternative approach, they knew $\frac{d\int_0^t uW_u du}{dt} = tW_t$, but incorrectly stated $\frac{d\int_0^t uW_u du}{dW} = t$, leading to partial credit. Also, some candidates, who used the alternative approach did not show the derivation process

2. Continued

(1) It's obvious that X_t is known, given I_t.

(2) $E[|X_t|] \leq t^2 E[|W_t|] + 2t \int_0^t E[|W_u|]du < \infty$

(3)

$$E[X_{t+s}|I_t] = E\left[W_{t+s}(t+s)^2 - 2\int_0^{t+s} uW_u du | I_t\right]$$

$$= E[W_{t+s}(t+s)^2|I_t]$$

$$- 2\left(E\left[\int_0^t uW_u du|I_t\right] + E\left[\int_t^{t+s} uW_u du|I_t\right]\right)$$

$$= W_t(t+s)^2 - 2\int_0^t uW_u du - 2E\left[\int_t^{t+s} uW_u du|I_t\right]$$

Since

$2E\left[\int_t^{t+s} uW_u du|I_t\right] = 2\int_t^{t+s} uE[W_u|I_t]du = 2\int_t^{t+s} uW_t du = 2W_t \int_t^{t+s} u du = W_t[(t+s)^2 - t^2]$,

we have

$E[X_{t+s}|I_t] = W_t(t+s)^2 - 2\int_0^t uW_u du - W_t[(t+s)^2 - t^2] = W_t t^2 - 2\int_0^t uW_u du = X_t$.

Therefore, X_t is a martingale.

Alternative solution for Condition 3:

From Ito's lemma

$$d(t^2 W_t) = 2tW_t dt + t^2 dW_t$$

Thus

$$t^2 W_t - 0^2 W_0 = \int_0^t 2uW_u du + \int_0^t u^2 dW_u$$

It follows that $X_t = \int_0^t u^2 dW_u$ and

$$E[X_{t+s}|I_t] = E\left[\int_0^{t+s} u^2 dW_u \Big| I_t\right] = \int_0^t u^2 dW_u + E\left[\int_t^{t+s} u^2 dW_u \Big| I_t\right] = X_t + 0$$

$$= X_t$$

3. **Learning Objectives:**
1. The candidate will understand the fundamentals of stochastic calculus as they apply to option pricing.

Learning Outcomes:
(1b) Understand the importance of the no-arbitrage condition in asset pricing.

(1c) Understand Ito integral and stochastic differential equations.

(1f) Demonstrate understanding of option pricing techniques and theory for equity and interest rate derivatives.

Sources:
Neftci, Chapters 3, 5, 9, 14, 17

Commentary on Question:
This question tested a candidate's knowledge of basic stochastic principles and application to option pricing. Some parts of this question were building off previous parts, and most candidates did well (even if they could not answer previous parts) by using the "proved" results stated in the question.

Solution:
(a) Show that your colleague's statement is correct.

Hint: Express
$\widetilde{W_t} = W_t - X_t$ and determine X_t.

Commentary on Question:
Most candidates did well on this part, though missing mention of Girsanov Theorem was a common mistake.

The stock follows geometric Brownian motion, so the following is true:

$$S_t = S_{t_0} e^{\tilde{\mu} t + \sigma W_t}$$

$$\tilde{\mu} = \mu - \frac{\sigma^2}{2}$$

By applying Girsanov Theorem to convert from real world measure $\mathbb{P}$ to risk-neutral measure $\mathbb{Q}$ to the solution of the SDE

$$d\widetilde{W_t} = dW_t - X_t dt$$

$$\widetilde{W_t} = W_t + \frac{\mu - r}{\sigma} t$$

3.   Continued

Leading to:

$$S_t = S_{t_0} e^{(r - \sigma^2/2)(t - t_0) + \sigma(\widetilde{W}_t - \widetilde{W}_{t_0})}$$

Taking the logarithm of the above equation

$$ln(S_t) = ln(S_{t_0}) + (r - \frac{\sigma^2}{2})(t - t_0) + \sigma(\widetilde{W}_t - \widetilde{W}_{t_0})$$

for $0 \le t_0 < t$

(b)   Calculate $E_{t_0}^{\mathbb{Q}}\left[\int_{t_0}^{t}(\widetilde{W}_s - \widetilde{W}_{t_0})ds\right]$.

Commentary on Question:
Again, most candidates did well. There are a variety of ways to derive the correct answer, one of which is shown below.

The mean is evaluated as

$$E_{t_0}^{\mathbb{Q}}\left[\int_{t_0}^{t}(\widetilde{W}_u - \widetilde{W}_{t_0})du\right] = \int_{t_0}^{t}E_{t_0}^{\mathbb{Q}}[\widetilde{W}_u - \widetilde{W}_{t_0}]du = 0$$

where the last equality follows by independent increments.

(c)   Show that

$$\mathrm{Var}_{t_0}^{\mathbb{Q}}\left[\int_{t_0}^{t}(\widetilde{W}_s - \widetilde{W}_{t_0})\,ds\right] = \frac{(t - t_0)^3}{3}$$

Commentary on Question:
Most candidates struggled to prove this correctly. The most common methods to prove this would be from first principles, or by Ito isometry (as shown below).

Note that:

$$(\widetilde{W}_u - \widetilde{W}_{t_0})du = d\big((\widetilde{W}_u - \widetilde{W}_{t_0})(u - t)\big) - (u - t)\,d\widetilde{W}_u$$

By taking integral we find
$$\int_{t_0}^{t}(\widetilde{W}_u - \widetilde{W}_{t_0})du = -\int_{t_0}^{t}(u - t)\,d\widetilde{W}_u$$

3. **Continued**

Then

$$\text{var}_{t_0}^{\mathbb{Q}}\left[\int_{t_0}^{t}(\tilde{W}_u - \tilde{W}_{t_0})du\right] = \text{var}_{t_0}^{\mathbb{Q}}\left[\int_{t_0}^{t}(u-t)\,d\tilde{W}_u\right]$$

Using Ito's isometry and part b) that the expectation is 0, we have

$$\text{var}_{t_0}^{\mathbb{Q}}\left[\int_{t_0}^{t}(u-t)\,d\tilde{W}_u\right] = \int_{t_0}^{t}(u-t)^2\,du = \frac{(t-t_0)^3}{3}$$

(d)   Show that for $t > t_0$,

$$\ln G_t = \frac{t_0}{t}\ln G_{t_0} + \frac{1}{t}\int_{t_0}^{t}\ln S_u du$$

Commentary on Question:
Most candidates did well, as this only involved standard calculus rules.

$$\ln G_t = \frac{1}{t}\int_{0}^{t}\ln S_u \ du$$

The integral can be split, and the first term can be multiplied by t_0/t_0 (which is equal to 1)

$$\ln G_t = \frac{t_0}{t} * \frac{1}{t_0}\int_{0}^{t_0}\ln S_u \ du + \frac{1}{t}\int_{t_1}^{t}\ln S_u \ du$$

$$\ln G_t = \frac{t_0}{t}\ln G_{t_0} + \frac{1}{t}\int_{t_0}^{t}\ln S_u \ du$$

(e)   Show that the distribution $\ln G_t$ for $t > t_0$ given information up through time t_0 under the measure $\mathbb{Q}$ is:

$$\ln G_t \sim N\left(\frac{t_0}{t}\ln G_{t_0} + \frac{t-t_0}{t}\ln S_{t_0} + \left(r - \frac{\sigma^2}{2}\right)\frac{(t-t_0)^2}{2t}, \frac{(t-t_0)^3\sigma^2}{3t^2}\right)$$

Commentary on Question:
Most candidates did well on this part as well. The most common mistake was not justifying the normality, and only evaluating mean and variance.

3.   Continued

From part d)

$$\ln G_t = \frac{t_0}{t} \ln G_{t_0} + \frac{1}{t} \int_{t_0}^{t} \ln S_u \ du$$

Expanding $\ln S_u$ and evaluating at time t_0:

$$\ln G_t = \frac{t_0}{t} \ln G_{t_0} + \frac{(t - t_0)}{t} \ln S_{t_0} + \frac{1}{t}\left(r - \frac{\sigma^2}{2}\right)\left(\frac{(t - t_0)^2}{2}\right)$$
$$+ \frac{1}{t}\int_{t_0}^{t} \sigma(\widetilde{W}_u - \widetilde{W}_{t_0}) \ du$$

The first three terms are constant. Only the last term is stochastic, and the evaluation of mean and variance of this term has been done in parts b) and c).

Since $\ln S$ is normal, $\ln G_t$ is a sum of normal variables, so is normal too.

In other words:

$$\ln G_t \sim N \left(\frac{t_0}{t} \ln G_0 + \frac{t - t_0}{t} \ln S_{t_0} + \frac{1}{t}\left(r - \frac{\sigma^2}{2}\right)\left(\frac{(t - t_0)^2}{2}\right), \frac{(t - t_0)^3 \sigma^2}{3t^2}\right)$$

(f)   Derive K such that the forward contract has zero value at time 0.

Commentary on Question:
This part tested a candidate's understanding of first principles around no-arbitrage, and most candidates did not do well. For candidates who did attempt, a common mistake was failing to take the calculation to completion and express K in a simplified form.

The forward has initial value

$$F_0 = e^{-rT} E_{t_0}^{\mathbb{Q}}(G_T - K)$$

Setting $F_0 = 0$ we have $K = E_{t_0}^{\mathbb{Q}}(G_T)$

3.   Continued

From parts d) and e), G_T is lognormally distributed. Using initial value at time 0 allows simplification:

$$\ln G_t \sim N\left(\ln S_0 + \left(r - \frac{\sigma^2}{2}\right)\left(\frac{t}{2}\right), \frac{t\sigma^2}{3}\right)$$

From the moment generating function of a lognormal distribution, we have:

$$E(X) = e^{\left(\mu + \frac{\sigma^2}{2}\right)}$$

Combining the above allows the evaluation of K:

$$K = e^{\left(\ln S_0 + \left(r - \frac{\sigma^2}{2}\right)\left(\frac{T}{2}\right) + \frac{T\sigma^2}{6}\right)}$$

$$K = S_0 e^{\left(\left(r - \frac{\sigma^2}{2}\right)\left(\frac{T}{2}\right) + \frac{T\sigma^2}{6}\right)}$$

$$K = S_0 e^{\left(r - \frac{\sigma^2}{6}\right)\frac{T}{2}}$$

4.   **Learning Objectives:**

1.   The candidate will understand the fundamentals of stochastic calculus as they apply to option pricing.

3.   The candidate will understand the quantitative tools and techniques for modeling the term structure of interest rates and pricing interest rate derivatives.

Learning Outcomes:

(1c)   Understand Ito integral and stochastic differential equations.

(1d)   Understand and apply Ito's Lemma.

(1e)   Understand and apply Jensen's Inequality.

(3c)   Understand and apply popular one-factor interest rate models including Vasicek, Cox-Ross-Ingersoll, Hull-White, Ho-Lee, Black-Derman-Toy, Black-Karasinski.

Sources:

Neftci Chapter 9, 10, 18

Wilmott FAQ Q23

Commentary on Question:
Commentary is listed underneath question component.

Solution:

(a)   Show that for $0 \le t \le T$

$$\int_t^T r_s ds = (T-t) r_t + \int_t^T (T-s) \theta_s ds + \sigma \int_t^T (W_s - W_t) ds$$

Commentary on Question:
This part is relatively easy, and candidates did well in general. Some candidates attempted to use alternative methods which are more difficult to carry through to the end result, but partial marks were given for the correct steps shown.

Under Ho-Lee model,

$$\int_t^s dr_v = \int_t^s \theta_v dv + \sigma \int_t^s dW_v$$
$$r_s = r_t + \int_t^s \theta_v dv + \sigma (W_s - W_t)$$
$$\int_t^T r_s ds = r_t(T-t) + \int_t^T \int_t^s \theta_v dv \, ds + \sigma \int_t^T (W_s - W_t) ds$$

4. **Continued**

Change of variable for the second term

$$\int_t^T \int_t^s \theta_v \, dv \, ds = \int_t^T \int_s^T \theta_s \, dv \, ds = \int_t^T \theta_s (T - s) ds$$

Thus

$$\int_t^T r_s \, ds = r_t (T - t) + \int_t^T \theta_s (T - s) ds + \sigma \int_t^T (W_s - W_t) ds$$

(b)   Derive the following formula using the result of part (a):

$$Z(t,T) = e^{A(t,T) - r_t(T-t)}$$

where

$$A(t,T) = -\int_t^T \theta_s (T - s) ds + \frac{1}{6} \sigma^2 (T - t)^3$$

Hint: The variance of $\int_t^T (W_s - W_t) ds$ at time t equals $\frac{1}{3}(T-t)^3$.

Commentary on Question:
This part is straightforward and is well done in general. Candidates are able to use the moments of normal distribution to calculate the expectation of a log normal distribution and hence derive the bond price.

From part a), $\int_t^T r_s ds$ is normally distributed with

$$E_t^Q \left[\int_t^T r_s \, ds \right] = (T - t) r_t + \int_t^T (T - s) \theta_s \, ds$$

$$Var_t^Q \left[\int_t^T r_s \, ds \right] = Var_t^Q \left[\sigma \int_t^T (W_s - W_t) \, ds \right] = \frac{\sigma^2}{3} (T - t)^3 \quad (Use\ hint)$$

Therefore, taking the expectation of a lognormal distribution,

$$Z(t,T) = E_t^Q \left[e^{-\int_t^T r_s ds} \right]$$

$$= e^{-E_t^Q \left[\int_t^T r_s ds \right] + \frac{1}{2} Var_t^Q \left[\int_t^T r_s ds \right]}$$

$$= e^{-r_t(T-t) - \int_t^T \theta_s(T-s) ds + \frac{1}{6} \sigma^2 (T-t)^3} = e^{A(t,T) - r_t(T-t)}$$

where

$$A(t,T) = -\int_t^T \theta_s (T - s) ds + \frac{1}{6} \sigma^2 (T - t)^3$$

4. Continued

(c) Show that

$$E_0^{\mathbb{Q}}\left[Z(t,T)\right] = e^{-\left[r_0(T-t)+\int_0^T \theta_s(T-s)ds - \int_0^t \theta_s(t-s)ds\right]+\frac{1}{6}(T-t)^2(T+2t)}$$

Commentary on Question:
This part is not done very well. Many candidates did not realize that the only variable at time 0 is r_t with respect to which the expectation should be taken. Some candidates attempted incorrectly to multiply a discounting factor from time t to 0 on Z(t, T), and some incorrectly assumed that E[Z(t,T)] = Z(0,T)/Z(0,t).

From part b), $Z(t,T) = exp(A(t,T) - r_t(T-t))$

From part a), $r_t - r_0 = \int_0^t \theta_v dv + \sigma W_t$

Thus $\mathbb{E}_0^{\mathbb{Q}}[r_t] = r_0 + \int_0^t \theta_s ds, Var_0^{\mathbb{Q}}[r_t] = \sigma^2 t,$

Since r_t is normally distributed random variable

$$\mathbb{E}_0^{\mathbb{Q}}[Z(t,T)] = exp\left(A(t,T) - (T-t)\mathbb{E}_0^{\mathbb{Q}}[r_t] + \frac{(T-t)^2}{2}Var_0^{\mathbb{Q}}[r_t]\right)$$

$$\mathbb{E}_0^{\mathbb{Q}}[Z(t,T)] = exp\left(A(t,T) - (T-t)(r_0 + \int_0^t \theta_s ds) + \frac{(T-t)^2}{2}\sigma^2 t\right)$$

Since $A(t,T) = -\int_t^T (T-s)\theta_s ds + \frac{\sigma^2}{6}(T-t)^3$
we have in the exponent

$$-\int_t^T (T-s)\theta_s ds - (T-t)\left(r_0 + \int_0^t \theta_s ds\right) + \frac{(T-t)^2}{2}\sigma^2 t + \frac{\sigma^2}{6}(T-t)^3$$

Now

$$\int_t^T (T-s)\theta_s ds + (T-t)\int_0^t \theta_s ds$$

$$= \int_0^T (T-s)\theta_s ds - \int_0^t (T-s)\theta_s ds + (T-t)\int_0^t \theta_s ds$$

$$= \int_0^T (T-s)\theta_s ds - \int_0^t (t-s)\theta_s ds$$

4. **Continued**

And
$$\frac{(T-t)^2}{2}\sigma^2 t + \frac{\sigma^2}{6}(T-t)^3 = \frac{\sigma^2}{6}(T-t)^2(T-t+3t) = \frac{\sigma^2}{6}(T-t)^2(T+2t)$$
Thus
$$\mathbb{E}_0^{\mathbb{Q}}[Z(t,\mathrm{T})] = exp\Bigg(-(T-t)r_0 - \int_0^T (T-s)\theta_s ds + \int_0^t (t-s)\theta_s ds$$
$$+ \frac{\sigma^2}{6}(T-t)^2(T+2t)\Bigg)$$

(d)   Show that $df\left(t,T_1,T_2\right) = \frac{1}{2}\left(T_1 + T_2 - 2t\right)\sigma^2 dt + \sigma dW_t.$

Commentary on Question:
Attempt rate for this part was not very high, but many candidates who attempted were able to obtain partial marks.

From the formula sheet or part b), price of a zero-coupon bond under the Ho-Lee model is:
$$Z(t,T) = e^{A(t,T) - r_t(T-t)}$$
where
$$A(t,T) = -\int_t^T \theta_s(T-s)ds + \frac{1}{6}\sigma^2(T-t)^3$$

From formula sheet,
$$f(t,T_1,T_2) = \frac{log[Z(t,T_1)] - log[Z(t,T_2)]}{T_2 - T_1}$$
$$= \frac{[A(t,T_1) - r_t(T_1 - t)] - [A(t,T_2) - r_t(T_2 - t)]}{T_2 - T_1}$$
$$= \frac{A(t,T_1) - A(t,T_2) + r_t(T_2 - T_1)}{T_2 - T_1}$$
$$= \frac{-\int_t^{T_1} \theta_s(T_1 - s)ds + \frac{1}{6}\sigma^2(T_1 - t)^3 + \int_t^{T_2} \theta_s(T_2 - s)ds - \frac{1}{6}\sigma^2(T_2 - t)^3}{T_2 - T_1} + r_t$$

4. **Continued**

Since

$$\frac{\partial f(t,T_1,T_2)}{\partial t} = \frac{\theta_t(T_1-t) - \frac{1}{2}\sigma^2(T_1-t)^2 - \theta_t(T_2-t) + \frac{1}{2}\sigma^2(T_2-t)^2}{T_2-T_1}$$

$$= \frac{\theta_t(T_1-T_2) - \frac{1}{2}\sigma^2[(T_1-T_2)*(T_1+T_2-2t)]}{T_2-T_1}$$

$$= -\theta_t + \frac{1}{2}\sigma^2\,(T_1+T_2-2t)$$

$$\frac{\partial f(t,T_1,T_2)}{\partial r_t} = 1$$

Thus

$$df(t,T_1,T_2) = \frac{\partial f(t,T_1,T_2)}{\partial t}dt + \frac{\partial f(t,T_1,T_2)}{\partial r_t}dr_t$$

$$= -\theta_t dt + \frac{1}{2}\sigma^2\,(T_1+T_2-2t)dt + \theta_t dt + \sigma dW_t$$

$$= \frac{1}{2}(T_1+T_2-2t)\sigma^2 dt + \sigma dW_t$$

Alternatively,

From the formula sheet or part b), price of a zero-coupon bond under the Ho-Lee model is:

$$Z(t,T) = e^{A(t,T)-r_t(T-t)}$$

where

$$A(t,T) = -\int_t^T \theta_s(T-s)ds + \frac{1}{6}\sigma^2(T-t)^3$$

Since

$$\frac{\partial Z}{\partial t} = Z\left(\frac{\partial A}{\partial t} + r_t\right) = Z\left(\theta_t(T-t) - \frac{1}{2}\sigma^2(T-t)^2 + r_t\right)$$

$$\frac{\partial Z}{\partial r} = -Z(T-t)$$

$$\frac{\partial^2 Z}{\partial r^2} = Z(T-t)^2$$

4. Continued

By Ito's Lemma,

$$dZ(t,T) = Z\left(\theta_t(T-t) - \frac{1}{2}\sigma^2(T-t)^2 + r_t\right)dt$$

$$-Z(T-t)dr + \frac{\sigma^2}{2}Z(T-t)^2dt$$

$$dZ(t,T) = \left[-(T-t)Z(t,T)\theta_t + \frac{1}{2}(T-t)^2Z(t,T)\sigma^2\right.$$

$$\left. + Z(t,T)\left(\theta_t(T-t) - \frac{1}{2}\sigma^2(T-t)^2 + r_t\right)\right]dt$$

$$- (T-t)Z(t,T)\sigma dW_t = r_tZ(t,T)dt - (T-t)Z(t,T)\sigma dW_t$$

From formula sheet,

$$f(t,T_1,T_2) = \frac{\log[Z(t,T_1)] - \log[Z(t,T_2)]}{T_2 - T_1}$$

By Ito's Lemma again,

$$d\log[Z(t,T)] = \left[\frac{r_tZ(t,T)}{Z(t,T)} - \frac{1}{2}\frac{1}{Z(t,T)^2}(T-t)^2Z(t,T)^2\sigma^2\right]dt$$

$$- \frac{1}{Z(t,T)}(T-t)Z(t,T)\sigma dW_t$$

$$= \left[r_t - \frac{1}{2}(T-t)^2\sigma^2\right]dt - (T-t)\sigma dW_t$$

Therefore,

$$df(t,T_1,T_2) = \frac{(T_2-t)^2 - (T_1-t)^2}{2(T_2-T_1)}\sigma^2 dt - \frac{T_1-T_2}{T_2-T_1}\sigma dW_t$$

$$= \frac{1}{2}(T_1 + T_2 - 2t)\sigma^2 dt + \sigma dW_t$$

(e)   Derive the convexity adjustment $\bar{F} - F$ using the result of part (d).

Commentary on Question:
Attempt rate for this part is low, but candidates who attempted did reasonably well.

By integrating the above we have

$$f(t,T_1,T_2) = f(0,T_1,T_2) + \frac{\sigma^2}{2}\left((T_1 + T_2)t - t^2\right) + \sigma W_t$$

Therefore

$$f(T_1,T_1,T_2) = f(0,T_1,T_2) + \frac{\sigma^2 T_1 T_2}{2} + \sigma W_{T_1}$$

4. **Continued**

Since W_{T_1} is a normal random variable with mean 0

$$\bar{F} = \mathbb{E}_0^{\mathbb{Q}}[f(T_1,T_1,T_2)] = f(0,T_1,T_2) + \frac{1}{2}\sigma^2 T_1 T_2$$

Alternatively,

$$\bar{F} = E_0^Q[f(T_1,T_1,T_2)] = E_0^Q\left[f(0,T_1,T_2) + \int_0^{T_1} df(s,T_1,T_2)\right]$$

$$= f(0,T_1,T_2) + E_0^Q\left[\int_0^{T_1} df(s,T_1,T_2)\right]$$

Integrating the result of part (d)

$$\int_0^{T_1} df(s,T_1,T_2) = \int_0^{T_1}\frac{1}{2}(T_1 + T_2 - 2s)\sigma^2 ds + \int_0^{T_1}\sigma dW_s$$

$$= \frac{1}{2}T_1 T_2 \sigma^2 + \int_0^{T_1}\sigma dW_s$$

which is normally distributed with mean $\frac{1}{2}T_1 T_2 \sigma^2$ and variance $\sigma^2 T_1$.
Therefore,

$$\bar{F} = f(0,T_1,T_2) + \frac{1}{2}T_1 T_2 \sigma^2$$

The convexity adjustment is $\frac{1}{2}T_1 T_2 \sigma^2$.

(f)   Determine and justify whether $E_0^{\mathbb{Q}}\left[Z(T_1,T_2)\right]$ is larger or smaller than $e^{-\bar{F}(T_2 - T_1)}$.

Commentary on Question:
Some candidates realized that this question involves Jensen's Inequality, but few candidates were able to show explicitly how it applies to the question. Some candidates confused it with the convexity adjustment from part e).

$\bar{F} = E_0^Q[f(T_1,T_1,T_2)]$

$f(T_1,T_1,T_2) = \frac{log[Z(T_1,T_1)] - log[Z(T_1,T_2)]}{T_2 - T_1} = \frac{-log[Z(T_1,T_2)]}{T_2 - T_1}$

Thus $log[Z(T_1,T_2)] = -(T_2 - T_1)f(T_1,T_1,T_2)$

By Jensen's inequality, if $f(x)$ is a convex function of variable x, then $f[E(X)] \le E[f(X)]$

4. **Continued**

Since $log[Z(T_1, T_2)]$ is a concave function of $Z(T_1, T_2)$

Then $-(T_2 - T_1)\bar{F} = E_0^Q\big[log[Z(T_1, T_2)]\big] \leq log\big(E_0^Q[Z(T_1, T_2)]\big)$

Therefore, we have $E_0^Q[Z(T_1, T_2)] \geq e^{-(T_2 - T_1)\bar{F}}$

Therefore, the expected value of $Z(T_1, T_2)$ is greater (or equal).

5. **Learning Objectives:**

2. The candidate will understand how to apply the fundamental theory underlying the standard models for pricing financial derivatives. The candidate will understand the implications for option pricing when markets do not satisfy the common assumptions used in option pricing theory such as market completeness, bounded variation, perfect liquidity, etc. The Candidate will understand how to evaluate situations associated with derivatives and hedging activities.

Learning Outcomes:

(2a) Identify limitations of the Black-Scholes pricing formula

(2d) Understand the different approaches to hedging.

(2e) Understand how to delta hedge and the interplay between hedging assumptions and hedging outcomes.

Sources:

Quantitative Finance, Wilmott, Paul, 2nd Edition, Chapter 2, 8 & 10

Commentary on Question:

Overall, candidates performed below expectation on this question. This question tests the concept of hedging with the Black-Scholes model, how to delta hedge and the interplay between hedging assumptions and hedging outcomes. It first tests to see if candidates can set up and calculate a static hedge based on market conditions. Then the question moves to discuss the limitations of setting up a static hedge. Finally, it asks the candidate to calculate performance after one day of market movement.

Solution:

(a) Construct a static hedge for the portfolio using a bear put spread and justify your strike levels.

Commentary on Question:

Few candidates could correctly identify a put strategy and recommend appropriate strike levels. If the candidate used 70 as the strike price for the put option that we should sell, full mark was given as long as otherwise the answer was in line with what is presented below.

A bear put spread consists of purchasing put options at a higher strike price while also selling the same number of puts at a lower strike price.

Buy put option with strike price = $100 given the minimum guarantee that protects the initial deposit amount at the current market level.

Sell put option with strike price = $100 * $e^{-0.30 \times 1}$ = $74.08 given the expected upcoming market turbulence.

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5. **Continued**

(b)   Calculate the number of options on the Russell 2000 Index needed to implement the bear put spread static hedge in part (a) that would match the delta of the liability using the Black-Scholes model.

Commentary on Question:
Few candidates could identify all the formulas needed to calculate the number of options. Full marks were also given based on the $70 strike price if the calculations were correct.

Calculation of $N(d_1)$ for long ATM put option, k=100
$S = K = 100$, r = 5%, d = 0%, T-t = 1, σ = 24%
$d_1 = [\ln(S/K) + (r\text{-}\delta + \sigma^2/2)(T\text{-}t)]/ [\sigma\sqrt{(T-t)}]$
$d_1 = [0 + (0.05 + .24^2/2)]/0.24 = 0.3283$
$N(d_1) = 0.62867$
Delta of higher strike Put $= N(d_1)\ e^{-d} - 1 = -0.37133$

Calculation of $N(d_1)$ for sell short out of the money put option, k = 74.08
    $S = 100$, $K = 100 * e^{-0.30} = 74.08$, r = 5%, d = 0%, t = 1%, σ = 24%
$d_1 = [\ln(S/K) + (r\text{-}\delta + \sigma^2/2)(T\text{-}t)]/ [\sigma\sqrt{(T-t)}]$
$d_1 = [\ln(100/74.08) + (0.05 + .24^2/2)]/0.24 = 1.5783$
$N(d_1) = 0.9427$
Delta of lower strike Put $= N(d_1)\ e^{-d} - 1 = -0.05724$

Delta of bear put spread = Sum of positions (i.e. + Long Positions – Short Position)
$$= -0.37133\text{- }-0.05724$$
$$= -0.31409$$
Delta of bear put spread per contract $= -0.3141 * 1000 = \$(314.09)$
Number to bear put spreads to neutralize $= \$(2.40)M / (314.09) = 7,641$
Therefore, need to buy 7,641 ATM Put contracts and sell 7,641 out of the money Put contracts.

(c)   Critique the delta-neutral approach used in part (b) and suggest an alternative.

Commentary on Question:
Candidates performed as expected. Most candidates recognized the limitations of a static hedge and were able to suggest an alternative.

5. Continued

Critique:
- Hedges position and tactically takes advantage of downward market expectation
- Static hedge would not immunize the economic mark-to-market loss of the liability.

Recommendation:
- The remaining mismatch could be hedged by dynamic hedging program or updating the static hedge to include gamma

(d) Estimate today's implied volatility (standard deviation) of the Russell 2000 Index underlying the bear put spread based on the information given above.

Commentary on Question:
Candidates performed as expected. Most candidates recognized the correct application of Vega.

Vega is the first derivative with respect to volatility.
Implied volatility (given) = 24%.
Vega (given) = 30.11
Initial market price of bear put spread (given) = 6.72
Actual Market price of bear put spread (given) = 6.93

$dO/d\sigma = Option_t – Option_{t-1} / (\sigma_t – \sigma_{t-1}) =$
$30.11 = (6.93 – 6.72) / (\sigma_t – 24\%)$
$\sigma_t = 24.697\%$

(e) Determine the one-day mark-to-market profit or loss on your delta-hedged portfolio of one contract of bear put spread based on historical data.

Commentary on Question:
Candidates performed as expected. Most candidates recognized the correct application of the MTM formula from section 10.5.

$S = 100$
$dt = 1/250$
Actual volatility = 0.28
Implied volatility = 0.24697
Gamma = 0.0125 (given)

1 day MTM $= 0.5 * [(\sigma^2_{\text{actual volatility}}- \sigma^2_{\text{implied volatility}}] * S(t)^2*Gamma * dt$
$= 0.5 * (0.28^2 – 0.24697^2) * 100^2 * 0.0125 * 1/250$
$= 0.004351$

Option multiplier is 1000 therefore one-day MTM= 1000* 0.004351

The one-day MTM profit is $4.35

6. **Learning Objectives:**
1. The candidate will understand the fundamentals of stochastic calculus as they apply to option pricing.

2. The candidate will understand how to apply the fundamental theory underlying the standard models for pricing financial derivatives. The candidate will understand the implications for option pricing when markets do not satisfy the common assumptions used in option pricing theory such as market completeness, bounded variation, perfect liquidity, etc. The Candidate will understand how to evaluate situations associated with derivatives and hedging activities.

Learning Outcomes:
(1f) Demonstrate understanding of option pricing techniques and theory for equity and interest rate derivatives.

(2d) Understand the different approaches to hedging.

(2e) Understand how to delta hedge and the interplay between hedging assumptions and hedging outcomes.

(2f) Appreciate how hedge strategies may go awry.

Sources:
Paul Wilmott Introduces Quantitative Finance, Wilmott, Paul, 2nd Edition, 2007 Ch. 8, 10

Paul Wilmott Introduces Quantitative Finance, Wilmott, Paul, 2nd Edition, 2007 Ch. 6

Commentary on Question:
The majority of the candidates attempted most parts in this question and received partial marks in them. The question intends to test the candidates' understanding of the replicating portfolio in a delta-hedge strategy, not just the application of the Black Scholes formula. Only a few candidates demonstrated understanding of the decomposition of the replicating portfolio. Most candidates demonstrated understanding of the evolution of the portfolio overtime which resulted in hedge error.

Solution:
(a) Calculate the implied volatility for the put option.

6. Continued

Commentary on Question:
Due to the $\sqrt{2}$ in the calculation, rounding errors in the calculated volatility are acceptable. Answers that were not exactly 20% but reasonably close, with proper demonstration of the calculation process earned full marks.
Many candidates adopted an alternative approach by solving the quadratic equation, which was more complicated and took longer time, with higher chance of calculation errors. This led to losing partial marks for not getting the correct final answer.

$N(d_1) = 0.7142$ and according to the cummulative Normal Distribution table, $d_1 = 0.565$

$N(d_2) = 0.6114$ and according to the cummulative Normal Distribution table, $d_2 = 0.283$

Using the Black Scholes formula, it follows that $\sigma = (d_1 - d_2)/\sqrt{2} = 0.2$

Alternative:

$N(d_1) = 0.7142$ and according to the cummulative Normal Distribution table, $d_1 = 0.565$

Using the Black Scholes formula,

$$d_1 = \frac{Ln\left(\frac{S}{K}\right) + \left(r + \frac{1}{2}\sigma^2\right)(T-t)}{\sigma\sqrt{T-t}} = 0.565$$

$$\left(0.06 + \frac{1}{2}\sigma^2\right) * 2 = 0.565 \, \sigma \, \sqrt{2}$$

By solving quadratic equation,
σ is 0.2.

(b) Calculate the amounts invested in the risk-free asset and the equity index in the replicating portfolio at time 0.

Commentary on Question:
Candidates performed poorly on this part. The question does not intend to be computational intensive but rather focuses on the understanding of the replicating portfolio. The key step is to identify the decomposition of the put option Black Scholes formula into the risk-free asset component and the equity investment component. Thus calculation of the put option is not a necessary step in solving for the risk-free asset amount and equity index investment.

6. **Continued**

Most candidates were able to identify the equity component with the delta and to try to solve for the risk-free assets by combining the put option value and equity component. However, a common mistake was failing to identify the short/long position of the equity component and put option, consequently getting the incorrect risk-free asset amount. In such situation, partial marks were given for setting up calculations correctly.

The replicating portfolio has the same initial value as the liability, which is a put option. The Black-Scholes put option pricing formula

$$P = K\,e^{-rT}N(-d_2) - S_0 e^{-\delta T}N(-d_1)$$

It can be decomposited into the equity component and the risk-free investment component, i.e.
P = B + ɸ S_0 where
$\Phi = -e^{-\delta T}N(-d_1)$ the number of units of equity index held in the delta-hedging strategy and B = K $e^{-rT}N(-d_2)$ the risk-free amount.

Since N(-d2) = 1-N(d2) = 1 − 0.6114 = 0.3886,
the risk-free amount is
= 100 * exp (-0.06 * 2) * N(-d₂)
= 34.47

And the equity component
Φ_0 S₀ = -N(-d₁) S₀ = -0.2858 * 100 = -28.58

(c)   Calculate the value of the hedging portfolio right before rebalancing.

Commentary on Question:
Most candidates performed well on this part. Candidates earned full marks for 1) correctly growing the risk-free portfolio, 2) applying the new equity index price with the correct units AND 3) using the consistent position (long/short) from part b), even if the amounts and positions in (b) were not correct.

Right before rebalancing, the replicating portfolio still holds -0.2858 units of equity index and the initial amount of B₀ has earned risk-free rate for a month

Therefore the value of the position becomes $B_1 + \phi_0\, S_1$
= B_0 * exp (r * 1/12) + Φ_0 S₁
= 34.47 * exp (0.06 *1/12) -0.2858 * 99.57
= 6.18 (or 6.181)

6. **Continued**

(d) Calculate the hedge error assuming that volatility of the stock and the risk-free interest rates have not changed.

Commentary on Question:
This is intended to be the computational intensive part of this question. It requires applying all the steps in Black Scholes formula and calculating the new put option value.
Several candidates failed to correctly update the term, which should be 23/12 (or 2-1/12). Some candidates mistakenly used 23/24.
Candidates earned full marks for 1) correctly calculating the new put option value AND 2) correctly taking difference from the replicating portfolio before rebalancing, even if the answer in part (c) is not correct.

After rebalancing, the replicating portfolio should have the same total value as the liability again.

Therefore the hedge cost after the rebalancing will be the value of a put option with S = 99.57 and T* = 2- 1/12

Applying Black-Scholes formula

$$d_1 = \frac{Ln\left(\frac{S}{K}\right) + \left(r + \frac{\sigma^2}{2}\right) * T}{\sigma\sqrt{T}}$$

$$= \frac{Ln\left(\frac{99.57}{100}\right) + \left(0.06 + \frac{0.2^2}{2}\right) * (2 - \frac{1}{12})}{0.2 * \sqrt{2 - \frac{1}{12}}}$$

= 0.5382 (or 0.538319)

$$d_2 = d_1 - \sigma\sqrt{T} = 0.5382 - 0.2 * \sqrt{2 - \frac{1}{12}}$$

= 0.2613 (or 0.26154)

And thus
P = 100 * exp (-0.06 * (2-1/12)) * N(-0.2613) – 99.57 * N(-0.5382)
  = 5.99 (or 5.982)

Since the hedge cost before rebalancing (6.18) is greater than the hedge cost after rebalancing, there will be income released from the rebalacning because the new liability position does not cost as much to hedge.
The income released is thus 6.18– 5.99 = 0.19 (or 0.1997)

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6. **Continued**

(e)   Explain two possible sources of the hedge error.

Commentary on Question:
Most candidates explained the sources of hedge error in a general sense, such as: no gamma hedge, no theta hedge, risk free rate/volatility not constant, transaction cost. Some included drawbacks of the Black Scholes formula, which are not intended as the answer for this particular situation (the hedge error has been defined in the question).

Cause 1: Discrete rebalancing
If the replicating portfolio is continuously rebalanced, the equity component and risk-free component would perfectly reflect the movement of liability with respect to the equity index move, in which case there would be no hedge error. Since the replicating portfolio is rebalanced discretely, there will be a hedge error.

Cause 2: use of hedging volatility
The hedging volatility may not be the same as the real volatility of the equity index. If the hedging volatility is greater than the actual volatility, the calculated hedge cost is larger than needed, i.e. there's over-hedge.

7. **Learning Objectives:**
   1. The candidate will understand the fundamentals of stochastic calculus as they apply to option pricing.

   3. The candidate will understand the quantitative tools and techniques for modeling the term structure of interest rates and pricing interest rate derivatives.

Learning Outcomes:
(1d) Understand and apply Ito's Lemma.

(1g) Demonstrate understanding of the differences and implications of real-world versus risk-neutral probability measures.

(1i) Understand and apply Girsanov's theorem in changing measures.

(3a) Understand and apply the concepts of risk-neutral measure, forward measure, normalization, and the market price of risk, in the pricing of interest rate derivatives.

(3c) Understand and apply popular one-factor interest rate models including Vasicek, Cox-Ross-Ingersoll, Hull-White, Ho-Lee, Black-Derman-Toy, Black-Karasinski.

Sources:
Paul Wilmott Introduces Quantitative Finance (2nd Edition, 2007) Chapter 16

An Introduction to the Mathematics of Financial Derivatives (Hirsa, Ali and Neftci, Salih N., 3rd Edition, 2014) Chapters 10 & 14

Problems and Solutions in Mathematical Finance: Stochastic Calculus (Chin, Eric, Nel, Dian and Olafsson, Sverrir, 2014) Chapter 3

Commentary on Question:
The question tests candidates' understanding and abilities of deriving relationships between bond prices and short rate model (e.g., applying Ito's Lemma on SDE, etc.), as well as applying Girsanov's theorem in changing the measures.

Solution:
(a) Show that $\zeta = \zeta(r,t,T)$ is independent of the maturity T by constructing a self-financing and instantaneously risk-free portfolio with Δ_1 units of a bond maturing at time T_1 and Δ_2 units of a bond maturing at time T_2.

   Commentary on Question:
   Most Candidates attempted this question, those who were able to show ζ is the same for both Vs got full mark.

7. **Continued**

$$\Pi = \Delta_1 V_1 + \Delta_2 V_2$$

Since the portfolio is self-financing $d\Pi = \Delta_1 dV_1 + \Delta_2 dV_2$

From Ito's lemma

$$dV_i = \frac{\partial V_i}{\partial t} dt + \frac{\partial V_i}{\partial r} dr + \frac{1}{2} \frac{\partial^2 V_i}{\partial r^2} (dr)^2, i = 1,2.$$

$$dV_i = \left(\frac{\partial V_i}{\partial t} + \mu \frac{\partial V_i}{\partial r} + \frac{1}{2} \sigma^2 \frac{\partial^2 V_i}{\partial r^2} \right) dt + \sigma \frac{\partial V_i}{\partial r} dW_t, i = 1,2.$$

To make the portfolio risk free Δ's has to be chosen such that

$$\Delta_1 \frac{\partial V_1}{\partial r} + \Delta_2 \frac{\partial V_2}{\partial r} = 0 \Rightarrow \frac{\Delta_2}{\Delta_1} = -\frac{\frac{\partial V_1}{\partial r}}{\frac{\partial V_2}{\partial r}}$$

By substituting this we have

$$d\Pi = \Delta_1 \left(\frac{\partial V_1}{\partial t} + \mu \frac{\partial V_1}{\partial r} + \frac{1}{2} \sigma^2 \frac{\partial^2 V_1}{\partial r^2} \right) dt - \Delta_1 \left(\frac{\frac{\partial V_1}{\partial r}}{\frac{\partial V_2}{\partial r}} \right) \left(\frac{\partial V_2}{\partial t} + \mu \frac{\partial V_2}{\partial r} + \frac{1}{2} \sigma^2 \frac{\partial^2 V_2}{\partial r^2} \right) dt$$

Since Π is instantaneously risk free $d\Pi = r\Pi dt = r\Delta_1 \left(V_1 - \left(\frac{\frac{\partial V_1}{\partial r}}{\frac{\partial V_2}{\partial r}} \right) V_2 \right) dt$, thus we have the following equality:

$$\left(\frac{\partial V_1}{\partial t} + \mu \frac{\partial V_1}{\partial r} + \frac{1}{2} \sigma^2 \frac{\partial^2 V_1}{\partial r^2} \right) - \left(\frac{\frac{\partial V_1}{\partial r}}{\frac{\partial V_2}{\partial r}} \right) \left(\frac{\partial V_2}{\partial t} + \mu \frac{\partial V_2}{\partial r} + \frac{1}{2} \sigma^2 \frac{\partial^2 V_2}{\partial r^2} \right)$$

$$= r \left(V_1 - \left(\frac{\frac{\partial V_1}{\partial r}}{\frac{\partial V_2}{\partial r}} \right) V_2 \right)$$

$$\left(\frac{\partial V_1}{\partial t} + \mu \frac{\partial V_1}{\partial r} + \frac{1}{2} \sigma^2 \frac{\partial^2 V_1}{\partial r^2} \right) \frac{\partial V_2}{\partial r} - \left(\frac{\partial V_2}{\partial t} + \mu \frac{\partial V_2}{\partial r} + \frac{1}{2} \sigma^2 \frac{\partial^2 V_2}{\partial r^2} \right) \frac{\partial V_1}{\partial r}$$

$$= r \left(V_1 \frac{\partial V_2}{\partial r} - V_2 \frac{\partial V_1}{\partial r} \right)$$

Collecting like terms i.e. terms with same subscripts of V we have

$$\frac{\frac{\partial V_1}{\partial t} + \frac{1}{2} \sigma^2 \frac{\partial^2 V_1}{\partial r^2} - rV_1}{\frac{\partial V_1}{\partial r}} = \frac{\frac{\partial V_2}{\partial t} + \frac{1}{2} \sigma^2 \frac{\partial^2 V_2}{\partial r^2} - rV_2}{\frac{\partial V_2}{\partial r}}$$

7. **Continued**

LHS is a function of T_1 and RHS is a function of T_2, so they both must be independent of T_1, T_2.

$$\frac{\frac{\partial V_1}{\partial t} + \frac{1}{2}\sigma^2 \frac{\partial^2 V_1}{\partial r^2} - rV_1}{\frac{\partial V_1}{\partial r}} = \frac{\frac{\partial V_2}{\partial t} + \frac{1}{2}\sigma^2 \frac{\partial^2 V_2}{\partial r^2} - rV_1}{\frac{\partial V_2}{\partial r}} = \zeta(r,t)$$

For any bond, we should have

$$\frac{\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 \frac{\partial^2 V}{\partial r^2} - rV}{\frac{\partial V}{\partial r}} = \zeta(r,t) = \zeta$$

which leads to (when re-writing ζ)

$$\zeta = (\lambda(r,t)\sigma - \mu) \Rightarrow \lambda(r,t) = \frac{\zeta + \mu}{\sigma}$$

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 \frac{\partial^2 V}{\partial r^2} + (\mu - \lambda\sigma)\frac{\partial V}{\partial r} - rV = 0.$$

(b) Show by using Ito's Lemma that $dZ = \sigma X \frac{\partial V}{\partial r}(\lambda dt + dW_t)$.

Commentary on Question:
Most candidates attempted the question, most lost marks due to incorrect derivation on partial derivatives

$$dZ = XdV + VdX + dVdX$$
$$dV = \left(\frac{\partial V}{\partial t} + \mu\frac{\partial V}{\partial r} + \frac{1}{2}\sigma^2 \frac{\partial^2 V}{\partial r^2}\right)dt + \sigma\frac{\partial V}{\partial r}dW_t$$

Using an earlier result,

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 \frac{\partial^2 V}{\partial r^2} + \mu\frac{\partial V}{\partial r} = \lambda\sigma\frac{\partial V}{\partial r} + rV$$
$$dV = \left(\lambda\sigma\frac{\partial V}{\partial r} + rV\right)dt + \sigma\frac{\partial V}{\partial r}dW_t$$

$$dX = -rXdt$$
$$dVdX = 0$$
$$dZ = X\left(\left(\lambda\sigma\frac{\partial V}{\partial r} + rV\right)dt + \sigma\frac{\partial V}{\partial r}dW_t\right) + V(-rXdt)$$
$$dZ = \sigma X\frac{\partial V}{\partial r}(\lambda dt + dW_t)$$

7. Continued

(c)   Show by using Girsanov's theorem that for an appropriate measure $\mathbb{Q}$ equivalent to $\mathbb{P}$:

$$V(r,t;T) = \mathbb{E}_t^{\mathbb{Q}}[\exp(-\int_t^T r(s)ds)].$$

Commentary on Question:
Most candidate did not attempt this question, those who did performed well on this part.

By Girsanov's theorem, $W_t^{\mathbb{Q}} = \int_0^t \lambda(s)ds + W_t$, or, $dW_t^{\mathbb{Q}} = \lambda dt + dW_t$, where $W_t^{\mathbb{Q}}$ is a standard Brownian motion under the $\mathbb{Q}$ measure.

$$dZ = \sigma X \frac{\partial V}{\partial r} dW_t^{\mathbb{Q}}$$

Therefore, $Z = \exp(-\int_0^t r(s)ds) V(r,t;T)$ is a martingale under the $\mathbb{Q}$ measure

$$\exp\left(-\int_0^t r(s)ds\right) V(r,t;T) = \mathbb{E}^{\mathbb{Q}}\left[\exp\left(-\int_0^T r(s)ds\right) V(r,T;T) | \mathcal{F}_t\right]$$

Since

$$V(r,T;\ T) = 1$$
$$V(r,t;\ T) = \mathbb{E}^{\mathbb{Q}}\left[\exp\left(-\int_t^T r(s)ds\right) | \mathcal{F}_t\right]$$

(d)   Show that $dr = -\zeta dt + \sigma\, dW_t^{\mathbb{Q}}$ where $W_t^{\mathbb{Q}}$ is a standard Wiener process under the measure $\mathbb{Q}$.

Commentary on Question:
Very simple question, most candidates did well.

$$dr = \mu dt + \sigma dW_t = \mu dt + \sigma(dW_t^{\mathbb{Q}} - \lambda dt)$$
$$dr = (\mu - \sigma\lambda)dt + \sigma dW_t^{\mathbb{Q}}$$
$$dr = -\zeta dt + \sigma\, dW_t^{\mathbb{Q}}$$

(e)   Determine the drift and the diffusion of the SDE for $r(t)$ under the measure $\mathbb{Q}$ in terms of α and β.

Commentary on Question:
Very tough question, most did not attempt, and most of those who did made errors.

7. Continued

$$\frac{\partial V}{\partial r} = -B(t,T)V$$

$$\frac{\partial^2 V}{\partial r^2} = B(t,T)^2 V$$

$$\frac{\partial V}{\partial t} = V\left(\frac{\partial A}{\partial t} - r\frac{\partial B}{\partial t}\right)$$

$$\frac{\partial A}{\partial t} = \alpha(T-t) - \frac{\beta^2}{2}(T-t)^2$$

$$\frac{\partial B}{\partial t} = -1$$

$$\frac{\partial V}{\partial t} = \left(\alpha(T-t) - \frac{\beta^2}{2}(T-t)^2 + r\right)V$$

$$\frac{\partial V}{\partial r} = -(T-t)V,$$

$$\frac{\partial^2 V}{\partial r^2} = (T-t)^2 V$$

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2\frac{\partial^2 V}{\partial r^2} - rV = \left(\alpha(T-t) - \frac{\beta^2}{2}(T-t)^2 + \frac{1}{2}\sigma^2(T-t)^2\right)V$$

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2\frac{\partial^2 V}{\partial r^2} - rV = -\frac{\partial V}{\partial r}\left(\alpha - \frac{\beta^2}{2}(T-t) + \frac{1}{2}\sigma^2(T-t)\right)$$

Since

$$\frac{\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2\frac{\partial^2 V}{\partial r^2} - rV}{\frac{\partial V}{\partial r}}$$

has to be independent of $T \Rightarrow \sigma = \beta$ that means $\zeta = -\alpha \Rightarrow \mu - \lambda\sigma = \alpha$, therefore SDE for $r(t)$ in the $\mathbb{Q}$ measure is

$$dr = \alpha dt + \beta dW_t^{\mathbb{Q}}$$

8. **Learning Objectives:**
1. The candidate will understand the fundamentals of stochastic calculus as they apply to option pricing.

3. The candidate will understand the quantitative tools and techniques for modeling the term structure of interest rates and pricing interest rate derivatives.

Learning Outcomes:
(1d) Understand and apply Ito's Lemma.

(3b) Apply the models to price common interest sensitive instruments including: callable bonds, bond options, caps, floors and swaptions.

(3c) Understand and apply popular one-factor interest rate models including Vasicek, Cox-Ross-Ingersoll, Hull-White, Ho-Lee, Black-Derman-Toy, Black-Karasinski.

Sources:
Introduces Quantitative Finance, Wilmott, Paul, 2nd Edition, Chapter 16

An Introduction to the Mathematics of Financial Derivatives, Hirsa and Neftci, 3rd Edition, Chapter 8, 10

Q#10 page 132 of Chin et al and Q#13, page 135 of Chin et al

Commentary on Question:
A straight forward question with most of candidates being able to get decent answers and many getting full mark.

Purpose of the question is to test the understanding of the two interest rate model: Vasicek and CIR model and their application with Ito's Lemma.

Solution:
(a) Describe the properties of the following two models for interest rates:

   (i) The one-factor Vasicek model

   (ii) The Cox-Ingersoll-Ross model

Commentary on Question:
Part (a) tests the characteristics and limitations of the two interest rate models. Most of the candidates could list out the more general characteristics. However, some candidates listed contradictory comments and thus didn't get any marks for that.

8. Continued

Vasicek -

The model is mean reverting. A mean-reverting process is one with time-dependent drift such that the model is reverting to a constant level.

It is time homogenous – future dynamics of r(t) only depend upon the current value of r(t) rather than what the present time t is.

It follows a normal distribution

The model allows negative interest rates

The model is one-factor equilibrium model (or has arbitrage opportunities)

Yield curve – upward sloping, downward sloping and slightly humped.

It is easy to implement since the characteristic functions of all related quantities are available.

Candidates only need to answer any three of the above for full mark of this part.

CIR -

The model is mean reverting to a constant level, with long rates being linearly dependent on the short rates.

It is time homogeneous.

It follows a lognormal distribution

Volatility depends on the level of the rates: it is high/low when rates are high/low

Impossibility of negative interest rates under certain conditions.

The model is one-factor equilibrium model (or has arbitrage opportunities)

Yield curve – upward sloping, downward sloping and slightly humped.

It is more involving to implement as it is linked to the non-central Chi-Squared distribution.

Candidates only need to answer any three of the above for full mark of this part.

8. Continued

(b) Show that the solution to the SDE is

$$r(t) = \theta + \left(r(0) - \theta\right)e^{-\alpha t} + \sigma e^{-\alpha t}\int_0^t e^{\alpha s}dW_s$$

Commentary on Question:
Most of the candidate got full mark on this part.

By taking $X(t) = e^{\alpha t}r(t)$,

$$dX = e^{\alpha t}dr + \alpha e^{\alpha t}r\,dt = \alpha\theta e^{\alpha t}dt + \sigma e^{\alpha t}dW_t$$

Taking integral of both sides from 0 to t :

$$X(t) - X(0) = \int_0^t \alpha\theta e^{\alpha t}dt + \int_0^t \sigma e^{\alpha s}dW_s$$

$$r(t) = e^{-\alpha t}\left(r(0) + \theta(e^{\alpha t} - 1) + \int_0^t \sigma e^{\alpha s}dW_s\right)$$

$$r(t) = \theta + (r(0) - \theta)e^{-\alpha t} + \sigma\int_0^t e^{-\alpha(t-s)}dW_s$$

(c) Determine the expectations of $r(t)$ and r_t, respectively.

Commentary on Question:
The solution is easy. But, some candidates didn't provide comment on the Ito's integral so they only got half the mark.

$$E[r(t)] = \theta + (r(0) - \theta)e^{-\alpha t} \text{ since } E\left\{\int_0^t e^{\alpha s}\,dW_s\right\} = 0$$

$$E[r_t] = \theta + (r_0 - \theta)e^{-\alpha t} \text{ since } E\left\{\int_0^t e^{\alpha s}\sqrt{r_s}dW_s\right\} = 0$$

(d) Determine the variances of $r(t)$ and r_t, respectively.

Commentary on Question:
The proof on Vasicek model is straightforward. For CIR, some candidates were not aware that they could re-use the solution from part (c) and failed to use Ito Isometry to complete the 2nd moment integral.

8. Continued

$$\text{V}ar[\text{r(t)}] = E[\text{r(t)} - E[\text{r(t)}]\}^2$$

$$= \sigma^2 e^{-2\alpha t} E\left[\left(\int_0^t e^{\alpha s}\, dW_s\right)^2\right]$$

using Ito's Isometry

$$= \sigma^2 e^{-2\alpha t} E\left[\int_0^t e^{2\alpha s}\, ds\right]$$

$$= \sigma^2 e^{-2\alpha t} \frac{e^{2\alpha t} - 1}{2\alpha}$$

$$Or\ \frac{\sigma^2}{2\alpha}(1 - e^{-2\alpha t})$$

$$\text{V}ar[r_t] = E[r_t - E[r_t]\}^2$$

$$= \sigma^2 e^{-2\alpha t} E\left[\left(\int_0^t e^{\alpha s} \sqrt{r_s}\, dW_s\right)^2\right]$$

using Ito's Isometry

$$= \sigma^2 e^{-2\alpha t} E\left[\int_0^t e^{2\alpha u}\, r_s ds\right]$$

$$= \sigma^2 e^{-2\alpha t} \int_0^t e^{2\alpha u}\, E[r_s] ds$$

Substituting $E[r_t]$ from part (c) gives:

$$= \sigma^2 e^{-2\alpha t} \int_0^t e^{2\alpha s}\, (\theta + (r_o - \theta))e^{-\alpha s}) ds$$

$$= (\sigma^2 \theta)\frac{1}{2\alpha}(1 - e^{-2\alpha t} - 2e^{-\alpha t} + 2e^{-2\alpha t}) + \frac{\sigma^2}{\alpha} r_o e^{-2\alpha t}(e^{\alpha t} - 1)$$

$$Or\ \frac{\sigma^2 \theta}{2\alpha}(1 - e^{-\alpha t})^2 + \frac{\sigma^2}{\alpha} r_o e^{-\alpha t}(1 - e^{-\alpha t})$$

8. **Continued**

(e) Show that the expectations and the variances in parts (c) and (d) have the following limits as $t \to \infty$

(i) $\lim\limits_{t \to \infty} E\big[r(t)\big] = \lim\limits_{t \to \infty} E\big[r_t\big] = \theta$

(ii) $\lim\limits_{t \to \infty} Var\big[r(t)\big] = \dfrac{\sigma^2}{2\alpha}$ and $\lim\limits_{t \to \infty} Var\big[r_t\big] = \dfrac{\sigma^2}{2\alpha}\theta$

Commentary on Question:
Simply application of the limit function. Most candidates get full mark if parts (c) and (d) are correct.

Noting that $\lim\limits_{t \to \infty} e^{-\alpha t} = 0$, from parts (c) and (d) we see that

$$\lim_{t \to \infty} E[r(t)] = \lim_{t \to \infty} E[r_t] = \theta$$

$$\lim_{t \to \infty} Var[r(t)] = \frac{\sigma^2}{2\alpha}$$
$$\lim_{t \to \infty} Var[r_t] = \frac{\sigma^2}{2\alpha}\theta$$

9.

Learning Objectives:
4.   The candidate will understand the concept of volatility and some basic models of it.

Learning Outcomes:
(4b)   Understand and apply various techniques for analyzing conditional heteroscedastic models including ARCH and GARCH.

Sources:
QFIC-109-15 Chapter 20 Risk Management and Financial Institutions
Tsay: Time Series, Chapter 3

Commentary on Question:
This question tests candidates' understanding of GARCH models with regards to model specification, parameter estimation, and volatility forecast.

Solution:
(a)   Identify a potential issue with the above model.

   Commentary on Question:
   The candidates performed as expected on this part. To receive credit, candidates must identify the specific issues of the given model. Credit was not given for merely stating the generic issues of the GARCH(1,1) model (such as the issue of giving equal weight to both positive and negative shocks).

   A potential issue is that the weight for long-run average variance is negative since the weight of $0.25 + 0.82 = 1.07 > 1$. Thus, this GARCH(1,1) model is unstable.

(b)   Propose a simpler alternative model that addresses the issues you identified in part (a) and outline the pros and cons of your proposed model vs. the original model.

   Commentary on Question:
   *The candidates performed below expectation on this part. To receive credit, candidates must propose a **simpler** alternative as asked. Credit was not given for proposing an alternative that is not simpler than the original model.*

   Estimate an EWMA model instead: $\sigma_t^2 = \theta u_{t-1}^2 + (1 - \theta)\sigma_{t-1}^2$
   Pros: Simpler than GARCH(1,1)
   Cons: No mean-reversion feature as with GARCH(1,1)

(c)   Determine the parameters of your proposed model in part (b) by using the maximum likelihood method and the available stock price data.

9. Continued

Commentary on Question:
The candidates performed poorly on this part. Partial credit was given for each step completed correctly.

t)	1	2	3	4
ABC Stock Price (P_t)	$10.0	$10.4	$9.8	$10.3
$u_t = \dfrac{P_t}{P_{t-1}} - 1$		4.0%	-5.8%	5.1%

For Day 3: $\sigma_3^2 = u_2^2$
For Day 4: $\sigma_4^2 = \theta u_3^2 + (1-\theta)\sigma_3^2 = \theta u_3^2 + (1-\theta)u_2^2$

Maximum Likelihood quantity $= \sum -\dfrac{u_t^2}{\sigma_t^2} - \ln\sigma_t^2$

$= \left[-\dfrac{u_3^2}{\sigma_3^2} - \ln\sigma_3^2\right] + \left[-\dfrac{u_4^2}{\sigma_4^2} - \ln\sigma_4^2\right]$

Since u_2^2, u_3^2, u_4^2 and σ_3^2 are known, solving for θ that maximizes the quantiy is equivalent to solving for σ_4^2 that maximizes $-\dfrac{u_4^2}{\sigma_4^2} - \ln\sigma_4^2$

Let $Y = -\dfrac{u_4^2}{\sigma_4^2} - \ln\sigma_4^2$, take the derivative with respect to σ_4^2, and set the derivative to 0. This gives $\sigma_4^2 = u_4^2$

Using Day 4 equation, this gives $\theta = \dfrac{u_4^2 - u_2^2}{u_3^2 - u_2^2} = \dfrac{5.1\%^2 - 4.0\%^2}{(-5.8\%)^2 - 4.0\%^2} = 0.57$

(d) Calculate the forecasted Day 8 annualized volatility as of Day 4 using the model you established in part (c).

Commentary on Question:
The candidates performed poorly on this part. Many candidates did not correctly estimate the model parameters in part (c).

For EWMA model, $E(\sigma_8^2) = \sigma_4^2 = 5.1\%^2$
Annual volatility forecast $= 5.1\% * \sqrt{252} = 81\%$

(e) Compare and contrast your model to IGARCH(1,1) model.

Commentary on Question:
The candidates performed poorly on this part. Many candidates either skipped this part or did not reflect on the key contrast between the EWMA model and the IGARCH(1,1) model. Partial credit was given for each step that was done correctly.

9.   Continued

Model specification (using Hull's notation in QFIC-109-15)
EWMA: $\sigma_t^2 = \beta\sigma_{t-1}^2 + (1-\beta)u_{t-1}^2$
IGARCH(1,1): $\sigma_t^2 = \alpha + \beta\sigma_{t-1}^2 + (1-\beta)u_{t-1}^2$
Compare and contrast: EWMA is a restricted form of IGARCH(1,1) with $\alpha = 0$

Volatility forcast for day n+t at the end of day n–1
Using Hull's notation in QFIC-109-15:
    EWMA: $E(\sigma_{n+t}^2) = \sigma_n^2$
    IGARCH(1,1): $E(\sigma_{n+t}^2) = \sigma_n^2 + t\alpha$

Using Tsay's notation:
    EWMA: $\sigma_{n-1}^2(t+1) = \sigma_{n-1}^2(1)$
    IGARCH(1,1): $\sigma_{n-1}^2(t+1) = \sigma_{n-1}^2(1) + t\alpha$

Compare and contrast: In IGARCH(1,1), the volatility forecast forms a straight line with a slope $\alpha \neq 0$. When $\alpha = 0$, the volatility forecast is the current forecast regardless of the forecast horizon, which is the same outcome as the EWMA model.

10. **Learning Objectives:**

5. The candidate will understand and identify the variety of fixed instruments available for portfolio management. This section deals with fixed income securities. As the name implies the cash flow is often predictable, however there are various risks that affect cash flows of these instruments. In general the candidates should be able to identify the cash flow pattern and the factors affecting cash flow for commonly available fixed income securities. Candidates should also be comfortable using various interest rate risk quantification measures in the valuation and managing of investment portfolios. Candidates should also understand various strategies of managing the portfolio against given benchmark.

Learning Outcomes:

(5c) Demonstrate an understanding of the characteristics of leveraged loans.

(5e) Demonstrate an understanding of the characteristics and mechanics of fixed income ETFs.

Sources:
QFIC-117-17: High-Yield Bond Market Primer

Chapter 21, The Handbook of Fixed Income Securities, Fabozzi, Frank, 8th Edition, 2012

Commentary on Question:
The candidates' performance is fair. But none of them receives full scores. This question aims at testing the candidates' knowledge of high yield bonds. Then further testing on their understanding of the characteristics of leveraged loans, fixed income ETFs and Open-ended mutual fund.

Solution:
(a) Define high yield bonds and explain their common features.

   Commentary on Question:
   The candidates performance is good. Most of them can define high yield bonds and explain some of the common features. Half of them only give one or two common features and fail to receive full scores.

   1) High yield bonds are non-investment grade bonds.
   2) They are generally higher risk
   3) Offer higher interest rates
   4) Usually issued by companies seeking money for growth or other cash flow purposes
   5) They usually have investor friendly covenants to attract buyers.

10. Continued

(b) Describe four different types of issuers of high yield bonds and their objectives in issuing high yield bonds.

Commentary on Question:
The performance is good. Most of the candidates receive full scores. A handful of them do not attempt this part.

1) Fallen angel
Companies that used to carry higher rating but have since been downgraded
They issue bonds to improve their balance sheet for an eventual upgrade

2) Start ups
New companies that need funding to grow its operations
Do not have an operational history strong enough to achieve investment grade ratings.

3) Cyclical businesses
Use high yield market to weather downturns

4) Leveraged buy outs
Use high yield bond to fund purchase of companies and to pay special dividends

5) Bankruptcy exits
Raise funds as part of the deal to exit bankruptcy

6) Capital-intensive businesses
Use high yield bond to fund capital

(c) You are considering two alternatives to invest in high yield bonds:

I.   Individual high yield bonds

II.   High yield bond exchange traded funds (ETF)

Explain the advantages of each alternative.

Commentary on Question:
The performance is fair. Almost of the candidates answer the pros of investment in high yield ETF correctly. However, only a few know the pros of investment in individual high yield bonds. Suggest the candidates to read the study materials in details.

10. Continued

- Pros of investment in individual high yield bonds:
1) Customizable: make tactical investments in bonds to earn above market return
2) Participate in bond issuance process - influence the terms during the underwriting process

- Pros of investments in high yield ETF:
1) Benefits of diversification (investment in wide portfolio of high yield bonds) leads to lower risk
2) Low cost (low transaction cost) to access this asset class.
3) Transparency
4) Tax efficiency
5) Risk Control
6) Generally better liquidity

(d) Compare the price execution of an ETF with transacting at the NAV of an open-ended mutual fund.

Commentary on Question:
The performance is very poor. None of the candidates receives full scores. A lot of candidates don't even attempt the question. Those candidates who attempt the question only address ETF portion of the question and hence only receive partial scores.

ETF:
1) In an ETF, each investor incurs the transaction costs created by their specific transaction, through the market price of the ETF
2) Price should reflect the cost of ETF share creation
3) Existing ETF investors are unaffected.

Open-ended mutual fund:
1) The securities purchased as a result of the new mutual fund investor's entry may cost more than the NAV.
2) The differential is paid for by existing investors in the fund (all transaction costs are shared by all investors).
3) Customary to value securities on the bid, so investor's entry may cost more than NAV

11. Learning Objectives:

2. The candidate will understand how to apply the fundamental theory underlying the standard models for pricing financial derivatives. The candidate will understand the implications for option pricing when markets do not satisfy the common assumptions used in option pricing theory such as market completeness, bounded variation, perfect liquidity, etc. The Candidate will understand how to evaluate situations associated with derivatives and hedging activities.

Learning Outcomes:

(2c) Compare and contrast various approaches for setting volatility assumptions in hedging.

Sources:

The Handbook of Fixed Income Securities, Fabozzi, Frank, 8th Edition, Ch. 12

QFIC-117-17: High-Yield Bond Market Primer

Commentary on Question:

The candidates' performance is not good. None of them gets full score. This question is to test the candidates' understanding about characteristics and cash flows of different corporate bonds, syndication, etc.

Solution:

(a) Assess each of the statements above and explain your answers.

Commentary on Question:

The candidates' performance is relatively good. Candidates only receive points by explaining why the statements are true or false ("assessing" the statements), so no point is given for simply stating "true" or "false".

(i) False – a corporate trustee represents interest of bondholders.

(ii) True– Because compounding is not as meaningful in low interest rate markets as when rates are higher.

(iii) False – high-yield has 20% of overall bond market **OR** high-yield market is larger than the US Treasury market or the municipal bond market

(iv) False – high-yield covenants are looser than on bank loans as they are unsecured obligations. This provides the issuer more operating flexibility / avoid need for quarterly compliance certification.

(v) True – As a principal amount of debt is paid off, they issue another series of bonds under the same mortgage. Easier to issue a series of bonds under one mortgage and one indenture than to create new bond issues.

(vi) False – collateral trust bonds can be secured by financial assets and real property.

11. Continued

(b)   Explain why companies would prefer to issue the following bonds:

   (i)   Pay-in-kind (PIK) notes

   (ii)   Bonds with incorporated poison puts

   Commentary on Question:
   The performance is fair. Some candidates receive the full mark for this question. Most candidates mention PIK notes pay extra bonds instead of cash and poison puts prevent the hostile takeover.

   (i)   Pay-in-kind (PIK) notes
   -    Issuer pay with additional bonds rather than cash as coupons for PIK notes
   -    This allows a company to borrow more money without immediate concerns about cash flow.
   -    With PIKs, cash interest payments are deferred at the issuer's option until some future date.
   -    The issuer has the option to pay cash or in-kind interest payments.
   -    An investemnt in this type of bond, requires careful analysis of the issuers' cash-flow prospects and ability to survive.

   (ii)   Bonds with incorporated poison puts
   -    Poison puts thwart unfriendly takeovers by making the target company less attractive / more expensive to the acquirer.
   -    Poison put provides that the bondholder can require the company to repurchase the debt under certain circumstances such as a change in control.

(c)   Describe two primary types of syndication that Best Cosmetics may use for the bond issuance.
   Commentary on Question:
   The performance is very poor. None of the candidates receives full scores. Many candidates don't even attempt the question. Candidates who mention the keywords "underwritten deal", or "back-stop deal", receive partial scores.

   (i)
    - Underwritten deal
   - Transaction marketed on a "best-effort" basis.
   - Underwriter has no legal obligation to the issuer regarding completion of the transaction.
   - Common for first-time issuers without a proven cash flow records.

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11. **Continued**

(ii)
- Back-stop deal
- Underwriter agrees to purchase the deal at a maximum interest rate for a well-defined period of time.
- Similar to a bought deal, but timeframe is longer, generally up to a week.

(d)   Calculate the call price of the bond.

Commentary on Question:
The performance is poor. Only a few candidates receive the full marks. Candidates who merely list the correct cash flows by using correct discount rates would receive partial marks. Most candidate are able to calculate the initial price of the bond.

First, calculate the initial price of the bond.
YTM = 6%
semi-annual coupons of 5%/2 = 2.5%

Period (0.5 year)	Cash Flow	Discounted CF @ 3%
1	2.5	2.43
2	2.5	2.36
3	2.5	2.29
4	2.5	2.22
5	2.5	2.16
6	102.5	85.84
Price		97.29

Second, calculate the call price, X by setting up the PV of cash flows to equal to initial bond price.
Since the bond is called at the end of year 2, there are only 4 cash flows.

Period (0.5 year)	Cash Flow	Discounted CF @ 3.5%
1	2.5	2.42
2	2.5	2.33
3	2.5	2.25
4	2.5 + X	$(2.5 + X) * 1.035^{-4}$
Equal to initial price		97.29

$2.42 + 2.33 + 2.25 + (2.5 + X) (1.035)^{-4} = 97.29$
Call price, X = $101.11

12.

Learning Objectives:

5. The candidate will understand and identify the variety of fixed instruments available for portfolio management. This section deals with fixed income securities. As the name implies the cash flow is often predictable, however there are various risks that affect cash flows of these instruments. In general the candidates should be able to identify the cash flow pattern and the factors affecting cash flow for commonly available fixed income securities. Candidates should also be comfortable using various interest rate risk quantification measures in the valuation and managing of investment portfolios. Candidates should also understand various strategies of managing the portfolio against given benchmark.

Learning Outcomes:

(5b) Describe the cash flow of various corporate bonds considering underlying risks such as interest rate, credit and event risk.

(5d) Demonstrate understanding of cash flow pattern and underlying drivers and risks of mortgage-backed securities, and commercial mortgage-backed securities.

Sources:
Fabozzi, Chapter 12, 24, 31, 32

Commentary on Question:
This question is a bit long in length, it appears that candidates do not have enough time, especially for part (b), which is a calculation question. In general, some candidates do relatively well on parts (a) and (c).

Solution:

(a)

(i) List the two principal risks inherent in a pool of residential mortgages that non-agency RMBS deals are structured to manage.

(ii) Explain how non-agency RMBS deals are structured to manage these two risks.

(iii) Compare non-agency RMBS and CMBS based on 3 key attributes.

Commentary on Question:
This question only has 2 points out of a total of 6 points.. Candidates should allocate their time accordingly. Some candidates spent too much time writing overdetailed responses.

12. **Continued**

- The two risks are: prepayment risk and credit risk
- The structure of nonagency RMBS deals separate cash flows from the udnerlying loans into principal and interest and redistributes the cash flows and associated losses to individual tranches according to rules specified in the deal prospectus (i.e. waterfall)
- Prepayment risk: Nonagency RMBS manage prepayment exposure and average life variability through time-tranching the cash flows. Each senior tranche has a different expected maturity and expected time window for principla repayment.
- Credit risks in nonagency RMBS deals were addressed through credit enhancements: external, internal through deal structure, or both.
- Collaterall Composition: collateral for non-agency RMBS is fairly **diversified**. It composes of large number of loans in a transaction. Loans are small in size and individual loans represent a small portion of the transaction.
- Collateral composition for CMBS is more **concentrated**. Larger sized loans with fewer loans in the trust – underlying collateral is more heterogeneous. Topp 15 loans comprising 30-50% of the transaction.
- Loan Types: Nonagency RMBS consists of **15- to 30-year** fixed rate and floating rate loans. CMBS consists of **shorter-term loans**, with 5-,7- or 10-year balloon matruities. Most loans are fixed-rate.
- Prepayment: Nonagency RMBS constins mostly mortgage loans that are f**reely prepayable**, while CMBS contains mortgage loans that typically **prohibit prepayment** until a few months prior to the matruity date.
- Transaction Structure: Nonagency RMBS has **a wide range** of principal interest, and loss cash-flow **distribution rules**, in addition to triggers and **multiple forms of credit enhancements**. CMBS typically pay principal **sequentially**, credit enhancement is in the form of **subordintion only**, and losses are allocated in **reverse-sequential** order.

(b)   Calculate expected amount for the following at the beginning of the second month post issuance, assuming both default and prepayment take place at the end of the month and that default takes place after prepayment.

	Principal Payment	Interest Payment	Remaining Par Value
Senior tranche			
Junior tranche			

Commentary on Question:
Many candidates do not have sufficient time to complete the calculation of this part of the question. Candidates who demonstrate an understanding of required steps and concepts to complete this question receive partial scores. A sample full solution is provided below for completeness.

12. Continued

- Monthly interest=$1.045^{(1/12)}-1=0.367\%$
- Monthly interest payment=367,481
- Scheduled payment= 100m *0.367% *$(1+0.367\%)^{60}$/ $((1+0.367\%)^{60}$ -1) =1,860,202
- Scheduled principal payment=1,860,202-367,481=1,492,721
- CPR (conditional prepayment rate) = 0.3%
- SMM (single monthly mortality) =$1-(1-CPR)^{(1/12)}$ = 0.02503%
- Expected mortgage balance = 100m – 1,492,721 = 98,507,279
- Prepayment=SMM*(100m – 1,492,721) = 24,661
- Principal before default=100m-1,492,721-24,661=98,482,618
- Default rate=$1-(1-0.01)^{(1/12)}$=0.0837%
- Default loss=principal before default*0.0837*0.4=32,979
- Default recovery= principal before default*0.0837*0.6=49,468
- Demonstrated understanding that default loss hits junior tranche first.
- Senior tranche interest =$1.03^{(1/12)}-1$=0.247%
- Senior tranche interest payment=0.247%*20m=49,325
- Senior tranche principal payment =all principal payment + voluntary prepayment + recovery from default=1,566,850
- Junior tranche interest=Junior tranche interest payment=total interest payment-senior interest payment=318,156
- Senior Par=20m – all principal payment=18,433,150
- Junior Par=80m-defualt loss=79,967,021

	Principal Payment	Interest Payment	Remaining Par Value
Senior tranche	1,566,850	49,325	18,433,150
Junior tranche	0	318,156	79,967,021

(c)

    (i)   Explain the similarities in risk exposure between callable bonds and mortgage backed securities;

    (ii)   Define a make-whole call provision and explain why it is preferable to fixed-price call provision.

Commentary on Question:
Candidates do well in this part. Many candidates understand the definition of a make-whole call provision.

12. Continued

- In a low interest rate environment, a callable bond means that the issuer is incentivised to call back the bond to substitute new, lower-cost debt for older, higher-cost issues. This is similar to prepayment risk embedded in mortgage backed securities where borrowers of underlying loans are incentivised to refinance their mortgages.

- With macro-economy outlook being gloomy, callable bonds, like other corporate bonds, face higher **credit-risk,** particularly, credit spread risk. Credit spead widening can result in further reduction in bond prices. Mortgage backed securities also face higher credit risks since underlying loans are faced with higher delinquencies

- A make-whole call provision is where the call price is calculated as the **present value of the bond's remaining cash flows** subject to a floor price equal to par value. The discount rate used to determine the PV is the yieald on a comparable-maturity Treasury security plus a contractually specificied **make-whole call premium**.

- The make-whole call price is a floating price that is inversely related to the level of interest rates. The pure refunding motive is virtually eliminated, reducing the bondholder's risk exposure to declining interest rate.

13.

Learning Objectives:

8.   The candidate will understand the theory and techniques of portfolio asset allocation.

Learning Outcomes:

(8b)   Propose and critique asset allocation strategies.

(8c)   Evaluate the significance of liabilities in the allocation of assets.

Sources:

Managing Investment Portfolios, Maginn & Tuttle, 3rd Edition (Ch 5).

QFIC-111-16: Stop Playing with Your Optimizer.

Commentary on Question:

The question tests the mean-variance surplus optimization (MVSO) approach applied to the relation between the assets and the liabilities of a pension plan, its surplus, including calculations based on the expected return, the standard deviation of those rates, the correlation factors between the asset classes and the liabilities, and the definition of the efficient surplus frontier.

It also includes references to some management concepts that are important to consider in the choice of the asset allocation, in addition to pure mathematical aspects and results of applying the MVSO.

Solution:

(a)   Determine α and β and calculate σ_L.

Commentary on Question:
In general, candidates perform well in part a..

Expected value of $R_L = \alpha + \beta * 4.5\% + 0 = 5.0\%$

$\beta = 15/12 = 1.25$

$\alpha = 5.0 - 1.25 * 4.5 = -0.625$

$\sigma_L = (((15/12) *.08)\,^2+.02^2)\,^{.5} = (.1^2+.02^2)\,^{.5}=.102$

13. Continued

(b)

Asset Classes	Expected return on Surplus $E[R_{S,1}]$	Standard deviation of Surplus σ_S
Domestic Equities (DE)	5.75%	17.68%
International Equities (IE)	W	X
Long-Term Bonds (LTB)	Y	Z
Real Estate (RE)	-0.25%	15.41%

Calculate W, X, Y, and Z in the table above.

Commentary on Question:
See comments in part c).

Asset	Return	STD	weight	Correlation with LTB	Covariance with Liability	Return on Surplus	Variance of Surplus	STD of Surplus
DE	12	15	0.3	0.2	30	5.75	312.5	17.68
IE	10	12	0.3	0	0	3.75	306.5	17.51
LTB	4.5	8	0.2	1	80	-1.75	26.5	5.15
RE	6	10	0.2	0.1	10	-0.25	237.5	15.41
Portfolio	8.7	8.98			27	2.45	175.73	13.26

$$W = 10 - (\tfrac{250}{200}) * 5 = 3.75$$

$$X = (12^2 + (\tfrac{250}{200})^2 * 10.2^2 - 2*(\tfrac{250}{200})*0)^{\frac{1}{2}} = 17.51$$

$$Y = 4.5 - (\tfrac{250}{200}) * 5 = -1.75$$

$$Z = (8^2 + (\tfrac{250}{200})^2 * 10.2^2 - 2*(\tfrac{250}{200})*1*(\tfrac{15}{12})*8*8)^{\frac{1}{2}} = 5.15$$

(c) Identify the position of current asset portfolio of ABC's DB plan in the expected return on surplus $(E[R_{S,1}])$ and standard deviation of return on surplus (σ_S) space (Surplus Efficient Frontier) below to determine if the portfolio is efficient.

13. Continued

Commentary on Question:
The calculation of σ^2_s at the portfolio level and to a less extent the calculation of the standard deviation of surplus for International equities [X] and for Long-Term Bonds [Z] in part b) produce mixed and low results. In particular, the candidates have difficulties in calculating or using the appropriate correlation factors, and/or the correct return and standard deviation figures by class of assets.

This results in additional difficulties in positioning the portfolio in function of the efficient surplus frontier to confirm whether the portfolio was efficient or not.

We need to calculate both of expected return on surplus and STD of return on surplus for the portfolio.

For expected return on surplus:
$$R_{S,1} = \frac{S_1 - S_0}{A_0} = \frac{(A_1 - L_1) - (A_0 - L_0)}{A_0}$$

$A_1 = 200*.3*(1+0.12) +200*.3*(1+0.1) +200*.2*(1+.045) +200*.2*(1+.06) =217.4$

$L_1 = 250*(1+.05) =262.5$

$S_1 - S_0 = (217.4-262.5) -(200-250) =4.9$

$R_{S,1} = 4.9/200=2.45\%$

Other approaches can also be used, such as:

Weighted average of expected return on surplus:

$R_{s,1}= 0.3*5.75\% + 0.3*W + 0.2*Y + 0.2*-.25\%$, where W, and Y were calculated in part b).

Expected return on assets-Expected return on liabilities:

$R_{s,1}= 0.3*12\% + 0.3\%10\% +0.2*4.5\% +0.2*6\% - (250/200) *5\% = 2.45\%$

For STD of return on surplus:
$$\sigma_S^2 = \sigma_A^2 + \left(\frac{L_0}{A_0}\right)^2 \sigma_L^2 - 2\left(\frac{L_0}{A_0}\right)\rho_{A,L}\sigma_A\sigma_L$$

Since
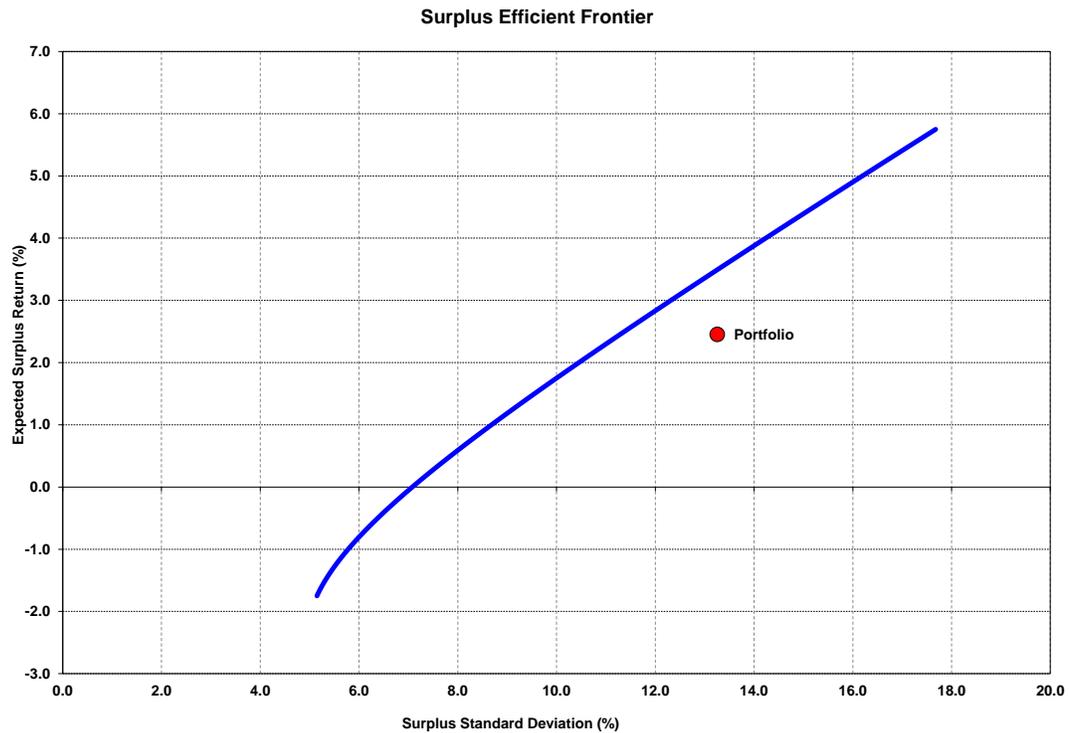$$\rho_{A,L}\sigma_A = .3*.2*.15 + .3*0*.12 + .2*1*.08 + .2*.1*.1 = .027$$
we have
$$\sigma_S^2 =.09^2 + \left(\frac{250}{200}\right)^2 *.102^2 - 2*\left(\frac{250}{200}\right)*.027*.102 = .0175 ==> \sigma_S = 13.23\%$$

13. Continued

Surplus Efficient Frontier:

The Portfolio is below the Efficient Frontier, so it is not efficient.

Surplus Efficient Frontier

(d)   Outline the limitations of the MVSO approach.

Commentary on Question:
Results are reasonable.

(1) MVSO approach is highly sensitive to a small change in input and estimation error.

(2) The most important inputs in MVSO are the expected surplus returns

(e)   Explain how optimal portfolio weights of ABC's DB plan asset portfolio will change under the Black-Litterman model.

Commentary on Question:
Results are reasonable.

13. Continued

(1) BL model starts with the equilibrium view that reflects the current market capitalization. Therefore, equilibrium returns are same as the given expected returns.

(2) BL model calculates the view-adjusted market equilibrium returns. According to the expressed views, view-adjusted return gap between RE and LTB will be wider than 1.5% since the view indicates a wider than existing gap in return between RE and LTB.

Similarly, view-adjusted return gap between DE and IE will be wider than 2% since the view indicates a wider than existing gap in return between DE and IE

(3) Run MVSO based on the view-adjusted market equilibrium returns, the resulting portfolio weight would be:

Asset Classes	Portfolio Weight (%)
Domestic Equities (D E)	>30
International Equities (I E)	<30
Long-Term Bonds (L T B)	<20
Real Estates (R E)	>20

(f) Describe additional considerations that may compensate relying on the MVSO approach in the asset portfolio allocation decision making process.

Commentary on Question:
Although the correct answer comes directly from the reading material (QFI-111-16), unfortunately no candidate makes a direct relation, even a partial relation, with this material.

It is an occasion to realize that interrelated aspects of a same question may come from another part of a material and it is a good occasion for the candidates to demonstrate that they have a global view and understanding of a subject or a topic.

(1) Lower tolerance for volatility may require smaller equity allocation.

(2) Higher percentage of young workers may enable greater equity allocation.

(3) Dividend tax credit on domestic dividends and withholding taxes on foreign dividends may encourage higher domestic equity allocation.

13. **Continued**

(4) Concentrating in countries with greater term or credit spreads vs. global diversification.

(5) Increased exposure to dimension of higher expected surplus return may result in periods of underperformance relative to the market.

14. **Learning Objectives:**
6.   The candidate will understand the variety of equity investments and strategies available for portfolio management.

Learning Outcomes:
(6b)   Demonstrate an understanding of the basic concepts surrounding passive, active, and semi active (enhanced index) equity investing, including managing exposures.

(6h)   Describe the core-satellite approach to portfolio construction with a completeness fund to control overall risk exposures.

(6i)   Explain alpha and beta separation as an approach to active management and demonstrate the use of portable alpha.

Sources:
Managing Investment Portfolios: Chapter 7, Equity Portfolio Management

Commentary on Question:
Candidates generally performed well on the problem. A significant portion of the candidates simply wrote the solution in numerical form without explaining anything. If a mistake was made in this situation there was no basis for granting partial credit.

Solution:
(a)   Calculate the information ratio of the XYZ equity portfolio.

Commentary on Question:
The most common mistake was the improper calculation of the tracking risk forgetting to square the weights in the formula.

The information ratio, by definition, is equal to the expected alpha divided by the tracking risk.

Expected Alpha Portfolio = Sum (Allocation * Expected Alpha Manager)

Tracking risk is equal to square root of sum of (Allocation squared times Tracking Risk squared)

14. Continued

The following chart shows the necessary calculations:

	Tracking Risk	Exp (Active Returns)	Alloc	Exp Alpha	square exp Tracking
Index	0.0%	0.0%	20.0%	0.00%	0.000%
Semiactive	2.0%	1.5%	40.0%	0.60%	0.006%
Active Manager A	3.0%	2.0%	20.0%	0.40%	0.004%
Active Manager B	5.0%	3.0%	10.0%	0.30%	0.003%
Active Manager C	8.0%	4.0%	10.0%	0.40%	0.006%
Sum	Port Exp α = Sum(Alloc * Exp α)			1.700%	0.019%
Sq Rt	Port Track Risk = Sq Rt (Sum Sq of (Alloc*Track Risk))				1.375%
Info Ratio = Port Exp Alpha / Port Tracking Risk				1.24	

(b) Evaluate whether the reallocation is justifiable as an alternative active strategy.

Commentary on Question:
Stating that Manager C was on the efficient frontier was not an acceptable answer as there was not enough information to calculate a Sharpe Ratio.

To evaluate whether the reallocation is justifiable, the information ratio for the alternative portfolio can be calculated as follows:

	Tracking Risk	Exp(Active Returns)	Alloc	Exp Alpha	square exp Tracking
Index	0.0%	0.0%	60.0%	0.00%	0.000%
Semiactive	2.0%	1.5%	0.0%	0.00%	0.000%
Active Manager A	3.0%	2.0%	0.0%	0.00%	0.000%
Active Manager B	5.0%	3.0%	0.0%	0.00%	0.000%
Active Manager C	8.0%	4.0%	40.0%	1.60%	0.102%
Sum				1.600%	0.102%
Sq Rt					3.200%
Info Ratio = Port Exp Alpha / Port Tracking Risk				0.50	

Given the decrease in the information ratio, this implies that the new strategy would not be justifiable.

(c) Characterize the structure of the portfolio of managers in parts (a) and (b), respectively.

Commentary on Question:
The most common issue that candidates have is to recognize that the semi-active strategy is part of the core for a Core Satellite portfolio.

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14. Continued

Both portfolios are core-satellite Portfolios because both have more than 50% allocated in Index and Semi-Active strategies.

Portfolio in part (a) has 3 satellites for Manager A, B and C.
The portfolio in part (b) just has a single satellite, and with a growth focus.

(d)

(i) Calculate Manager B's "true" active return

(ii) Calculate Manager C's "misfit" return

Commentary on Question:
Full credit required having a definition of the formula for "True" Active and "Misfit" return.

"True" Active Return = Manager Return less than the Normal Manager Benchmark.

"Misfit" Return = Normal Manager Benchmark less than Portfolio Benchmark.

"True" Active return (Manager B) = (Exp Alpa + Index) – Manager Benchmark = (10%+3%) – 14% = -1%

Misfit Return (Manager C) = 15% - 10% = 5%

(e) Describe one advantage and one disadvantage of using the Completeness Fund approach to build a portfolio of multiple active managers.

Commentary on Question:
Better answers to this question include a description of the Completeness Fund approach.

The advantage of a completeness fund approach is to capture the stock selecting ability of active managers, while making the overall portfolio risk profile like its benchmark.

The disadvantage of the Completeness Fund approach is that it essentially seeks to eliminate misfit risk, but a non-zero amount of misfit risk may be optimal.

15.

Learning Objectives:

6.   The candidate will understand the variety of equity investments and strategies available for portfolio management.

Learning Outcomes:

(6a)   Explain the nature and role of equity investments within portfolios that may include other asset classes.

(6c)   explain the basic active equity selection strategies including value, growth and combination approaches.

(6d)   Demonstrate an understanding of equity indices and their construction, including distinguishing among the weighting schemes and their biases.

(6f)   compare techniques for characterizing investment style of an investor.

Sources:
Managing Investment Portfolios (Chapter 7)

Commentary on Question:
This question is to test candidate's understanding of equity indices and its construction, including weighting schemes and bias within each weighting scheme. Candidates are expected to understand different investment styles and the associated characteristics.

Solution:

(a)   Describe four choices that determine an equity index's characteristics.

Commentary on Question:
Majority of the candidates can list the 4 characteristics, but only a few can provide detailed description and thus obtain full credits.

-Boundaries of the index's universe: this is important in determining how well the index represent a specific population of stocks. The greater number of stocks in the index, the better the index will measure broad market performance.

- Criteria for inclusion in the index: this establishes characteristics desired for stocks to be selected for the index.

- How stock index is weighted: 3 weighting: pricing weighting, value weighting, or equal weighting.

- How return is calculated: return could be measured as price only return, or total return that includes reinvestment and dividend.

15. Continued

(b)

  (i)   Calculate the 1-year price-only return as of December 2017 under the price weighting, value weighting, float weighting, and equal weighting.

  (ii) Describe the potential bias under each weighting method and recommend the appropriate index weighting method.

Commentary on Question:
For (i), majority of the candidates can perform the return calculation for at least one or two of the weighting methods. Calculation is relatively simple and straight forward.

For (ii), most candidates can describe the biases for each weighting, but only a few of them can describe why the float weighting is the optimal choice. Credit is also given to candidates with other preferred weightings, if candidates can describe the reason properly.

(i)
- Price weighting: V0 = 12+8+10+40=70, V1 = 20+15+10+10 = 55, so the 1-year only return = (55-70)/70-1 =-21.43%.

- Value weighting: V0 = 12x100+8*50++10x10+40x20=2500. V1 = 20x100+15x50++10x10+10x20=3050,
1-year return = (3050-2500)/2500-1 = 22%.
- Equal weighting: 20/12 = 1.667, 15/8 = 1.875, ,10/10=1, 10/40 = 0.25.
(1.667+1.875+1+0.25)/ (1+1+1+1)-1=19.8%.
-Float weighting: V0 = 12x100+8x50x1+10x10x0.5+40x20x0.5=2050,
V1=20x100+15x50x1++10x10x0.5+10X20x0.5=2900, 1-year return = (2900-2050)/2050-1=41.5%.
(ii)
- Price Weighting: Bias toward the company with the highest share price (Company E).
- Value weighting: Bias toward the company with largest market capitalization (Company A).

- Equal weighting: Bias toward small company (Company B).
- Recommend using the Float weighting because it reflects the average dollar invested in the 5 companies and reflects the supply of shares available in the market place.

Note that credit is granted to candidate for answer of other weighting, if the reason is appropriate. For example, for Value weighting, large cap companies are frequently traded in the market and thus there is less trading cost.

15. Continued

(c)   Identify the fund manager's investment style and comment on the performance.

Commentary on Question:
Most candidates can identify the investment style with appropriate justification. But none can describe that the return is only on price-only.

Fund manager is following Growth Investment Style.
The portfolio's P/E, P/B are higher than benchmark.
Dividend is lower than the benchmark.
EPS growth is higher than benchmark.
Sectors are focusing on IT/Health non-traditional sectors, which reflects a growth investment style.

Fund manager's active return is 4.5% (46 %-41.5% =4.5%), which indicates fund manager's positive contribution in managing the portfolio.
However, the active return is calculated on a price-only base. The fund manager's performance is lower when considering other return such as dividend (1% lower than the benchmark).

16.

Learning Objectives:
7.   The candidate will understand how to develop an investment policy including governance for institutional investors and financial intermediaries.

Learning Outcomes:
(7a)   Explain how investment policies and strategies can manage risk and create value.

(7c)   Determine how a client's objectives, needs and constraints affect investment strategy and portfolio construction. Include capital, funding objectives, risk appetite and risk-return trade-off, tax, accounting considerations and constraints such as regulators, rating agencies, and liquidity.

(7d)   Incorporate financial and non-financial risks into an investment policy, including currency, credit, spread, liquidity, interest rate, equity, insurance product, operational, legal and political risks.

Sources:
A Dynamic Process, Maginn & Tuttle, 3rd Edition

The Portfolio Management Process and the Investment Policy Statement (Chapter 1)

Managing Institutional Investor Portfolios (Chapter 3)

Commentary on Question:
This question is to test candidates' fundamental understanding in asset liability management. This includes risk concern and return objective in asset management; investment policy concerns such as liquidity, interest rate and tax issues.

Solution:
(a)   Assess the impact of the above information on the risk objectives of XYZ with respect to the following considerations:

- Valuation concerns
- Reinvestment risk
- Cash-flow volatility

Commentary on Question:
In general candidates did not do well in this part. Some candidates do not understand what valuation concern is: the potential impact on surplus of current asset and liability holdings. Same issue for reinvestment risk and cashflow volatility. Few candidates mentioned potential impact of each issue on risk tolerance of the company, as the question asked for risk objectives in the context of IPS.

16. Continued

For part a(i), candidates received full credit if they recognized XYZ's asset liability duration mismatch and its potential impact on surplus; Candidates were also given partial credit if they recognized potential impact on liability valuation of low interest rate environment and/or of increase in annuity crediting rates.

For part a(ii), candidates received full credit if they recognized the need to reinvest coupon, maturing principle, or prepayments in low interest rate environment. Candidates also received full credit if they explained how low interest rate could impact XYZ's earned spreads.

For part a(iii), candidates received full credit if they discussed two out of the three related issues: 1. Asset CFs volatility as outlined in the model solution; 2. Liability CFs volatility as outlined in the model solution. 3. Impact on the risk tolerance.

(i) **Valuation concerns**:

Asset duration is shorter than Liability duration.

When interest rate decreases, the mismatch between the asset and liability duration can lead to liability increases more than asset. This could cause erosion of surplus and potential insolvency due to small amount of surplus for capital.

The valuation concerns should limit XYZ's risk tolerance.

(ii) **Reinvestment Risk**:

XYZ faces the reinvestment risk in that it has to reinvestment coupon income and mature principal, or other bond-liked assets at lower rates due to the significant decline in the interest rate.

For annuity contracts, cash flows are reinvested at a lower rate leading less investment income. At the same time, due to competition, the crediting rates are set higher. The combination of lower investment return and higher crediting rates could lead to a decrease of interest spread and can jeopardize the profitability of these contracts.

Manaing and controlling the reinvestment risk is an important risk objective.

(iii) **Cash-flow volatility**:

Cash-flow volatility arises from asset cashflow and liability cashflow volatilities:

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16. Continued

       Asset CFs: lower interest rate generates lower investment return; lower tax creates tax saving and increases after tax net income; Both of them increase the asset CFs volatility.

       Liability CFs: To support business expansion, there are potential more expense and commission related marketing outflows; prepayment and refinance may increase due to the lower interest rate for the mortgage loan investment due to decrease of interest rate; higher cash outflow for the annuity products due to higher crediting rates. All of these increase the liability CFs volatility.

       In addition, duration mis-match causes asset and liability changes at different magnitude, which increases the cashflow volatility.

       The increase of asset and liability cashflow volatility limits XYZ's risk objective.

(b)   Recommend return objectives for XYZ in response to the recent developments.

Commentary on Question:
In general candidates can answer the objective of generating enough return to back liability and maintain surplus growth, but not many candidates mention about the need to maintain the margin for the interest-sensitive products.

The credit was split into two halves:

1. Half of the total credit is given if candidates recognized the need to earn higher return to support higher credited rate on annuities (or in general to meet liability obligations).

2. Half of the total credit is given if the candidates acknowledged company's objective to grow surplus.

XYZ must establish higher return objectives for its investment portfolio so as to maintain an adequate margin between its investment portfolio return and the recently increased rates of return being credited to its interest-rate-sensitive products.

Also, XYZ must establish higher return objectives for its investment portfolio to back its liability, grow surplus and support expanding business volume.

16. Continued

(c)   Recommend two ways to achieve the return objectives of XYZ that you recommended in part (b).

Commentary on Question:
Answers are open, and credits are given to candidates with reasonable responses. In general, most candidates get half grading points, only a portion of candidates get full credits.

For part c(i), any of the answers listed below has received full credit.

(i)   To maintain margin spread for the interest-sensitive product:

- Segmentation: XYZ can choose to establish sub-portfolio return objectives to promote competitive crediting rates for groups of contracts and XYZ IPS can incorporate multiple return objectives.

- Hedge the interest rate such as interest swap, interest rate floor and other interest derivatives to hedge the interest rate movement.

- Asset Liability management to monitor and manage the interest-rate sensitive business block.

For part c(ii), any of the answers listed below received full credit. In general, any reasonable technique to increase earned yield received full credit.

(ii)   To grow surplus and support expanding business volume:

- XYZ may consider investing in liquid common stocks, equity investment in real estate, and private equity.

- XYZ can also consider capital appreciation strategies, such as financial leverage, to help achieve surplus growth.

- Increase the duration of the asset income assets to generate higher yield.

Credits are also given to candidates if they answer: Active/Semi-active manager approach to gain higher return; Core-satellite fund management approach to gain alpha; Contingent immunization approach; Surplus efficient frontier to maximum growth.

16. Continued

(d)   Assess XYZ's liquidity requirements with respect to asset marketability risk.

Commentary on Question:
In general, candidates did poorly on this question. Most candidates answered the importance of liquidity (from liability side) but did not answer the question on the marketability of the asset. Recognizing either side received half credit for each.

The company may face need to liquidate assets in order to fund expected and unexpected cashflows (e.g. due to the recent or future market condition changes). The marketability of investment is important to ensure ample liquidity.

Due to a small portion of assets being invested in less liquid investment such as private placement bonds and commercial mortgage loans, XYZ has exposure to risk of not being able to exit from these investments in order to meet cash flow requirement.

(e)   Assess XYZ's tax concerns.

Commentary on Question:
Most of the candidates did not answer this question well. Question was asking for concern, but most candidates only answered the advantage of the tax cut change. Each answer listed below earned one fourth of the total credit.

- XYZ is tax-paying entity and subject to income, capital gains, and other types of taxes in the U.S.

- The types and application of tax differ by county, but in all cases, taxes mean that insurance company must focus on after-tax return in their investment activities.

- XYZ investment income can be divided into two parts for tax purposes: the policyholder's share and the corporate share (the balance that is transferred to surplus). Only the latter portion is taxed.

- The recent change in corporate tax rate from 35% to 21% might have an impact on the deferral of taxes on the XYZ's accumulation of cash values of its life and annuity contracts. The IPS must consider the impact of the tax rate change on its IPS.

- Tax-exempt assets may be less appealing due to the lower tax rate.